ON ABSOLUTE RIESZ SUMMABILITY FACTORS

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1. In this note sufficient conditions are given for a series $\sum a_n\epsilon_n$ to be summable $|R, \lambda, \kappa|$ whenever $\sum a_n$ is bounded (R, λ, κ). We shall restrict ourselves to integer values of κ. The non-integer case appears to present considerable difficulties—it is hoped to deal with it in a further note. When $\lambda_n = n$, our result is an alternative version of a theorem on absolute Cesàro summability factors (see Pati and Ahmad [7]); the equivalence of (R, n, κ), (C, κ) and $|R, n, \kappa|$, $|C, \kappa|$ summability being well known (Hobson [3], 90-98; Hyslop [4]).

In Section 4 we shall prove the following

THEOREM. *Suppose that the sequence of positive numbers* $\{\lambda_n\}$ *increases to infinity, and that*

(a) $0 < a \leqq \Delta\lambda_n/\Delta\lambda_{n-1} \leqq A$, *$a$,$A$ constants. If* $A^\kappa(\omega) = O(\omega^\kappa)$, $\kappa = 0,1,2,\cdots$, *where* $A^\kappa(\omega)$ *is defined in* §2, *and*

(i) $\sum |\epsilon_n| < \infty$,

(ii) *there exists a function* $g(u)$, *defined for* $u \geqq \lambda_0$, *and a number* α, *such that for* $\nu = 0,1,\cdots$,

$$\epsilon_\nu = \alpha + \int_{\lambda_\nu}^{\infty} (u - \lambda_\nu)^\kappa \, dg(u) \qquad \textit{with} \int_{\lambda_0}^{\infty} u^\kappa |dg(u)| < \infty,$$

then $\sum a_n\epsilon_n$ *is summable* $|R, \lambda, \kappa|$.

Although we shall be concerned only with the sufficiency of conditions (i) and (ii), we remark that (ii) is necessary without any restriction on λ_n. For it has been shown that (ii) is necessary for $\sum a_n\epsilon_n$ to be summable (R, λ, μ) whenever $\sum a_n$ is summable (R, λ, κ), $\mu \geqq 0$, $\kappa \geqq 0$ (see Maddox [5]). Since $|R, \lambda, \mu|$ implies (R, λ, μ) summability, the result follows. We note also that $\alpha = 0$. For by (i) $\epsilon_n = o(1)$, and by (ii) $\epsilon_n = \alpha + o(1)$. Some remarks have been made on the condition (a) by Maddox [5]. The necessity of (i) has been discussed by Maddox in a recent note [6].

2. We shall write for $\kappa > -1$,

$$A^{\kappa}(\omega) = \sum_{\lambda_\nu<\omega} (\omega - \lambda_\nu)^{\kappa} a_\nu = \int_0^{\omega} (\omega - t)^{\kappa}\, dA(t),$$

where $A(t) = A^0(t)$. The same notation is used for $B^{\kappa}(\omega)$, where b_ν replaces a_ν. If $A^{\kappa}(\omega) = O(\omega^{\kappa})$ we say that $\sum a_n$ is bounded (R, λ, κ), and if $\omega^{-\kappa}A^{\kappa}(\omega)$ is of bounded variation over (λ_0, ∞) we say that $\sum a_n$ is summable $|R, \lambda, \kappa|$.

For $\mu > 0$, $\kappa > -1$, $\kappa + \mu > 0$, we have

$$A^{\kappa+\mu}(\omega) = \frac{\Gamma(\kappa + \mu + 1)}{\Gamma(\kappa + 1)\,\Gamma(\mu)} \int_0^{\omega} (\omega - t)^{\mu-1} A^{\kappa}(t)\, dt.$$

Thus if $b_\nu = \lambda_\nu a_\nu$, and $A^{\kappa}(\omega) = O(\omega^{\kappa})$, then $B^{\kappa}(\omega) = O(\omega^{\kappa+1})$.

3. We shall require the following lemmas.

LEMMA 1. *If* $0 < \mu \leqq 1$, $\kappa \geqq 0$, $0 \leqq \xi \leqq \omega$, *then*

$$\frac{\Gamma(\kappa + \mu + 1)}{\Gamma(\kappa + 1)\Gamma(\mu)} \left| \int_0^{\xi} (\omega - t)^{\mu-1} A^{\kappa}(t)\, dt \right| \leqq \max_{0\leqq t\leqq \xi} |A^{\mu+\kappa}(t)|.$$

See Hardy and Riesz [2], Lemma 8.

LEMMA 2. *If* $\kappa \geqq 0$, $\kappa + q \geqq 0$ *and* $A^{\kappa}(\omega) = O(\omega^{\kappa+q})$, *then, for* $\mu = 0,1, \cdots, [\kappa]$ *and* $\lambda_n < \omega \leqq \lambda_{n+1}$,

$$A^{\mu}(\omega) = O\{\omega^{\mu}\lambda_n^{q}\Lambda_n^{\kappa-\mu}\}, \text{ where } \Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n).$$

See Borwein [1], Lemma 2.

LEMMA 3. *Let* $\kappa = 1,2,\cdots$, *and* ϵ_n *be given by* (ii) *of the theorem with* $\alpha = 0$. *If* $0 < a \leqq \Delta\lambda_n/\Delta\lambda_{n-1} \leqq A$, *then for* $\lambda_n \leqq \omega \leqq \lambda_{n+1}$, *we have*

$$G_{\kappa}(\omega) = \int_{\omega}^{\infty} (u - \omega)^{\kappa} dg(u) = O(1) \int_{\lambda_n}^{\lambda_{n+\kappa+1}} u^{\kappa} |dg(u)| + O(1) \sum_{\nu=0}^{\kappa} |\epsilon_{n+\nu}|.$$

For integers $\kappa \geqq 1$, and $u \geqq \lambda_{n+\kappa+1}$ we have (Maddox [5], 349),

$$(u - \omega)^{\kappa} = \sum_{\nu=0}^{\kappa} c_\nu(\omega)(u - \lambda_{n+\nu})^{\kappa} \qquad \text{where } c_\nu(\omega) = O(1).$$

Hence

$$G_{\kappa}(\omega) = \int_{\omega}^{\lambda_{n+\kappa+1}} (u - \omega)^{\kappa} dg(u) +$$

$$+\sum_{\nu=0}^{\kappa} c_\nu(\omega) \times \left\{ \int_{\lambda_{n+\nu}}^{\infty} (u-\lambda_{n+\nu})^\kappa dg(u) - \int_{\lambda_{n+\nu}}^{\lambda_{n+\kappa+1}} (u-\lambda_{n+\nu})^\kappa dg(u) \right\}.$$

The result of the lemma now follows.

4. We proceed to the proof of the theorem. The case $\kappa = 0$ is trivial (and well known). For $\kappa > 0$, we have

$$I = \int_{\lambda_0}^{\infty} |d\{\omega^{-\kappa} \sum_{\lambda\nu<\omega} (\omega-\lambda_\nu)^\kappa a_\nu \epsilon_\nu\}| = \kappa \int_{\lambda_0}^{\infty} \omega^{-\kappa-1} d\omega \left| \sum_{\lambda\nu<\omega} (\omega-\lambda_\nu)^{\kappa-1} \lambda_\nu a_\nu \epsilon_\nu \right|$$

$$= \kappa \int_{\lambda_0}^{\infty} \omega^{-\kappa-1} d\omega \left| \sum_{\lambda\nu<\omega} (\omega-\lambda_\nu)^{\kappa-1} \lambda_\nu a_\nu \left\{ \int_{\lambda\nu}^{\omega} (u-\lambda_\nu)^\kappa dg(u) + \int_{\omega}^{\infty} (u-\lambda_\nu)^\kappa dg(u) \right\} \right|$$

$$\leqq \kappa I_1 + \kappa I_2, \tag{1}$$

where

$$I_1 = \int_{\lambda_0}^{\infty} \omega^{-\kappa-1} d\omega \left| \int_{\lambda_0}^{\omega} dg(u) \int_0^u (\omega-t)^{\kappa-1}(u-t)^\kappa dB(t) \right|, \tag{2}$$

$$I_2 = \int_{\lambda_0}^{\infty} \omega^{-\kappa-1} d\omega \left| \int_{\omega}^{\infty} dg(u) \int_0^{\omega} (\omega-t)^{\kappa-1}(u-t)^\kappa dB(t) \right|, \tag{3}$$

and $b_\nu = \lambda_\nu a_\nu$. To prove that $\sum a_n \epsilon_n$ is summable $|R, \lambda, \kappa|$ we shall show that $I_1 < \infty$ and $I_2 < \infty$.

We first show that $I_1 < \infty$. The argument is valid whether κ is an integer or not. Let p be the integer such that $0 < \kappa - p \leqq 1$ (for integer κ, $p = \kappa - 1$). Take the inner integral in I_1 and integrate by parts $p+1$ times:

$$\int_0^u (\omega-t)^{\kappa-1}(u-t)^\kappa dB(t) = c \int_0^u B^p(t) \left(\frac{d}{dt}\right)^{p+1} \{(\omega-t)^{\kappa-1}(u-t)^\kappa\} dt$$

$$= \sum_{r=0}^{p+1} c_r \int_0^u (\omega-t)^{\kappa-r-1} (u-t)^{\kappa-p+r-1} B^p(t) dt, \tag{4}$$

where c, c_r are non-zero constants.

Now if $\kappa \geqq 1$, $(\omega-t)^{\kappa-r-1}(u-t)^r$ decreases in $(0, u)$. Also $B^\kappa(\omega) = O(\omega^{\kappa+1})$. Hence by the second mean-value theorem and Lemma 1, the integral in (4) is equal to

$$\omega^{\kappa-r-1} u^r \int_0^\xi (u-t)^{\kappa-p-1} B^p(t) dt, \quad (0 \leqq \xi \leqq u),$$

$$= \omega^{\kappa-1}(u/\omega)^r O(u^{\kappa+1}) = O(\omega^{\kappa-1} u^{\kappa+1}). \tag{5}$$

If $0 < \kappa < 1$, $(\omega-t)^{\kappa-1}$ increases and $(u-t)^r(\omega-t)^{-r}$ decreases and is less than or equal to 1. Hence by the second mean-value theorem and Lemma 1, the integral in (4) is equal to

$$(\omega - u)^{\kappa-1} \int_{\xi}^{u} (u-t)^{r}(\omega - t)^{-r}(u-t)^{\kappa-p-1} B^{p}(t)dt, \qquad (0 \leqq \xi \leqq u),$$

$$= (\omega - u)^{\kappa-1} \left(\frac{u-\xi}{\omega-\xi}\right)^{r} \int_{\xi}^{\xi'} (u-t)^{\kappa-p-1} B^{p}(t)dt, \qquad (\xi \leqq \xi' \leqq u),$$

$$= O\{(\omega - u)^{\kappa-1} u^{\kappa+1}\}. \tag{6}$$

Thus by (2), (4), (5) and (6) we obtain

$$I_1 = O(1) \int_{\lambda_0}^{\infty} u^{\kappa+1} |dg(u)| \int_{u}^{\infty} \{\omega^{-2} + \omega^{-\kappa-1}(\omega - u)^{\kappa-1}\} d\omega$$

$$= O(1) \int_{\lambda_0}^{\infty} u^{\kappa} |dg(u)| < \infty. \tag{7}$$

Now consider I_2. The inner integral, after repeated partial integration, is equal to

$$\int_{0}^{\omega} (\omega - t)^{\kappa-1}(u-t)^{\kappa} dB(t) = c\, B^{\kappa-1}(\omega)(u-\omega)^{\kappa} + \sum_{r=1}^{\kappa} c_r J_r, \tag{8}$$

where

$$J_r = \int_{0}^{\omega} (\omega - t)^{r-1}(u-t)^{\kappa-r} B^{\kappa-1}(t)dt, \tag{9}$$

and c, c_r are non-zero constants.

By Lemma 2, since $\kappa - 1$ is a positive integer or zero, and $B^{\kappa}(\omega) = O(\omega^{\kappa+1})$, we have for $\lambda_n < \omega \leqq \lambda_{n+1}$,

$$B^{\kappa-1}(\omega) = O\{\omega^{\kappa-1} \lambda_n \Lambda_n\}. \tag{10}$$

Hence by (10), and Lemma 3,

$$\int_{\lambda_0}^{\infty} \omega^{-\kappa-1} d\omega\, |B^{\kappa-1}(\omega) \int_{\omega}^{\infty} (u-\omega)^{\kappa} dg(u)|$$

$$= O(1) \sum_{n=0}^{\infty} \lambda_n \Lambda_n \int_{\lambda_n}^{\lambda_{n+1}} \omega^{-2} d\omega \left\{ \int_{\lambda_n}^{\lambda_{n+\kappa+1}} u^{\kappa} |\, dg(u)| + \sum_{\nu=0}^{\kappa} |\epsilon_{n+\nu}| \right\}$$

$$= O(1) \int_{\lambda_0}^{\infty} u^{\kappa} |dg(u)| + O(1) \sum_{n=0}^{\infty} |\epsilon_n| < \infty. \tag{11}$$

Finally consider the integral in (9). Since $(\omega - t)^{r-1}(u-t)^{\kappa-r}$ decreases in

$(0, \omega)$, we have by the second mean-value theorem and Lemma 1,

$$J_r = \omega^{r-1} u^{\kappa-r} \int_0^{\xi} B^{\kappa-1}(t)\, dt \qquad (0 \leqq \xi \leqq \omega),$$

$$= O\{\omega^{r-1} u^{\kappa-r} \omega^{\kappa+1}\} = O\{u^{\kappa-1} \omega^{\kappa+1}\}, \quad \text{since } r \geqq 1,\ \omega \leqq u. \tag{12}$$

Hence it follows that

$$\int_{\lambda_0}^{\infty} \omega^{-\kappa-1} d\omega \left| \int_{\omega}^{\infty} dg(u) J_r \right| = O(1) \int_{\lambda_0}^{\infty} u^{\kappa-1} |dg(u)| \int_{\lambda_0}^{u} d\omega$$

$$= O(1) \int_{\lambda_0}^{\infty} u^{\kappa} |dg(u)| < \infty. \tag{13}$$

Combining (3), (8), (11) and (13) we see that $I_2 < \infty$. Hence, by (7), we have shown that $I < \infty$. This proves the theorem.

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