SOME TRANSFORMATIONS ON MANIFOLDS WITH ALMOST CONTACT AND CONTACT METRIC STRUCTURES, II

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(Received July 2, 1963)
(Revised September 10, 1963)

For the notations we conform to the previous paper [15][1] which is the part I of the present paper. By Φ we denote the group of all diffeomorphisms which leave ϕ of the structure tensors (ϕ, ξ, η) invariant in an almost contact manifold M. We assume always that the dimension of the manifold is greater or equal to 3. In §4, we shall see that Φ is a Lie transformation group if M is a contact Riemannian manifold, and some structures of this group are considered. In §6, for an arbitrary point x of a contact Riemannian manifold M and an element μ of Φ, we shall search for the relation between the scalar curvature R_x at x and $R_{\mu x}$ at μx by lengthy calculations. As applications, in §7, we treat some contact Riemannian manifolds which are supposed to satisfy certain conditions, for examples, being of constant scalar curvature, or being an Einstein space, etc.. Then, with some exceptions, it is shown that Φ coincides with the group of all automorphisms.

The auther wishes to thank Professor S.Sasaki and Mr. A.Morimoto for their valuable criticisms.

4. The group Φ on contact Riemannian manifolds.

THEOREM 4-1. *For every contact Riemannian manifold, the group Φ is a Lie transformation group.*

PROOF. We denote by L_ϕ the Lie algebra of all infinitesimal transformations leaving ϕ invariant and by $\mathfrak{A}$ that of all infinitesimal automorphisms. Then, it has been shown [16] that L_ϕ is finite dimensional. In fact, $L_\phi = \mathfrak{A} + L$ (direct sum), where L is a 1-dimensional Lie algebra generated by a vector field $Z \in L_\phi$ such that $\mathfrak{L}(Z)\eta = \eta$. Moreover, we have

$$[\mathfrak{A},\mathfrak{A}] \subset \mathfrak{A},\ [\mathfrak{A},L] \subset \mathfrak{A}. \tag{4. 1}$$

1) Correction of [15]; we must insert the underlined part into the conclusion of the statement of Corollary (p. 141) as follows:
μ is necessarily an isometry. Therefore, $\underline{\text{if } \mu\xi \neq -\xi,}$ μ is an automorphism of this almost contact metric structure.
This was communicated by Mr. Y. Tashiro, whom the auther should like to express his gratitude.

Let L_ϕ^* be the set of all proper vector fields in L_ϕ, that is of all vector fields each of which generates a global 1-parameter group of transformations. The work of R.S.Palais ([14], Chapter IV) teaches us that the group Φ is a Lie transformation group with infinitesimal group L_ϕ^*. Of course, in this case, L_ϕ^* as the set coincides with a Lie algebra generated by L_ϕ^*.

LEMMA 4-1. *Let μ be an element of Φ, then we have some positive constant α such that*

$$(\mu^*g)(X,Y) = \alpha g(X,Y) + \alpha(\alpha-1)\eta(X)\cdot\eta(Y), \tag{4. 2}$$

for any $X, Y \in \mathfrak{X}(M)$. This is valid also for μ which is local, if μ leaves ϕ invariant in its connected domain of definition.

PROOF. Putting $-\phi Y$ in place of Y of the relation (3.3) of [15], we get

$$(\mu^*g)(X, Y-\eta(Y)\xi) = \alpha g(X, Y-\eta(Y)\xi) \tag{4. 3}$$

for some positive constant α. On the other hand, we see that

$$(\mu^*g)(X, \eta(Y)\xi) = \eta(Y)\cdot g(\mu X, \mu\xi) = \alpha^2\eta(X)\cdot\eta(Y). \tag{4. 4}$$

Hence by (4. 3) and (4. 4), we have (4. 2).

THEOREM 4-2. *If M is a complete contact Riemannian manifold, the infinitesimal group of the Lie transformation group Φ is L_ϕ itself.*

PROOF. It suffices to show that any element X of L_ϕ generates a global 1-parameter group of transformations which leave ϕ invariant. As our differentiable manifold is a Hausdorff space any X is univalent, namely it generates a maximum local 1-parameter group $\exp(tX)$ of transformations which leave ϕ invariant. Let x be an arbitrary point of M, then we have uniquely two positive (finite or infinite) ε_1 and ε_2 depending on x such that $\exp(tX)$ is defined for t: $-\varepsilon_1 < t < \varepsilon_2$. Our purpose is to show that $\varepsilon_1 = \varepsilon_2 = \infty$. First we suppose that $\varepsilon_2 \neq \infty$, and we verify that $\exp(\varepsilon X)x$, $\varepsilon = \varepsilon_2$, is definite, since the case of $\varepsilon = \varepsilon_1$ is quite similar. We put

$$x_m = \exp(t_m X)x, \qquad t_m = \sum_{k=1}^{m} \frac{\varepsilon}{2^k}, \ m = 1,2,\cdots$$

and

$$y_m = \exp(t'_m X)x, \qquad t'_m = t_m - \frac{\varepsilon}{2}, \ m = 1,2,\cdots.$$

Further we define

$$E = [\max_p \eta_p(X)^2 : p = \exp(tX)x,\ 0 \leqq t \leqq \frac{\varepsilon}{2}],$$

$$F = [\max_p g_p(X, X) : p = \exp(tX)x,\ 0 \leqq t \leqq \frac{\varepsilon}{2}],$$

then we have a finite number $E^* = \max(\sqrt{E}, \sqrt{F})$. Now, by virtue of Lemma 4-1, we get a positive constant α such that

$$((\exp \frac{\varepsilon}{2} X)^* g)(X, X) = \alpha g(X, X) + \alpha(\alpha - 1)\eta(X)^2. \tag{4. 5}$$

At first, we assume that α is greater than 1. Let s_m be the length of the segment $[x_m, x_{m+1}]$ contained in $(\exp(tX)x, -\varepsilon_1 < t < \varepsilon_2)$. Then by (4.5) we have

$$\begin{aligned} s_m &= \int_{t_m}^{t_{m+1}} \sqrt{g_{\exp(tX)x}(X, X)}\, dt \\ &= \int_{t_m}^{t_{m+1}} \sqrt{g_{\exp(tX)x}\left[\exp\left(\frac{\varepsilon}{2} X\right)X, \exp\left(\frac{\varepsilon}{2} X\right)X\right]}\, dt \\ &= \int_{t'_m}^{t'_{m+1}} \sqrt{\left(\exp\left(\frac{\varepsilon}{2} X\right)^* g\right)_{\exp(t'X)x}(X, X) dt'} \qquad t' = t - \frac{\varepsilon}{2} \\ &= \int_{t'_m}^{t'_{m+1}} \sqrt{\alpha g_{\exp(t'X)x}(X, X) + \alpha(\alpha - 1)\eta_{\exp(t'X)x}(X)^2}\, dt' \\ &\leqq \int_{t'_m}^{t'_{m+1}} \sqrt{\alpha F + \alpha(\alpha - 1)E}\, dt' \leqq \int_{t'_m}^{t'_{m+1}} \alpha E^* dt'. \end{aligned}$$

Consequently, we get

$$s_m \leqq \frac{\alpha\varepsilon}{2^{m+1}} E^*. \tag{4. 6}$$

On the other hand, if α is smaller or equal to 1,

$$s_m \leqq \int_{t'_m}^{t'_{m+1}} \sqrt{\alpha F}\, dt'.$$

Hence, we obtain

$$s_m \leqq \frac{\sqrt{\alpha}\,\varepsilon}{2^{m+1}} E^*. \tag{4. 7}$$

Since the distance $d(x_m, x_{m+1})$ between x_m and x_{m+1} is smaller or equal to s_m, we see that the sequence $\{x_m\}$ is nothing but a Cauchy sequence by virtue of (4. 6) and (4. 7). We denote its limit point by $\exp(\varepsilon X)x$, namely we have

$$\lim_{m\to\infty} \exp\Big(\sum_{k=1}^{m} \frac{\varepsilon}{2^k} X\Big)x = \exp(\varepsilon X)x. \qquad \text{Q.E.D.}$$

Now, we suppose that L in (4.1) is not zero and that its generator Z generates a global 1-parameter group $R^* = (\gamma_t = \exp(tZ); -\infty < t < \infty)$. By A we understand the group of all automorphisms of the contact metric structure. Needless to say that A is a closed subgroup of the group $I(M)$ of isometries as well as of Φ, hence it is a Lie transformation group.

THEOREM 4-3. *In a contact Riemannian manifold, if* $Z \in L_\phi$ *satisfying* $\mathfrak{L}(Z)\eta = \eta$ *generates a global* 1*-parameter group* R^*, *then* $\Phi = (A, R^*)$, *where the multiplication in* (A, R^*) *is defined by*

$$(a, \gamma_t) \times (b, \gamma_s) = (a\gamma_t b\gamma_t^{-1}, \gamma_t\gamma_s)$$

for any $a, b \in A$ *and* $\gamma_t, \gamma_s \in R^*$.

PROOF. It is clear that $a\gamma_t b\gamma_t^{-1}$ belongs to A. Let R^+ be the multiplicative group of positive numbers. Then the homomorphism $h{:}R^* \to R^+$ is (at least continuous and hence) an analytic onto and one-to-one map. In fact, it will be seen that $h(\gamma_s) = e^s$. By the way, the map $(a,\gamma_t) \to a\cdot\gamma_t$ gives an isomorphism of (A, R^*) into Φ. And this map is onto, because for $h(\mu) \in R^+$, we have γ_t such that $h(\gamma_t) = h(\mu)$. Then it is evident that

$$\mu = \mu\gamma_t^{-1} \cdot \gamma_t, \quad \mu\gamma_t^{-1} \in A.$$

5. The group Φ_c on a compact manifold with an almost contact structure. For $\mu \in \Phi$, we have a scalar field σ such that $\mu\xi = \sigma\xi$, the totality of μ for which σ is constant constitutes a subgroup of Φ and we denote it by Φ_c. Then it can be shown with slight modification of A.Morimoto's method [11] that, if an almost contact manifold is compact, Φ_c is a Lie transformation group. However, when we deliberate its Lie algebra, it may be remarked that in a compact almost contact Riemannian manifold such that ξ leaves η and the volume element invariant, there does not exist any infinitesimal transformation X such that $\mathfrak{L}(X)\phi = 0$ and $\mathfrak{L}(X)\xi = \beta\xi$ ($\beta = \text{constant} \neq 0$).

6. A relation between the scalar curvature. We take an arbitrary point x in a contact Riemannian manifold and let (x^k, U) be the local coordinate system about x. For $\mu \in \Phi$, we set $\bar{g} = \mu^* g$, then by Lemma 4-1, we see that

$$\bar{g}_{ij} = \alpha g_{ij} + \alpha(\alpha - 1)\eta_i\eta_j \tag{6. 1}$$

for some constant α. We write the Christoffel's symbols and curvature tensors $\bar{\Gamma}^i_{jk}$ and $\bar{R}^i{}_{jkl}$, Γ^i_{jk} and $R^i{}_{jkl}$ according to the metrics $\bar{g}, g$ respectively. The notation 'bar' will be applied similarly to the geometric objects for $\bar{g}$.

Define Δ_{jki} as follows:

$$\Delta_{jki} = \partial_j(\eta_k\eta_i) + \partial_k(\eta_j\eta_i) - \partial_i(\eta_j\eta_k).$$

Then we get

$$2\overline{[jk,i]} = 2\alpha[jk,i] + \alpha(\alpha - 1)\Delta_{jki}, \tag{6. 2}$$

where

$$2[jk,i] = \partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}.$$

On the other hand, the inverse matrix $\bar{g}^{is}$ of $\bar{g}_{ij}$ is seen to be

$$\bar{g}^{is} = \frac{1}{\alpha} g^{is} - \frac{\alpha-1}{\alpha^2} \xi^i\xi^s. \tag{6. 3}$$

Consequently, we have

$$(6.4)\qquad \bar{\Gamma}^s_{jk} = \bar{g}^{is}\overline{[jk,i]} = \Gamma^s_{jk} + \frac{\alpha-1}{2} g^{is}\Delta_{jki} - \frac{\alpha-1}{\alpha}\Gamma^p_{jk}\eta_p\xi^s - \frac{(\alpha-1)^2}{2\alpha}\xi^i\xi^s\Delta_{jki}.$$

In order to calculate the scalar curvature $\bar{R}$, we prepare some relations. Since $\mu^*(\eta\wedge w^n) = \alpha^{n+1}(\eta\wedge w^n)$. We see that $\sqrt{|\bar{g}|} = \alpha^{n+1}\sqrt{|g|}$ where $|g|$ is the determinant of g_{ij}, and so $\partial_k \log\sqrt{|\bar{g}|} = \partial_k \log\sqrt{|g|}$, thereby we have

$$(6.5)\qquad \bar{\Gamma}^l_{lk} = \Gamma^l_{lk}, \qquad \text{(summed for } l).$$

Now we put for brevity

$$\Gamma^i_j = \Gamma^i_{jk}\,\xi^k, \qquad \Gamma_{jk} = \Gamma^i_{jk}\eta_i,$$
$$\Gamma_j = \Gamma_{jk}\xi^k, \qquad \Gamma = \Gamma_j\xi^j.$$

Then the following relations hold good,

$$(6.6)\qquad \xi^r\partial_j\eta_r = \xi^r\partial_r\eta_j = -\partial_j\xi^r\cdot\eta_r = \Gamma_j,$$

$$(6.7)\qquad g^{jk}\partial_j\eta_k = \Gamma_{jk}g^{jk}.$$

And as Δ_{jki} is rewritten as follows:

$$\Delta_{jki} = (\partial_j\eta_k + \partial_k\eta_j)\eta_i + \eta_j\phi_{ki} + \eta_k\phi_{ji},$$

we have

$$(6.8)\qquad \xi^i\Delta_{jki} = \partial_j\eta_k + \partial_k\eta_j,\ \xi^k\xi^i\Delta_{jki} = 2\Gamma_j,$$

$$(6.9)\qquad \xi^j\Delta_{jki} = 2\Gamma_k\eta_i + \phi_{ki},\ \xi^j\xi^k\Delta_{jki} = 2\Gamma\eta_i,$$

$$(6.10)\qquad g^{ij}\Delta_{jki} = 2\Gamma_k,$$

$$(6.11)\qquad g^{jk}\Delta_{jki} = 2g^{jk}\Gamma_{jk}\eta_i.$$

Ricci curvature tensor for $\bar{g}$ is by definition

$$(6.12)\qquad \bar{R}_{jk} = \partial_l\bar{\Gamma}^l_{jk} - \partial_k\bar{\Gamma}^l_{lj} + \bar{\Gamma}^l_{ls}\bar{\Gamma}^s_{jk} - \bar{\Gamma}^l_{ks}\bar{\Gamma}^s_{jl}.$$

We put as follows:

$$\begin{cases} A_1 = \dfrac{1}{\alpha} g^{jk}\partial_l\bar{\Gamma}^l_{jk}, \\ A_2 = -\dfrac{1}{\alpha} g^{jk}\partial_k\Gamma^l_{lj}, \\ A_3 = \dfrac{1}{\alpha} g^{jk}\Gamma^l_{ls}\bar{\Gamma}^s_{jk}, \\ A_4 = -\dfrac{1}{\alpha} g^{jk}\bar{\Gamma}^l_{ks}\bar{\Gamma}^s_{jl}, \end{cases}$$

and

$$\Big\{\ B_1 = \frac{(1-\alpha)}{\alpha^2}\xi^j\xi^k\partial_l\,\bar{\Gamma}^l_{jk},$$

$$\left\{\begin{aligned} B_2 &= \frac{(\alpha-1)}{\alpha^2}\,\xi^j\xi^k\partial_k\Gamma^l_{lj},\\ B_3 &= \frac{(1-\alpha)}{\alpha^2}\,\xi^j\xi^k\Gamma^l_{ls}\overline{\Gamma}^s_{jk},\\ B_4 &= \frac{(\alpha-1)}{\alpha^2}\,\xi^j\xi^k\overline{\Gamma}^l_{ks}\overline{\Gamma}^s_{jl}.\end{aligned}\right.$$

Then by (6.3) and (6.12), we have

$$\overline{R} = A_1 + A_2 + A_3 + A_4 + B_1 + B_2 + B_3 + B_4.$$

By straightforward calculations we obtain

$$\begin{aligned} A_1 &= \frac{1}{\alpha}\,g^{jk}\partial_l\Gamma^l_{jk} - \frac{\alpha-1}{\alpha}\,(\Gamma_{jk}g^{jk})^2 + \frac{\alpha-1}{2\alpha}\,g^{it}\partial_t\Delta_{jki}g^{jk}\\ &\quad - \frac{\alpha-1}{\alpha^2}\,\partial_l\Gamma^s_{jk}g^{jk}\eta_s\xi^l - \frac{\alpha-1}{\alpha^2}\,\Gamma^s_{jk}g^{jk}\Gamma_s - \frac{(\alpha-1)^2}{\alpha^2}\,\xi^l\partial_l(\partial_j\eta_k)g^{jk},\\ A_3 &= \frac{1}{\alpha}\,g^{jk}\Gamma^l_{ls}\Gamma^s_{jk},\\ A_4 &= -\frac{1}{\alpha}\,g^{jk}\Gamma^l_{ks}\Gamma^s_{jl} - \frac{\alpha-1}{\alpha}\,\Gamma^s_{jl}g^{lr}\Delta_{ksr}g^{jk} + \frac{2(\alpha-1)}{\alpha^2}\,\Gamma^s_j g^{jk}\Gamma_{sk}\\ &\quad + \frac{(\alpha-1)^2}{\alpha^2}\,\Gamma^s_j(\partial_k\eta_s + \partial_s\eta_k)g^{jk} - \frac{(\alpha-1)^2}{4\alpha}\,g^{st}\Delta_{jlt}g^{lr}\Delta_{ksr}g^{jk}\\ &\quad + \frac{(\alpha-1)^2}{\alpha}\,\Gamma_j\Gamma_k g^{jk},\\ B_1 &= -\frac{\alpha-1}{\alpha^2}\,\partial_l\Gamma^l_{jk}\xi^j\xi^k + \frac{(\alpha-1)^2}{\alpha^2}\,(\Gamma_{jk}g^{jk})\Gamma - \frac{(\alpha-1)^2}{2\alpha^2}\,g^{rt}\xi^j\xi^k\partial_t\Delta_{jkr}\\ &\quad + \frac{(\alpha-1)^2}{\alpha^3}\,\partial_l\Gamma^s_{jk}\xi^j\xi^k\eta_s\xi^l + \frac{(\alpha-1)^2}{\alpha^3}\,\Gamma^s\Gamma_s + \frac{(\alpha-1)^3}{\alpha^3}\,\xi^l\partial_l(\partial_j\eta_k)\xi^j\xi^k,\\ B_3 &= -\frac{\alpha-1}{\alpha^2}\,\Gamma^l_{ls}\Gamma^s,\\ B_4 &= \frac{\alpha-1}{\alpha^2}\,\Gamma^l_s\Gamma^s_l - \frac{(\alpha-1)^2}{\alpha^2}\,\phi^l_s\Gamma^s_l - \frac{(\alpha-1)^3 n}{2\alpha^2}\ .\end{aligned}$$

As for the above equations, we denote by $A_4(5)$, for example, the 5-th term of A_4. Then we have

$$A_1(1) + A_2 + A_3(1) + A_4(1) = \frac{1}{\alpha}\,R$$

and

$$B_1(1) + B_2 + B_3(1) + B_4(1) = \frac{1-\alpha}{\alpha^2}\,R_{jk}\xi^j\xi^k.$$

Furthermore, we have

$$A_1(3) = -A_1(2) + \frac{\alpha-1}{\alpha}[\phi^r_j\partial_r\xi^j + \xi^r\partial_r(\partial_j\eta_k)g^{jk} - n],$$

$$A_4(2) = \frac{\alpha-1}{\alpha}[\phi^r_j\Gamma^j_r - \Gamma^s_j(\partial_k\eta_s + \partial_s\eta_k)g^{jk}],$$

$$A_4(5) + A_4(6) = \frac{(\alpha-1)^2 n}{2\alpha}.$$

Thus we get

$$A_1(2) + A_1(3) + A_1(6) + A_4(2) + A_4(4) = \frac{\alpha-1}{\alpha}\phi^r_j\xi^j_{,r}$$
$$+ \frac{\alpha-1}{\alpha^2}\xi^r\partial_k(\partial_r\eta_j)g^{jk} + \frac{1-\alpha}{\alpha^2}\Gamma^s_j(\partial_k\eta_s + \partial_s\eta_k)g^{jk} - \frac{(\alpha-1)n}{\alpha}$$
$$= \frac{1-\alpha}{\alpha^2}[\Gamma^s_j\partial_s\eta_k - \partial_k\Gamma^s_{jl}\eta_s\xi^l - \Gamma_{jl}\partial_k\xi^l + \partial_r\eta_j\partial_k\xi^r]g^{jk},$$

where a comma such as in $\xi^j_{,r}$ denotes the covariant derivation. Consequently,

$$A_1 + A_2 + A_3 + A_4 = \frac{1}{\alpha}R + \frac{1-\alpha}{\alpha^2}\eta_s\xi^l g^{jk}[\partial_l\Gamma^s_{jk} - \partial_k\Gamma^s_{jl} + \Gamma^s_{tl}\Gamma^t_{jk} \tag{6.13}$$
$$- \Gamma^s_{kt}\Gamma^t_{jl}] + \frac{1-\alpha}{\alpha^2}g^{jk}[\Gamma^s_j\partial_s\eta_k + \partial_r\eta_j\partial_k\xi^r - \Gamma_{ks}\Gamma^s_j - \Gamma_{jl}\partial_k\xi^l]$$
$$+ \frac{(\alpha-1)^2 n}{2\alpha}$$
$$= \frac{1}{\alpha}R + \frac{1-\alpha}{\alpha^2}R_{sl}\xi^s\xi^l + \frac{1-\alpha}{\alpha^2}\xi^r_{,k}\xi^k_{,r} + \frac{(\alpha-1)^2 n}{2\alpha}.$$

On the other hand,

$$B_1(4) + B_1(5) + B_1(6) = \frac{(\alpha-1)^2}{\alpha^2}[\Gamma^s\Gamma_s + \partial_l\Gamma^s_{kj}\xi^k\xi^j\eta_s\xi^l],$$

$$B_1(3) = -\frac{(\alpha-1)^2}{2\alpha^2}g^{rt}[\partial_t(2\Gamma\eta_r) - \Delta_{jkr}\partial_t(\xi^j\xi^k)]$$
$$= -\frac{(\alpha-1)^2}{\alpha^2}[\partial_l\Gamma^s_{jk}\xi^l\eta_s\xi^j\xi^k + \Gamma^s\Gamma_s + \phi^r_k\partial_r\xi^k] - B_1(2).$$

Hence, we have

$$B_1 + B_2 + B_3 + B_4 = \frac{1-\alpha}{\alpha^2}R_{jk}\xi^j\xi^k - \frac{(\alpha-1)^2}{\alpha^2}(\phi^r_k\partial_r\xi^k + \phi^r_k\Gamma^k_r) \tag{6.14}$$
$$- \frac{(\alpha-1)^3 n}{2\alpha^2} = \frac{1-\alpha}{\alpha^2}R_{jk}\xi^j\xi^k - \frac{(\alpha-1)^2(\alpha+1)n}{2\alpha^2}.$$

Moreover, we see that

$$\xi^r_{,k}\xi^k_{,r} = -\xi^r{}_{,kr}\,\xi^k = -(\xi^r{}_{,rk} + R^r{}_{jkr}\xi^j)\xi^k = -R_{jk}\xi^j\xi^k. \tag{6.15}$$

Summerizing (6.13), (6.14) and (6.15), we have

$$\overline{R}=\frac{1}{\alpha}R+\frac{1-\alpha}{\alpha^2}R_{jk}\xi^j\xi^k-\frac{(1-\alpha)^2n}{2\alpha^2}.$$

However, as $\overline{R}_x$ is equal to $R_{\mu x}$, we obtain

$$R_{\mu x}=\frac{1}{\alpha}R_x+\frac{1-\alpha}{\alpha^2}(R_{jk}\xi^j\xi^k)_x-\frac{(1-\alpha)^2n}{2\alpha^2}. \tag{6.16}$$

7. Applicatoins of (6.16). First we assume that ξ is a Killing vector field, that is to say, M is a K-contact manifold. Then Ricci curvature tensor satisfies

$$R_{jk}\xi^k=\frac{n}{2}\eta_j. \tag{7. 1}$$

This follows from the Ricci identities and the relations

$$\xi^i_{,j}=-\frac{1}{2}\phi^i_j, \qquad \phi^i_{j,i}=-n\eta_j. \tag{7. 2}$$

Substituting (7.1) into (6.16), we have

$$R_{\mu x}=\frac{1}{\alpha}R_x+\frac{(1-\alpha)n}{2\alpha}. \tag{7. 3}$$

Therefore, we get the following

THEOREM 7-1. *Let M be a K-contact manifold such that the scalar curvature is constant and differs from $-2^{-1}n$. Then $\Phi\subset I(M)$ and $\Phi=A$.*

THEOREM 7-2. *In a K-contact manifold M, we assume that the scalar curvature R is not constant and that R is bounded on M. Then we have $\Phi\subset I(M)$ and $\Phi=A$.*

PROOF. As R is not constant, there exists a point x in M such that $R_x\neq-2^{-1}n$. (7.3) may be rewritten as follows:

$$R_{\mu x}=\frac{1}{2\alpha}(2R_x+n)-\frac{n}{2}.$$

Similarly, by induction we see that

$$R_{\mu^k x}=\frac{1}{2\alpha^k}(2R_x+n)-\frac{n}{2}, \quad k=1,2,\cdots.$$

Clearly, if $\alpha<1$, $R_{\mu^k x}\to+\infty$ or $-\infty$ as $k\to\infty$. From this we have $\mu\in I(M)$, consequently $\Phi=A$.

THEOREM 7-3. *If a contact Riemannian manifold is an Einstein space, then $\Phi=A$.*

PROOF. As $R_{ij}=\dfrac{R}{2n+1}g_{ij}$, we have by (6.16)

$$(\alpha-1)\left[\frac{R}{2n+1}-\frac{n}{2}+\frac{\alpha(2R+n)}{2}\right]=0. \tag{7. 4}$$

First we consider the case of $\frac{R}{2n+1} = \frac{n}{2}$. Then we have $2R + n \neq 0$ and $\alpha = 1$. Next we suppose that $\frac{R}{2n+1} \neq \frac{n}{2}$. Then if $2R + n = 0$, clearly $\alpha = 1$, and if $2R + n \neq 0$, then (7.4) is written as:

$$(\alpha - 1)\left[\alpha - \frac{2n^2 + n - 2R}{(2n+1)(2R+n)}\right] = 0.$$

And so if $R = 0$, $h(\mu) = 1$. The only possible case for μ not to belong to A is $h(\mu) = \frac{2n^2 + n - 2R}{(2n+1)(2R+n)}$, $R \neq 0$. However, as $h(\mu^2) = h(\mu)^2$ and $h(\mu^2)$ must be equal to 1, this case can not happen.

COROLLARY 7-1. *A K-contact manifold with parallel Ricci tensor is an Einstein space. And so, $\Phi = A$. Especially, in a K-contact symmetric space, we have $\Phi = A$.*

PROOF. Taking the covariant derivatives of (7.1), we get

$$R_{jk}\phi_l^k = \frac{n}{2}\phi_{jl} \tag{7. 5}$$

by the relation (7.2). Multiplying ϕ_i^l and summing over l, we have $R_{ij} = \frac{n}{2} g_{ij}$. Then $\Phi = A$ follows from Theorem 7-3.

COROLLARY 7-2. *Every conformally flat K-contact manifold M (dim $M > 3$) is necessarily normal and hence of constant curvature, therefor $\Phi = A$.*

PROOF. It has been shown [12] that a conformally flat normal contact manifold is of constant curvature. Here we shall show that a conformally flat K-contact manifold is a normal contact manifold. In K-contact manifold, we have

$$\phi_{ij,k} = 2R_{ijkl}\xi^l. \tag{7. 6}$$

As Weyl's conformal curvature tensor vanishes,

$$R_{ijkl} = \frac{1}{2n-1}(g_{il}R_{jk} - g_{ik}R_{jl} + g_{jk}R_{il} - g_{jl}R_{ik}) - \frac{R}{2n(2n-1)}(g_{il}g_{jk} - g_{jl}g_{ik}).$$

Consequently, we have

$$2R_{ijkl}\xi^l = \frac{1}{2n-1}\eta_i\left(2R_{jk} + \frac{n^2 - R}{n} g_{jk}\right) - \frac{1}{2n-1}\eta_j\left(2R_{ik} + \frac{n^2 - R}{n} g_{ik}\right). \tag{7. 7}$$

Transvecting (7.7) with ξ^i and using (7.6) we have

$$\phi_{ij.k}\xi^i = \frac{1}{2n-1}(2R_{jk} + \frac{n^2 - R}{n} g_{jk}) - \frac{2n^2 - R}{n(2n-1)}\eta_j\eta_k. \tag{7. 8}$$

And as

$$\phi_{ij,k}\xi^i = \frac{1}{2} g_{jk} - \frac{1}{2}\eta_j\eta_k,$$

it follows from (7.8) that

$$2R_{jk} = \frac{2R - n}{2n} g_{jk} + \frac{2n^2 + n - 2R}{2n}\eta_j\eta_k. \tag{7. 9}$$

Substituting (7.9) in (7.7), we get

$$4R_{ijkl}\xi^l = \eta_i g_{jk} - \eta_j g_{ik}.$$

Hence, our manifold is a normal contact manifold [10].

BIBLIOGRAPHY

[10] HATAKEYAMA, Y., Y. Ogawa and S. Tanno, Some properties of manifolds with contact metric structure, Tôhoku Math. Journ., 15(1963), 42–48.

[11] MORIMOTO, A., On normal almost contact structures, (to appear in Journ. Math. Soc. Japan).

[12] OKUMURA, M., Some remarks on space with a certain contact structure, Tôhoku Math. Journ., 14(1962), 135–145.

[13] ———, On infinitesimal conformal and projective transformation of normal contact spaces, Tôhoku Math. Journ., 14(1962), 398–412.

[14] PALAIS, R. S., A global formulation of the Lie theory of transformation groups, Memoirs of Amer. Math. Soc., 22(1957).

[15] TANNO, S., Some transformations on manifolds with almost contact and contact metric structures I, Tôhoku Math. Journ., 15(1963), 140–147.

[16] ———, On fiberings of some non-compact contact manifolds, Tôhoku Math. Journ., 15(1963), 290–297.

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