ON ABSOLUTE RIESZ SUMMABILITY FACTORS (II)

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1. In this note we give conditions for a series $\sum a_n\epsilon_n$ to be summable $|R,\lambda,\kappa|$ whenever $\sum a_n$ is bounded (R,λ,κ), $\kappa\geqq 0$. The case κ a positive integer or zero was dealt with by the author in a recent note [9]. Our object is to consider the truth of the theorem in [9] when κ is non-integral. The same conditions are required whether κ is an integer or not. As usual the proof for non-integral orders is much harder than for integral orders.

Our theorem generalizes theorems on absolute Cesàro summability factors (Chow [6], Ahmad [1]). Also the result is closely related to theorems on the abscissae of summability (R,λ,κ), $|R,\lambda,\kappa|$ of the Dirichlet series $\sum a_n e^{-\lambda_n s}$ (Hardy and Riesz [7], Obrechkoff [10], Bosanquet [4], Austin [2], Borwein [3]).

2. Let $\lambda=\{\lambda_n\}$ be an increasing unbounded sequence of positive numbers. We write $\lambda\in\Lambda$ if λ satisfies

(a) $$0<a\leqq\frac{\Delta\lambda_n}{\Delta\lambda_{n-1}}\leqq A,\qquad a, A \text{ constants,}$$

(b) $$\frac{\lambda_{n+1}}{\lambda_n}\quad\text{decreases to 1.}$$

In a certain sense the set of sequences Λ consists of increasing sequences which are 'reasonably regular'. By $\Delta\lambda_n$ we mean $\lambda_n-\lambda_{n+1}$.

For $\kappa>-1$ we define

$$A^\kappa(\omega)=\sum_{\lambda_\nu<\omega}(\omega-\lambda_\nu)^\kappa a_\nu=\int_0^\omega(\omega-t)^\kappa\,dA(t),$$

where $A(t)=A^0(t)$. Similarly for $B^\kappa(\omega)$. If $\omega^{-\kappa}A^\kappa(\omega)$ is bounded (of bounded variation) over (λ_0,∞) we say $\sum a_n$ is bounded (absolutely summable) (R,λ,κ). In the latter case it is usual to say $\sum a_n$ is summable $|R,\lambda,\kappa|$. When $\mu>0$, $\kappa>-1$, $\kappa+\mu>0$,

$$A^{\kappa+\mu}(\omega)=\frac{\Gamma(\kappa+\mu+1)}{\Gamma(\kappa+1)\Gamma(\mu)}\int_0^\omega(\omega-t)^{\mu-1}A^\kappa(t)\,dt.$$

Hence if $b_\nu = \lambda_\nu a_\nu$, $\kappa \geqq 0$ and $A^\kappa(\omega) = O(\omega^\kappa)$, then $B^\kappa(\omega) = O(\omega^{\kappa+1})$.

3. Some lemmas will be needed.

LEMMA 1. *If* $0 < \mu \leqq 1$, $\kappa \geqq 0$, $0 \leqq \xi \leqq \omega$, *then*

$$\frac{\Gamma(\kappa+\mu+1)}{\Gamma(\kappa+1)\Gamma(\mu)} \left| \int_0^\xi (\omega - t)^{\mu-1} A^\kappa(t) dt \right| \leqq \max_{0 \leqq t \leqq \xi} \left| A^{\mu+\kappa}(t) \right| .$$

See Hardy and Riesz [7], Lemma 8. We shall refer to this lemma as 'the Riesz mean-value theorem'.

LEMMA 2. *If* $\kappa \geqq 0$, $\kappa + q \geqq 0$, $A^\kappa(\omega) = O(\omega^{\kappa+q})$, *then, for* $\mu = 0, 1, [\kappa]$ *and* $\lambda_n < \omega \leqq \lambda_{n+1}$,

$$A^\mu(\omega) = O\{\omega^\kappa \lambda_n^q \Lambda_n^{\kappa-\mu}\}, \qquad \textit{where } \Lambda_n = \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_n} .$$

See Borwein [3], Lemma 2.

LEMMA 3. *Let* $\Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n)$ *be increasing,* $\kappa \geqq 0, q \geqq 0$ *and* $A^\kappa(\omega) = O(\omega^{\kappa+q})$. *Then for* $\lambda_n < \omega \leqq \lambda_{n+1}$, $0 \leqq \mu < \kappa$,

$$A^\mu(\omega) = O\{\omega^{\mu+q} \Lambda_n^{\kappa-\mu}\}.$$

This follows from [5], Theorem 1.61, on taking $\phi(\omega) = \omega^{\kappa+q}$, and noting that $\lambda_{n+1} = O(\lambda_n)$.

LEMMA 4. *Let* $\kappa > 0$, $\lambda \in \Lambda$, $\epsilon_n = G_\kappa(\lambda_n)$, *then for* $\lambda_n < \omega \leqq \lambda_{n+1}$,

$$G_\kappa(\omega) \equiv \int_\omega^\infty (u - \omega)^\kappa dg(u)$$

$$= O(1) \int_{\lambda_n}^{\lambda_{n+[\kappa]+1}} u^\kappa |dg(u)| + O(1) \sum_{\nu=0}^{[\kappa]} |\epsilon_{n+\nu}|$$

$$+ O(1)(\lambda_{n+1} - \lambda_n)^{[\kappa]+1} \int_{\lambda_{n+[\kappa]+1}}^\infty (u - \lambda_n)^{\kappa-[\kappa]-1} |dg(u)| \tag{1}$$

where we suppose that

$$\int_{\lambda_0}^\infty u^\kappa |dg(u)| < \infty.$$

If κ *is an integer, the final integral in* (1) *may be omitted.*

This lemma follows from the proof of Lemma 9 (Maddox [8]). For completeness we outline the method. It is possible to determine functions $c_\nu(\omega) = O(1)$ such that

$$(u-\omega)^{\kappa}=\sum_{\nu=0}^{[\kappa]} c_{\nu}(\omega)\,(u-\lambda_{n+\nu})^{\kappa}+O\{(\lambda_{n+1}-\lambda_{n})^{[\kappa]+1}(u-\lambda_{n})^{\kappa-[\kappa]-1}\}$$

uniformly for $u \geqq \lambda_{n+[\kappa]+1}$. This is equation (7) Maddox [8].

We now have

$$G_{\kappa}(\omega)=\int_{\omega}^{\lambda_{n+[\kappa]+1}}(u-\omega)^{\kappa}dg(u)+\sum_{\nu=0}^{[\kappa]} c_{\nu}(\omega)\left\{\varepsilon_{n+\nu}-\int_{\lambda_{n+\nu}}^{\lambda_{n+[\kappa]+1}}(u-\lambda_{n+\nu})^{\kappa}dg(u)\right\}$$

$$+O(1)(\lambda_{n+1}-\lambda_{n})^{[\kappa]+1}\int_{\lambda_{n+[\kappa]+1}}^{\infty}(u-\lambda_{n})^{\kappa-[\kappa]-1}|dg(u)|.$$

The result follows.

LEMMA 5. *If* $\kappa>1$ *and non-integral,* $\mu=\kappa-1$,

$$J=\int_{0}^{\omega}(\omega-t)^{\kappa-1}(u-t)^{\kappa}dB(t),$$

$$H(\alpha,\beta,\gamma)\equiv H(\alpha,\beta,\gamma;\omega,u)=\int_{0}^{\omega}(\omega-t)^{\alpha}(u-t)^{\beta}B^{\gamma}(t)dt,$$

then

$$J=\sum_{\nu=0}^{[\mu]} cH(\mu-\nu,\ \kappa-p-1+\nu,\ p)$$

$$+\sum_{\nu=[\mu]}^{q} c\,H(\mu-[\mu]-1,\ \kappa-p+\nu,\ p-\nu+[\mu])\equiv J_{1}+J_{2},$$

where p *is the integer such that* $0<\kappa-p\leqq 1$, *and* c *denotes a non-zero constant, possibly different at each occurrence.*

This lemma is established by suitable partial integrations; see Maddox [8], equation (38).

4. We now prove

THEOREM 1. *If* $\kappa\geqq 0$, $\lambda\in\Lambda$, $A^{\kappa}(\omega)=O(\omega^{\kappa})$ *and*

(i) $\sum|\epsilon_{n}|<\infty$,

(ii) *there exists a function* $g(u)$, *defined for* $u\geqq\lambda_{0}$, *such that for* $\nu=0,1,\ldots\ldots,$

$$\epsilon_{\nu}=\int_{\lambda_{\nu}}^{\infty}(u-\lambda_{\nu})^{\kappa}\,dg(u)\qquad \textit{with}\ \int_{\lambda_{0}}^{\infty}u^{\kappa}|dg(u)|<\infty,$$

then $\sum a_{n}\epsilon_{n}$ *is summable* $|R,\lambda,\kappa|$.

PROOF. When $\kappa=0$ the result follows trivially from (i). Now consider two cases.

CASE 1. $0 < \kappa < 1$. We have

$$I = \kappa \int_{\lambda_0}^{\infty} \omega^{-\kappa-1} d\omega \left| \sum_{\lambda_\nu < \omega} (\omega - \lambda_\nu)^{\kappa-1} \lambda_\nu a_\nu \varepsilon_\nu \right|$$

$$= \kappa \int_{\lambda_0}^{\infty} \omega^{-\kappa-1} \, d\omega \left| \int_0^{\omega} (\omega - t)^{\kappa-1} G_\kappa(t) \, dB(t) \right|, \tag{2}$$

where $G_\kappa(t) = \int_t^\infty (u - t)^\kappa \, dg(u)$, $B(t) = \sum_{\lambda_\nu < t} \lambda_\nu a_\nu$.

We note that

$$G_\kappa'(t) = - \kappa \int_t^\infty (u - t)^{\kappa-1} \, dg(u) \text{ p. p. in } (\lambda_0, \infty). \tag{3}$$

Integrating the inner integral in (2) by parts, we have for $\lambda_n < \omega \leqq \lambda_{n+1}$,

$$(\omega - \lambda_n)^{\kappa-1} \epsilon_n B(\lambda_n) + c \int_0^{\lambda n} (\omega - t)^{\kappa-2} G_\kappa(t) B(t) \, dt + c \int_0^{\lambda n} (\omega - t)^{\kappa-1} \, G_\kappa'(t) B(t) dt. \tag{4}$$

By Lemma 2, since $B^\kappa(\omega) = O(\omega^{\kappa+1})$, we have $B(\lambda_n) = O(\lambda_n \Lambda_n^\kappa)$.

Thus

$$\int_{\lambda_0}^{\infty} \omega^{-\kappa-1} (\omega - \lambda_n)^{\kappa-1} | \epsilon_n B(\lambda_n) | \, d\omega$$

$$= \sum_{n=0}^{\infty} \int_{\lambda n}^{\lambda n+1} \omega^{-\kappa-1} (\omega - \lambda_n)^{\kappa-1} | \epsilon_n | O(\lambda_n \Lambda_n^\kappa) \, d\omega$$

$$= O(1) \sum_{n=0}^{\infty} | \epsilon_n | \lambda_n \Lambda_n^\kappa \int_{\lambda n}^{\lambda n+1} \omega^{-\kappa-1} (\omega - \lambda_n)^{\kappa-1} \, d\omega$$

$$= O(1) \sum_{n=0}^{\infty} | \epsilon_n | < \infty. \tag{5}$$

Consider the contribution of the second term in (4) to I:

$$\int_{\lambda_0}^{\infty} \omega^{-\kappa-1} \, d\omega \left| \int_0^{\lambda n} (\omega - t)^{\kappa-2} G_\kappa(t) B(t) \, dt \right|$$

$$\leqq \sum_{n=1}^{\infty} \int_{\lambda n}^{\lambda n+1} \omega^{-\kappa-1} \, d\omega \sum_{\nu=0}^{n-1} \int_{\lambda \nu}^{\lambda \nu+1} (\omega - t)^{\kappa-2} | G_\kappa(t) B(t) | \, dt$$

$$= \sum_{\nu=0}^{\infty} \sum_{n=\nu+1}^{\infty} \int_{\lambda n}^{\lambda n+1} \omega^{-\kappa-1} \, d\omega \int_{\lambda \nu}^{\lambda \nu+1} (\omega - t)^{\kappa-2} | G_\kappa(t) B(t) | \, dt$$

$$= \sum_{\nu=0}^{\infty} \int_{\lambda\nu+1}^{\infty} \omega^{-\kappa-1}\, d\omega \int_{\lambda\nu}^{\lambda\nu+1} (\omega - t)^{\kappa-2} |G_\kappa(t) B(t)|\, dt$$

$$\leqq \sum_{\nu=0}^{\infty} \lambda_{\nu+1}^{-\kappa-1} \int_{\lambda\nu}^{\lambda\nu+1} |G_\kappa(t)|\, O(\lambda_\nu \Lambda_\nu^\kappa)(\lambda_{\kappa+1} - t)^{\kappa-1} dt,$$

by Lemma 2.

Now by Lemma 4, since $[\kappa] = 0$, we have for $\lambda_\nu < t < \lambda_{\nu+1}$,

$$G_\kappa(t) = O(1) \int_{\lambda\nu}^{\lambda\nu+1} u^\kappa |dg(u)| + O(|\epsilon_\nu|) + O(1)(\lambda_{\nu+1} - \lambda_\nu) \int_{\lambda\nu+1}^{\infty} (u - \lambda_\nu)^{\kappa-1} |dg(u)|.$$

Hence

$$O(1) \sum_{\nu=0}^{\infty} \lambda_{\nu+1}^{-\kappa-1} \lambda_\nu \Lambda_\nu^\kappa \left(\int_{\lambda\nu}^{\lambda\nu+1} u^\kappa |dg(u)| + |\epsilon_\nu| \right) \int_{\lambda\nu}^{\lambda\nu+1} (\lambda_{\nu+1} - t)^{\kappa-1} dt$$

$$= O(1) \sum_{\nu=0}^{\infty} \lambda_{\nu+1}^{-1} \lambda_\nu \left(\int_{\lambda\nu}^{\lambda\nu+1} u^\kappa |dg(u)| + |\epsilon_\nu| \right)$$

$$= O(1) \left(\int_{\lambda_0}^{\infty} u^\kappa |dg(u)| + \sum_{\nu=0}^{\infty} |\epsilon_\nu| \right) < \infty. \tag{6}$$

Also

$$O(1) \sum_{\nu=0}^{\infty} \lambda_{\nu+1}^{-\kappa-} \lambda_\nu \Lambda_\nu^\kappa (\lambda_{\nu+1} - \lambda_\nu) \int_{\lambda\nu+1}^{\infty} (u - \lambda_\nu)^{\kappa-1} |dg(u)| \int_{\lambda\nu}^{\lambda\nu+1} (\lambda_{\nu+1} - t)^{\kappa-1} dt$$

$$= O(1) \sum_{\nu=0}^{\infty} (\lambda_{\nu+1} - \lambda_\nu) \int_{\lambda\nu+1}^{\infty} (u - \lambda_\nu)^{\kappa-1} |dg(u)|$$

$$= O(1) \sum_{\nu=0}^{\infty} \int_{\lambda\nu}^{\lambda\nu+1} dt \int_t^{\infty} (u - t)^{\kappa-1} |dg(u)|$$

$$= O(1) \int_{\lambda_0}^{\infty} dt \int_t^{\infty} (u - t)^{\kappa-1} |dg(u)|$$

$$= O(1) \int_{\lambda_0}^{\infty} u^\kappa |dg(u)| < \infty. \tag{7}$$

Finally, consider the contribution of the third term in (4) to I. By (3) we have

$$c \int_0^{\lambda n} (\omega - t)^{\kappa-1} B(t)\, dt \int_t^{\infty} (u - t)^{\kappa-1}\, dg(u)$$

$$= c \int_0^{\lambda n} (\omega - t)^{\kappa-1} B(t) dt \left(\int_t^{\omega} + \int_{\omega}^{\infty} \right)$$

$$= c\int_0^{\omega} dg(u)\int_0^{m} (\omega - t)^{\kappa-1}(u - t)^{\kappa-1}B(t)dt$$

$$+ c\int_{\omega}^{\infty} dg(u)\int_0^{\lambda_n} (\omega - t)^{\kappa-1}(u - t)^{\kappa-1}B(t)dt. \tag{8}$$

where $m = \min(\lambda_n, u)$. Now $(\omega - t)^{\kappa-1}$ increases in $(0, m)$.

Hence, by the second mean-value theorem and the Riesz mean-value theorem,

$$\int_0^{m} (\omega - t)^{\kappa-1}(u - t)^{\kappa-1}B(t)dt = (\omega - m)^{\kappa-1}\int_{\xi}^{m} (u - t)^{\kappa-1}B(t)dt \qquad (0 \leqq \xi \leqq m)$$

$$= O((\omega - u)^{\kappa-1}u^{\kappa+1}). \tag{9}$$

Also $(u - t)^{\kappa-1}$ increases in $(0, \lambda_n)$, so that

$$\int_0^{\lambda_n} (\omega - t)^{\kappa-1}(u - t)^{\kappa-1}B(t)dt = (u - \lambda_n)^{\kappa-1}\int_{\xi}^{\lambda_n} (\omega - t)^{\kappa-1}\, B(t)dt \qquad (0 \leqq \xi \leqq \lambda_n)$$

$$= O((u - \omega)^{\kappa-1}\omega^{\kappa+1}). \tag{10}$$

By (8) and (9) it follows that

$$O(1)\int_{\lambda_0}^{\infty} \omega^{-\kappa-1}\, d\omega\int_0^{\omega} (\omega - u)^{\kappa-1}u^{\kappa+1}\, |dg(u)|$$

$$= O(1)\int_{\lambda_0}^{\infty} u^{\kappa+1}\, |dg(u)|\int_u^{\infty} \omega^{-\kappa-1}(\omega - u)^{\kappa-1}\, d\omega$$

$$= O(1)\int_{\lambda_0}^{\infty} u^{\kappa}|dg(u)| < \infty. \tag{11}$$

Also (8) and (10) yield

$$O(1)\int_{\lambda_0}^{\infty} \omega^{-\kappa-1}\, d\omega\int_{\omega}^{\infty} (u - \omega)^{\kappa-1}\omega^{\kappa+1}|dg(u)|$$

$$= O(1)\int_{\lambda_0}^{\infty} |dg\,(u)|\int_{\lambda_0}^{u} (u - \omega)^{\kappa-1}d\omega$$

$$= O(1)\int_{\lambda_0}^{\infty} u^{\kappa}|dg(u)| < \infty. \tag{12}$$

If we now combine the relevant equations (2)—(12), we see that $I < \infty$, i.e. $\sum a_n\epsilon_n$ is summable $|R, \lambda, \kappa|$. This proves the theorem when $0 < \kappa < 1$.

CASE 2. $\kappa \geqq 1$. I have already dealt with the case when κ is a positive integer in a recent note [9]. Suppose then that $\kappa > 1$ and non-integral. We have

$$I = \kappa \int_{\lambda_0}^{\infty} \omega^{-\kappa-1}\, d\omega \left| \sum_{\lambda_\nu < \omega} (\omega-\lambda_\nu)^{\kappa-1}\lambda_\nu a_\nu \epsilon_\nu \right|$$

$$= \kappa \int_{\lambda_0}^{\infty} \omega^{-\kappa-1}\, d\omega \left| \sum_{\lambda_\nu < \omega} (\omega - \lambda_\nu)^{\kappa-1}\lambda_\nu a_\nu \left(\int_{\lambda_\nu}^{\omega} + \int_{\omega}^{\infty} \right) \right|$$

$$\leqq \kappa I_1 + \kappa I_2,$$

where

$$I_1 = \int_{\lambda_0}^{\infty} \omega^{-\kappa-1}\, d\omega \left| \int_{\lambda_0}^{\omega} dg(u) \int_0^u (\omega - t)^{\kappa-1}(u - t)^{\kappa} dB(t) \right|,$$

$$I_2 = \int_{\lambda_0}^{\infty} \omega^{-\kappa-1}\, d\omega \left| \int_{\omega}^{\infty} dg(u) \int_0^{\omega} (\omega - t)^{\kappa-1}(u - t)^{\kappa}\, dB(t) \right|.$$

It will be shown that $I_1, I_2 < \infty$. In [9] it was proved that $I_1 < \infty$ for any $\kappa > 0$. For completeness we indicate the argument. If p is the integer such that $0 < \kappa - p \leqq 1$, we integrate the inner integral in I_1 by parts $p + 1$ times to obtain a sum of integrals of the form

$$c \int_0^u (\omega - t)^{\kappa-r-1} (u - t)^{\kappa-p+r-1}\, B^p(t)\, dt \qquad (0 \leqq r \leqq p + 1).$$

Since $\kappa > 1$, $(\omega - t)^{\kappa-r-1}(u - t)^r$ decreases in $(0, u)$. Applying the second mean-value theorem and the Riesz mean value theorem, we find that each integral is $O(\omega^{\kappa-1}u^{\kappa+1})$. Hence

$$I_1 = O(1) \int_{\lambda_0}^{\infty} u^{\kappa+1} |dg(u)| \int_u^{\infty} \omega^{-2}\, d\omega = O(1) \int_{\lambda_0}^{\infty} u^{k} |dg(u)| < \infty. \tag{13}$$

Consider now the inner integral in I_2. In the notation of Lemma 5, we may write this as

$$J = J_1 + J_2. \tag{14}$$

Putting $\mu = \kappa - 1$, $0 < \kappa - p \leqq 1$, p integral, we have $[\mu] = p - 1$. Since $(\omega - t)^{\mu-\nu}(u - t)^{\nu}$ decreases in $(0, \omega)$ for $0 \leqq \nu \leqq [\mu]$, the second mean-value theorem and the Riesz mean-value theorem give

$$J_1 = \sum_{\nu=0}^{[\mu]} c\omega^{\mu-\nu}\, u^{\nu} \int_0^{\xi} (u - t)^{\kappa-p-1}\, B^p(t)\, dt \qquad (0 \leqq \xi \leqq \omega)$$

$$= \sum_{\nu=0}^{[\mu]} O(\omega^{\mu-\nu}u^{\nu}\omega^{\kappa+1}) = O(u^{\kappa-1}\omega^{\kappa+1}). \tag{15}$$

Hence

$$\int_{\lambda_0}^{\infty} \omega^{-\kappa-1}\, d\omega \left| \int_{\omega}^{\infty} J_1 dg(u) \right|$$

$$= O(1) \int_{\lambda_0}^{\infty} u^{\kappa-1} |dg(u)| \int_{\lambda_0}^{u} d\omega = O(1) \int_{\lambda_0}^{\infty} u^{k} |dg(u)| < \infty. \tag{16}$$

Now consider J_2:

$$J_2 = cH(\kappa - p - 1,\ \kappa - 1, p) + cH(\kappa - p - 1,\ \kappa, p - 1). \tag{17}$$

Since $(u - t)^{\kappa-1}$ decreases in $(0, \omega)$, the second mean-value theorem and the Riesz mean-value theorem show that the first term in (17) is equal to

$$u^{\kappa-1} \int_0^{\xi} (\omega-t)^{\kappa-p-1} B^{p}(t)\, dt \qquad (0 \leqq \xi \leqq \omega)$$

$$= O(u^{\kappa-1}\omega^{\kappa+1}). \tag{18}$$

Take the second term in (17). Partial integration gives

$$H(\kappa - p - 1,\ \kappa, p - 1) = c(u - \omega)^{\kappa} B^{\kappa-1}(\omega)$$

$$+ c \int_0^{\omega} (u - t)^{\kappa-1}\, dt \int_0^{t} (\omega - x)^{\kappa-p-1} B^{p-1}(x)\, dx$$

$$= c(u - \omega)^{\kappa} B^{\kappa-1}(\omega) + cH(\kappa - p - 1,\ \kappa - 1, p)$$

$$+ c \int_0^{\omega} (u - t)^{\kappa-1}\, dt \int_0^{t} (\omega - x)^{\kappa-p-2} B^{p}(x)\, dx. \tag{19}$$

We have already dealt with the term $H(\kappa - p - 1,\ \kappa - 1, p)$, (see (18)). Consider the repeated intəgral in (19). Changing the order of integration, and nothing that

$$(\omega - x)^{-1} \int_x^{\omega} (u-t)^{\kappa-1}\, dt$$

is a decreasing function of x in $(0, \omega)$, the second mean-value theorem and the Riesz mean-value theorem show that the repeated integral is equal to

$$\omega^{-1}\int_0^{\omega}(u-t)^{\kappa-1}\,dt\int_0^{\xi}(\omega-x)^{\kappa-p-1}\;B^p(x)\,dx\;\;(0\leqq\xi\leqq\omega)$$
$$=\omega^{-1}O(u^{\kappa-1}\omega^{\kappa+2})=O(\,u^{\kappa-1}\omega^{\kappa+1}). \tag{20}$$

The contribution to I_2 of the terms in (18) and (20) is

$$O(1)\int_{\lambda_0}^{\infty}\omega^{-\kappa-1}\,d\omega\int_{\omega}^{\infty}u^{\kappa-1}\omega^{\kappa+1}|\,dg(u)|=O(1)\int_{\lambda_0}^{\infty}u^{\kappa}|\,dg(u)|\;<\infty. \tag{21}$$

Finally, we must find the contribution to I_2 of the first term in (19). We have

$$\int_{\lambda_0}^{\infty}\omega^{-\kappa-1}\;d\omega\left|\int_{\omega}^{\infty}(u-\omega)^{\kappa}B^{\kappa-1}(\omega)\,dg(u)\right|$$
$$=\int_{\lambda_0}^{\infty}\omega^{-\kappa-1}d\omega\,|\,B^{\kappa-1}(\omega)\;G_{\kappa}(\omega)|\,d\omega$$
$$=T_1+T_2+T_3,$$

where by Lemmas 3 and 4, and the fact that $\lambda_{n+1}/\lambda_n=O(1)$,

$$T_1=O(1)\sum_{n=0}^{\infty}\Lambda_n\int_{\lambda_n}^{\lambda_{n+1}}\omega^{-1}\,d\omega\int_{\lambda n}^{\lambda n+[\kappa]+1}u^{\kappa}|\,dg(u)|$$
$$=O(1)\sum_{n=0}^{\infty}\frac{\lambda_{n+1}}{\lambda_n}\int_{\lambda_n}^{\lambda n+[\kappa]+1}u^{\kappa}|\,dg(u)|\;<\infty, \tag{22}$$

$$T_2=O(1)\sum_{n=0}^{\infty}\frac{\lambda_{n+1}}{\lambda_n}\sum_{\nu=0}^{[\kappa]}|\,\epsilon_{n+\nu}|\;<\infty, \tag{23}$$

and

$$T_3=O(1)\sum_{n=0}^{\infty}\frac{\lambda_{n+1}}{\lambda_n}(\lambda_{n+1}-\lambda_n)^{[\kappa]+1}\int_{\lambda n+[\kappa]+1}^{\infty}(u-\lambda_n)^{\kappa-[\kappa]-1}|\,dg(u)|$$
$$=O(1)\sum_{n=0}^{\infty}\int_{\lambda n}^{\lambda_{n+1}}(\omega-\lambda_n)^{[\kappa]}\;d\omega\int_{\lambda_n+[\kappa]+1}^{\infty}(u-\lambda_n)^{\kappa-[\kappa]-1}|\,dg(u)|$$
$$=O(1)\sum_{n=0}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}}\omega^{[\kappa]}\;d\omega\int_{\omega}^{\infty}(u-\omega)^{\kappa-[\kappa]-1}|\,dg(u)|$$
$$=O(1)\int_{\lambda_0}^{\infty}\omega^{[\kappa]}\;d\omega\int_{\omega}^{\infty}(u-\omega)^{\kappa-[\kappa]-1}|\,dg(u)|$$
$$=O(1)\int_{\lambda_0}^{\infty}|\,dg(u)|\int_{\lambda_0}^{u}\omega^{[\kappa]}(u-\omega)^{\kappa-[\kappa]-1}\,d\omega$$

$$= O(1) \int_{\lambda_0}^{\infty} u^k |dg(u)| < \infty. \tag{24}$$

If we combine the relevant equations (13)-(24) we see that $I < \infty$ in the case $\kappa > 1$ and non-integral. The theorem is thus proved for all $\kappa \geqq 0$.

5. If we write l_n instead of λ_n (in line with the notation of Bosanquet [4] and Borwein [3]), we obtain from our Theorem 1 a theorem due to these authors on the abscissae of summability of the Dirichlet series $\sum a_n l_n^{-s}$. This result can be stated as

$$\bar{\sigma}_\kappa - \sigma_\kappa \leqq D = \overline{\lim_{n\to\infty}} \frac{\log n}{\log l_n} < \infty, \tag{25}$$

where σ_κ, $\bar{\sigma}_\kappa$ are respectively the abscissae of summability (R, l, κ), $|R, l, \kappa|$ of the Dirichlet series $\sum a_n l_n^{-s}$.

The proof of (25) was made to depend on the following theorem (Bosanquet [4], Borwein [3]), which we deduce from Theorem 1.

THEOREM 2. *If* $\sum a_n$ *is bounded* (R, l, κ), $\kappa \geqq 0$, *then for* $\sigma > D$, $\sum a_n l_n^{-\sigma}$ *is summable* $|R, l, \kappa|$.

PROOF. In Theorem 1 take $g(u) = -\dfrac{\Gamma(\kappa+\sigma+1)}{\Gamma(\sigma)\Gamma(\kappa+1)} u^{-\kappa-\sigma}$. Since $\sigma > D \geqq 0$, we have

$$\int_{l_0}^{\infty} u^\kappa |dg(u)| < \infty,$$

and $\epsilon_n = l_n^{-\sigma}$. Also, since $\sigma > D$, $\sum |\epsilon_n| = \sum l_n^{-\sigma} < \infty$. The hypotheses of Theorem 1 are satisfied, so that $\sum a_n l_n^{-\sigma}$ is summable $|R, l, \kappa|$.

As it stands Theorem 2 is weaker than Borwein's result, since I suppose $l \in \Lambda$, whereas Borwein does not. However $G_\kappa(\omega) = \omega^{-\sigma}$ in this case, so that my argument is considerably simplified. Lemma 4 may be avoided, and it is easy to check that the theorem is true without restriction on l_n.

We may now generalize Theorem 1 to some extent if we replace the hypothesis that $\sum a_n$ is bounded (R, λ, κ) by $\omega^{-\kappa} A^\kappa(\omega) = O(\phi(\omega))$. To ensure the existence of the Stieltjes integrals we suppose that $\phi(\omega)$ is a positive, non-decreasing function of ω which has no common points of discontinuity with $g(u)$ in (λ_0, ∞). Also we suppose that $\phi(\lambda_{n+1}) = O(\phi(\lambda_n))$, i. e. ϕ does not increase too rapidly. Lemma 2 is then replaced by

LEMMA 2'. *If* $\kappa \geqq 0$, $\kappa + q \geqq 0$, $A^{\kappa}(\omega) = O(\omega^{\kappa+q}\phi(\omega))$, $\phi(\omega) > 0$ *and non-decreasing, with* $\phi(\lambda_{n+1}) = O(\phi(\lambda_n))$, *then for* $\mu = 0, 1, \ldots [\kappa]$, *and* $\lambda_n < \omega \leqq \lambda_{n+1}$,
$$A^{\mu}(\omega) = O\{\omega^{\mu}\lambda_n^q\phi(\lambda_n)\ \Lambda_n^{\kappa-\mu}\}.$$

The proof of Theorem 1 goes through as before, if we suppose now that

$$\sum \phi(\lambda_n)|\epsilon_n| < \infty,$$

$$\epsilon_\nu = \int_{\lambda\nu}^{\infty} (u - \lambda_\nu)^{\kappa} dg(u), \text{ with } \int_{\lambda_0}^{\infty} u^{\kappa}\phi(u)|\, dg(u)| < \infty.$$

Hence we have

THEOREM 3. *Suppose that* $\phi(\omega) > 0$, *non-decreasing, with* $\phi(\lambda_{n+1}) = O(\phi(\lambda_n))$, *and that* ϕ *has no common points of discontinuity with* $g(u)$ *in* (λ_0, ∞). *If* $\omega^{-\kappa} A^{\kappa}(\omega) = O(\phi(\omega))$, $\kappa \geqq 0$, $\lambda \in \Lambda$, *and*

(i) $$\sum \phi(\lambda_n)|\epsilon_n| < \infty,$$

(ii) *there exists* $g(u)$, *such that*

$$\epsilon_\nu = \int_{\lambda\nu}^{\infty} (u - \lambda_\nu)^{\kappa}\, dg(u) \textit{ with } \int_{\lambda_0}^{\infty} u^{\kappa}\phi(u)|dg(u)| < \infty,$$

then $\sum a_n\epsilon_n$ *is summable* $|R, \lambda, \kappa|$.

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