# TOPOLOGICAL REPRESENTATION OF C*-ALGEBRAS

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Let $\boldsymbol{A}$ be a $C^*$-algebra, $\Omega$ the structure space of $\boldsymbol{A}$, i.e. the space of all primitive ideals in $\boldsymbol{A}$ with hull-kernel topology. At every point $P$ of $\Omega$ we associate a primitive $C^*$-algebra $\boldsymbol{A}/P$ (which we denote by $\boldsymbol{A}(P)$) and we may associate for any element $a \in \boldsymbol{A}$ the function $a(P)$ whose value at $P$ is the homomorphic image of $a$ in $\boldsymbol{A}(P)$. Then the most difficult parts of the non-commutative structure theory of $C^*$-algebras are the restrictions such as to destroy the main feature of the commutative case——the Gelfand representation of $\boldsymbol{A}$ by the *continuous* function $a(P)$ on $\Omega$. Even if $\Omega$ is a Hausdorff space, it has long been observed hopeless to discuss the continuity of the function $a(P)$ since Kaplansky [7] proposed a method to study the structure of general $C^*$-algebras and instead of these discussions the continuity of the function $\|a(P)\|$ was studied. Unfortunately this property does not give directly the suitable topological representation of algebras.

On the other hand, in [11], in the case that $\boldsymbol{A}$ satisfies the condition that any irreducible representation of $\boldsymbol{A}$ is $n$-dimensional (such a $C^*$-algebra is called $n$-dimensionally homogeneous) we have defined a topology in the set $\mathfrak{B} = \bigcup_{P \in \Omega} \boldsymbol{A}(P)$ and represented $\boldsymbol{A}$ as the algebra of all $\mathfrak{B}$-valued functions $a(P)$ on $\Omega$ with $a(P) \in \boldsymbol{A}(P)$ which is continuous in this topology (we call these functions the (continuous) cross-sections of $\mathfrak{B}$).

Now the above treatment offers a non-commutative model of the classical Gelfand representation theorem in the case that the structure space $\Omega$ is a Hausdorff space. Is it always possible to define a natural topology in the set $\mathfrak{B} = \bigcup_{P \in \Omega} \boldsymbol{A}(P)$ so that $\boldsymbol{A}$ is represented as the algebra of all continuous cross-sections of $\mathfrak{B}$ vanishing at infinity ? It is the main purpose of this paper to give a positive answer for this question and to analyse the algebras by their topological representations.

§1 and §2 are devoted to define a suitable topology in $\mathfrak{B}$ in somewhat general situations and to discuss the general structure theory of algebras of cross-sections. Some fundamental results corresponding to the algebras of continuous functions are proved here, including the Stone-Weierstrass theorem and as a direct consequence of their results we can settle the problems remained unsolved in Kaplansky [7].

In §3 we treat the above mentioned problem stating our result in rather

general form so that it may be applicable to the case where $\Omega$ is not a Hausdorff space. Roughly speaking, the result (Theorem 3.1) is the following one; if there exists an appropriate decomposition of $\Omega$ (called a continuous decomposition), then we get a locally compact Hausdorff space $X$ at each point of which a suitable $C^*$-algebra $\boldsymbol{A}(x)$ is given and, setting $\mathfrak{B} = \bigcup_{x \in X} \boldsymbol{A}(x)$, $\boldsymbol{A}$ is represented as the algebra of all cross-sections of $\mathfrak{B}$, continuous in a suitable topology in $\mathfrak{B}$ and vanishing at infinity on $X$. The case where $\Omega$ is a Hausdorff space is the one where every classes in the decomposition reduce to one point.

In [ 5 ], Kaplansky defined a class of $C^*$-algebras, central $C^*$-algebras, to which commutative methods are applicable to some extent. The structure spaces of these are always Hausdorff spaces. However, the above result shows that there are no distinctions between the centrality and the Hausdorff property of the structure spaces of $C^*$-algebras and we get, as a direct consequence of our representation theorem, the following: If the center of a $C^*$-algebra $\boldsymbol{A}$ is not contained in any primitive ideal in $\boldsymbol{A}$ then $\boldsymbol{A}$ is central if and only if the structure space of $\boldsymbol{A}$ is a Hausdorff space.

In the last section, we show the case where there exists always the non-trivial (or rather finest) continuous decomposition. Theorem 4. 1. is an another interpretation of the decomposition considered in Glimm [ 3 ] and we prove later more sharpened results for this decomposition than those of [ 3 ].

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## 1. Algebras of cross-sections.

Let $X$ be a Hausdorff topological space at each point $x$ of which a Banach algebra $\boldsymbol{A}(x)$ is given. All $\boldsymbol{A}(x)$'s are considered to be different each other. Put $\mathfrak{B} = \bigcup_{x \in X} \boldsymbol{A}(x)$. We suppose that, for each element $b \in \mathfrak{B}$, there exists uniquely a point $x \in X$ such as $b \in \boldsymbol{A}(x)$. The projection mapping $\pi$ from $\mathfrak{B}$ to $X$ is defined by $\pi(b)=x$ and $\boldsymbol{A}(x)$ is called the fibre over the point $x \in X$. A function $a(x)$ on $X$ is called a cross-section of $\mathfrak{B}$[1] if $a(x) \in \boldsymbol{A}(x)$ for each $x \in X$.

Let $f(x)$ be a complex-valued function on $X$ and $a(x)$ a cross-section of $\mathfrak{B}$. We denote by $f \cdot a$ the cross-section of $\mathfrak{B}$ defined by $f \cdot a(x) = f(x)a(x)$.

DEFINITION. *Let $\boldsymbol{A}$ be a family of cross-sections of $\mathfrak{B}$. $\boldsymbol{A}$ is said to be closed under multiplication by $f(x)$ if $f \cdot a \in \boldsymbol{A}$ for every $a \in \boldsymbol{A}$.*

We consider an arbitrary fixed family $F$[2] of cross-sections $a(x)$ of $\mathfrak{B}$ satisfying the following condition:[2]

1) Operator field in terms of Fell's recent work in Acta Math., 106(1961) (cf. concluding remark of the present paper).

2) This corresponds to the definition of the continuity structure in Fell's paper, ibidem.

(i) $\| a(x) \|$ *is continuous and bounded on* $X$,
(ii) *at each point* $x \in X$, $F(x)$ *fills out the algebra* $\boldsymbol{A}(x)$,
(iii) $F$ *forms an algebra under pointwise operations.*
Then we get the following

THEOREM 1.1. *The family* $F$ *defines a Hausdorff topology* $\mathscr{T}_F$ *in* $\mathfrak{B}$ *and the algebra of all bounded* $\mathscr{T}_F$*-continuous cross-sections of* $\mathfrak{B}$ *becomes a Banach algebra, which is closed under multiplication by* $C(X)$, *the algebra of all bounded complex-valued continuous functions on* $X$.

PROOF. Take an arbitrary element $b_0 \in \mathfrak{B}$, an element $a \in F$ with $a(x_0) = b_0$, and a neighborhood $U$ of $x_0 = \pi(b_0)$. Put $\mathcal{U}(b_0,U,\varepsilon,a(x)) = \bigcup_{x \in U} \{ b \in \mathfrak{B} \mid b \in \boldsymbol{A}(x)$ and $\|b - a(x)\| < \varepsilon \} = \{ b \in \mathfrak{B} \mid \pi(b) = x \in U$ *and* $\|b - a(x)\| < \varepsilon \}$,
where $\varepsilon$ is an arbitrary positive number. Then a straight-forward calculation shows that the family $\{ \mathcal{U}(b_0,U,\varepsilon,a(x)) \mid b_0 \in \mathfrak{B} \}$ forms a neighborhood system of $\mathfrak{B}$ and defines a topology $\mathscr{T}_F$ in $\mathfrak{B}$.

Besides, one sees that $\mathscr{T}_F$ is a Hausdorff topology and the relative topology of $\mathscr{T}_F$ in $\boldsymbol{A}(x)$ coincides with the original norm topology of $\boldsymbol{A}(x)$.

Let $\tilde{C}_F(X, \mathfrak{B})$ be the set of all bounded cross-sections of $\mathfrak{B}$ continuous in $\mathscr{T}_F$-topology. We notice that the function $\|a(x)\|$ is a continuous function on $X$ for each $a \in \tilde{C}_F(X,\mathfrak{B})$. In fact, let an arbitrary positive number $\varepsilon$ and a point $x_0 \in X$ be given. Take an element $a_0 \in F$ with $a_0(x_0) = a(x_0)$. Since each of the functions of $F$ is norm continuous, we can find a neighborhood $U$ of $x_0$ such as

$$| \, \|a_0(x)\| - \|a_0(x_0)\| \, | < \varepsilon/2 \text{ for every } x \in U.$$

On the other hand, the continuity of $a(x)$ in $\mathscr{T}_F$ implies that there exists a neighborhood $V$ of $x_0$ such as

$$a(x) \in \mathcal{U}(a(x_0), U, \frac{\varepsilon}{2}, a_0(x)) \text{ for every } x \in V.$$

Hence we have

$$| \, |\|a(x)\| - \|a(x_0)\| \, | \leqq | \, \|a(x)\| - \|a_0(x)\| \, | + | \, \|a_0(x)\| - \|a_0(x_0)\| \, |$$
$$\leqq \|a(x) - a_0(x)\| + \frac{\varepsilon}{2} < \varepsilon$$

at eaoh point $x \in V$.

Now, it is not difficult to see that $\tilde{C}_F(X, \mathfrak{B})$ is closed under pointwise addition, multiplication and scalar multiplication. Define the norm $\|a\| = \sup \|a(x)\|$ for $a \in \tilde{C}_F(X, \mathfrak{B})$, then $\tilde{C}_F(X, \mathfrak{B})$ becomes a Banach algebra. The one non-trivial point here is the completeness of $\tilde{C}_F(X, \mathfrak{B})$. Let $\{a_n\}$ be a Cauchy sequence in $\tilde{C}_F(X, \mathfrak{B})$. One easily verifies that the sequences $\{a_n(x) \mid x \in X\}$ are uniformly Cauchy sequences and,as A(x)'s are complete, $\{a_n(x) \mid x \in X\}$ define a cross-section

$a(x) = \lim_{n\to\infty} a_n(x)$. Clearly $a(x)$ is a bounded cross-section of $\mathcal{B}$. We assert that this is continuous in $\mathcal{T}_F$. Let $x_0$ be an arbitrary point of $X$ and $\mathcal{U}(a(x_0), U_0, \varepsilon, a'(x))$ a neighborhood of $a(x_0)$ There exists a number $n_0$ such that

$$\|a(x)-a_n(x)\|<\varepsilon/3 \text{ for every } n\geqq n_0 \text{ and } x\in X.$$

Let $a''\in F$ be an element with $a''(x_0) = a_{n_0}(x_0)$. Since

$$\|a'(x_0) - a''(x_0)\| = \|a(x_0) - a_{n_0}(x_0)\| <\varepsilon/3,$$

there exists a neighborhood $U_1$ of $x_0$ such as

$$\|a'(x) - a''(x)\|<\varepsilon/3. \text{ for every } x\in U_1.$$

Moreover $a''(x_0) = a_{n_0}(x_0)$ and $a''$, $a_{n_0}\in \widetilde{C}_F(X, \mathcal{B})$ imply that we can find a neighborhood $U_2$ of $x_0$ such as

$\|a''(x) - a_{n_0}(x)\|<\varepsilon/3$ for every $x\in U_2$. Then at each point $x$ in the neighborhood $U$ of $x_0$ which is contained in all of $U_0, U_1$ and $U_2$, we have

$$\begin{aligned}\|a(x) - a'(x)\| &\leqq \|a(x) - a_{n_0}(x)\| + \|a_{n_0}(x) - a''(x)\| + \|a''(x) - a'(x)\| \\ &<\varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

That is, $a(x)\in \mathcal{U}(a(x_0), U_0, \varepsilon, a'(x))$. Thus the first half part of the theorem is proved.

Now let $f(x)$ be an arbitrary bounded complex-valued continuous function on $X$ and take a cross-section $a\in \widetilde{C}_F(X, \mathcal{B})$. It is clear that $f\cdot a$ is a bounded cross-section of $\mathcal{B}$. Let $x_0$ be a point of $X$ and consider a neighborhood $\mathcal{U}(f(x_0)$ $a(x_0), U_0, \varepsilon, a_0(x))$ of $f(x_0)\, a(x_0)$. Take an element $a_1\in F$ with $a(x_0) = a_1(x_0)$. Since $a(x)$ is continuous in $\mathcal{T}_F$ we can find a neighborhood $U_1$ of $x_0$ such that

$$\|a(x) - a_1(x)\|<\varepsilon/3m \text{ for every } x\in U_1,$$

where $m = \sup_{x\in X}|f(x)|$. On the other hand, the continuity of $f(x)$ implies that there exists a neighborhood $U_2$ of $x$ such as

$$|f(x) - f(x_0)|<\varepsilon/3\|a_1\| \text{ for every } x\in U_2.$$

Finally, as $f(x_0)a_1\in F$ and $f(x_0)a_1(x_0) = f(x_0)a(x_0) = a_0(x_0)$ there exists a neighborhood $U_3$ of $x_0$ at each point $x$ of which

$$\|f(x_0)a_1(x) - a_0(x)\|<\varepsilon/3.$$

Therefore, at each point $x$ of the neighborhood $U$ of $x_0$ which is contained in all of the above neighborhoods, we have

$$\begin{aligned}\|f(x)a(x) - a_0(x)\| &\leqq \|f(x)a(x) - f(x)a_1(x)\| \\ &\quad + \|f(x)a_1(x) - f(x_0)a_1(x)\| + \|f(x_0)a_1(x) - a_0(x)\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

Hence $f\cdot a(x)$ is a bounded continuous cross-section of $\mathcal{B}$. That is, $f\cdot a \in \widetilde{C}_F(X,\mathcal{B})$. This completes the proof.

Now we assume, for the rest of the discussions, that $X$ is a locally compact Hausdorff space and $\boldsymbol{A}(x)$'s are $C^*$-algebras. We consider a fixed family $F$ of cross-sections $a(x)$ of $\mathcal{B}$ satisfying the following conditions;

(a) $\|a(x)\|$ *is continuous on* $X$ *and vanishes at infinity,*

(b) *at each point* $x \in X$, $F(x)$ *fills out the algebra* $\boldsymbol{A}(x)$,

(c) $F$ *forms a self-adjoint algebra under pointwise operations.*

Denote by $C_F(X,\mathcal{B})$[3] the algebra of all cross-sections of $\mathcal{B}$, continuous in $\mathcal{T}_F$-topology and vanishing at infinity of $X$. (Here we mean a cross-section $a(x)$ vanishing at infinity if the function $\|a(x)\|$ vanishes at infinity). We notice that the proof of Theorem 1.1. can be applicable to the algebra $C_F(X, \mathcal{B})$ and we see that $C_F(X, \mathcal{B})$ is a $C^*$-algebra. Moreover for any cross-section $a(x)$ in $C_F(X, \mathcal{B})$ and any bounded complex-valued continuons function $f(x)$, the cross-scction $f\cdot a(x)$ is $\mathcal{T}_F$-continuous and vanishes at infinity. It follows that $C_F(X, \mathcal{B})$ is closed under multiplication by $C(X)$, the algebra of all bounded complex-valued continuous function on $X$.

If $X$ is compact and all $\boldsymbol{A}(x)$'s are isomorphic to a fixed $C^*$-algebra $\boldsymbol{A}$ and $F$ is a family of so-called constant cross-sections, then $C_F(X, \mathcal{B})$ is isomorphic to the usual $\boldsymbol{A}$-valued continuous function algebra $C(X, \boldsymbol{A})$. Moreover it is not difficult to see that in this case the space $\mathcal{B}$ with $\mathcal{T}_F$-topology is homeomorphic with the product space $X\times\boldsymbol{A}$. But generally the situation is not so simple as we shall see from the discussions in section **3** and Tomiyama-Takesaki [11].

The next theorem shows that the cross-section algebra $C_F(X, \mathcal{B})$ satisfies the condition corresponding to the regularity in commutative function algebras.

THEOREM 1.2. *For any closed set* $G$ *in* $X$, *any point* $x_0 \notin G$ *and an arbitrary element* $b$ *in* $\boldsymbol{A}(x_0)$, $C_F(X, \mathcal{B})$ *contains a cross-section* $a(x)$ *such that* $a(x_0)=b$ *and* $a(x)=0$ *for every* $x \in G$.

PROOF. Let $a'(x)$ be an element of $C_F(X, \mathcal{B})$ with $a'(x_0) = b$ and $f(x)$ a bounded complex-valued continuous function on $X$ with $f(x_0) = 1$ and $f(G) = 0$. Then $a=f\cdot a' \in C_F(X, \mathcal{B})$ satisfies the property.

LEMMA 1.1. *Let* $P$ *be a primitive ideal in* $C_F(X, \mathcal{B})$. *Then there exists uniquely a point* $x_0$ *in* $X$ *and a primitive ideal* $\mathrm{P}(x_0)$ *in* $\boldsymbol{A}(x_0)$ *such that*

$$P = \{a \in C_F(X, \mathcal{B}) \mid a(x_0) \in P(x_0)\}.$$

PROOF. Let $X_0$ be the one-point compactification of $X$. Adding new fibre $\boldsymbol{A}(x_\infty) = 0$ at the exceptional point $x_\infty$, $C_F(X, \mathcal{B})$ may be considered to be the algebra of all cross-sections of $\mathcal{B}' = \mathcal{B} \cup \boldsymbol{A}(x_\infty)$ continuous in $\mathcal{T}_F$-topology.

(3) This definition is the same as the maximal full algebra of operator fields in Fell's paper in Acta Math., 106 (1961).

Hence, by Lemma 3.2 in Kaplansky [7], we see that the algebra $C_F(X, \mathcal{B})$ on $X_0$ satisfies all the conditions (a) to (d) in [7 : p.225]. Thus, coming back to the algebra $C_F(X,\mathcal{B})$ on $X$ one easily see that we can freely use Therem 3.1 in [7] on $X$.

By Theorem 3.1 in [7] we have

$$P = \{a \in C_F(X, \mathcal{B}) \mid a(x) \in P(x) \text{ for every } x \in X\}$$

where $P(x)$ means the closed ideal in $\boldsymbol{A}(x)$ consisting of all $a(x)$'s for $a \in P$. Suppose that there exist different points $x_1, x_2$ such that $P(x_1)$ and $P(x_2)$ are proper closed ideals in $\boldsymbol{A}(x_1)$ and $\boldsymbol{A}(x_2)$ respectively. Let $U(x_1)$ and $U(x_2)$ be disjoint neighborhoods of $x_1$ and $x_2$, and put

$$P_1 = \{a \in C_F(X, \mathcal{B}) \mid a(x) \in P(x) \text{ for } x \in U(x_2)^c\},$$

$$P_2 = \{a \in C_F(X, \mathcal{B}) \mid a(x) \in P(x) \text{ for } x \in U(x_1)^c\},$$

where $U(x_1)^c$ and $U(x_2)^c$ mean the complements of $U(x_1)$ and $U(x_2)$. $P_1$ and $P_2$ are proper closed ideals in $C_F(X, \mathcal{B})$ and since $U(x_1)^c \cup U(x_2)^c = X$ we have $P_1 \cap P_2 = P$. On the other hand, by Theorem 1.2. $C_F(X, \mathcal{B})$ contains a cross-section $a(x)$ such that $a(x_2) \notin P(x_2)$ and $a(x) = 0$ for $x \in U(x_2)^c$. Hence we get $P_1 \supsetneqq P$ and similarly $P_2 \supsetneqq P$, which is a contradiction. Therefore there exists only one point $x_0 \in X$ where $P(x_0)$ is a proper ideal in $\boldsymbol{A}(x_0)$. We have

$$P = \{a \in C_F(X, \mathcal{B}) \mid a(x_0) \in P(x_0)\}.$$

It is not difficult to see that the ideal $P(x_0)$ is a primitive ideal in $\boldsymbol{A}(x_0)$. This completes the proof.

Now let $\Omega$ be the structure space of $C_F(X. \mathcal{B})$, i.e. the space of all primitive ideals in $C_F(X, \mathcal{B})$ with hull-kernel topology.

$$I_x = \{a \in C_F(X, \mathcal{B}) \mid a(x) = 0\}.$$

Clearly $I_x$ is a closed ideal in $C_F(X, \mathcal{B})$. We denote by $h(I_x)$ the hull of $I_x$ in $\Omega$, that is, $h(I_x) = \{P \in \Omega \mid P \supset I_x\}$.

The following lemma is almost clear, so we omit the proof.

LEMMA. 1.2. *$h(I_x)$ is homeomorphic with the structure space of $\boldsymbol{A}(x)$.*

Then we get the structure theorem for $\Omega$.

THEOREM 1.3. *$\Omega = \bigcup_{x \in X} h(I_x)$ is a decomposition of $\Omega$ into closed sets $h(I_x)$ and the space $X$ is homeomorphic with the quotient space of this decomposition. In particular, if all $\boldsymbol{A}(x)$'s are simple $C^*$-algebras, $X$ is homeomorphic with $\Omega$, hence in this case $\Omega$ is a Hausdorff space.*

PROOF. By Lemma 1.1 we see that $\bigcup_{x \in X} h(I_x)$ is a decomposition of $\Omega$.

Let $O$ be an open set in $X$ and put $\widetilde{O} = \bigcup_{x \in O} h(I_x)$. We show that $\widetilde{O}$ is an open set in $\Omega$. Let $P$ be a primitive ideal in $C_F(X, \mathcal{B})$ such as $P \supset k(\widetilde{O}^c)$, where $k(\widetilde{O}^c)$ means the kernel of the complement of $\widetilde{O}$ in $\Omega$. $P$ belongs to some $h(I_{x_0})$ by Lemma 1.1. Suppose that $P \in \widetilde{O}$, then $x_0 \in O$. By Theorem 1.2, there exists a cross-section $a(x)$ in $C_F(X, \mathcal{B})$ satisfying the condition that $a(x_0) \notin P(x_0)$ and $a(x) = 0$ for each $x \in \widetilde{O}^c$, then $a \in k(\widetilde{O}^c)$ and $a \notin P$. This is a contradiction. Hence $P \in \widetilde{O}^c$ and $\widetilde{O}$ is an open set in $\Omega$.

Conversely let $\widetilde{O} = \bigcup_{x \in O} h(I_x)$ be an open set in $\Omega$ and $x_0$ be an arbitrary point of the closure of $O^c$, the complement of $O$ in $X$. We must show that $x_0 \in O^c$. Suppose on the contrary that $x_0 \in O$, then for an ideal $P \in h(I_{x_0})$ we can find a cross-section $a \in C_F(X, \mathcal{B})$ such as $a \in k(\widetilde{O}^c)$ and $a \notin P$ because $P$ does not belong to the closed set $\widetilde{O}^c$. Since $\widetilde{O}^c = \bigcup_{x \in O} h(I_x)$, this means that $a(x) = 0$ for every $x \in O^c$ and $a(x_0) \neq 0$. However this contradicts the continuity of $a(x)$. Thus $x_0 \in O^c$ and $O$ is an open set in $X$.

Since there is one-to-one correspondence between $X$ and the quotient space of the decomposition $\Omega = \bigcup_{x \in X} h(I_x)$, we have shown that this correspondence is bicontinuous.

## 2. Subalgebras of algebras of cross-sections.

In order to prove the non-commutative Stone-Weierstrass theorem for cross-section algebras, we need the following theorem which is a direct consequence of Glimm's strengthened non-commutative Stone-Weierstrass theorem of pure state type (cf. Glimm [3]).

THEREM 2.1. *Let $\boldsymbol{A}$ be a C*-algebea and $\boldsymbol{B}$ a C*-subalgebra of $\boldsymbol{A}$. Suppose that $\boldsymbol{B}$ separates the $w^*$-closure of the pure states of $\boldsymbol{A}$. Then $\boldsymbol{A} = \boldsymbol{B}$ if both $\boldsymbol{A}$ and $\boldsymbol{B}$ have a unit or $\boldsymbol{A}$ has no unit. If $\boldsymbol{A}$ has a unit and $\boldsymbol{B}$ has not, $\boldsymbol{A}$ coincides with the algebra generated by $\boldsymbol{B}$ and a unit.*

PROOF. Let $\boldsymbol{A}_1$ be a C*-algebra obtained by adjoining a unit to $\boldsymbol{A}$, then the algebra $\boldsymbol{B}_1$ obtained also by adjoining a unit to $\boldsymbol{B}$ is naturally considered to be a C*sub-algebra of $\boldsymbol{A}_1$. Let $\varphi$ be an element of the $w^*$-closure of the pure states of $\boldsymbol{A}_1$ and $\{\varphi_\alpha\}$ a net of pure states of $\boldsymbol{A}_1$ converging weakly to $\varphi$. If $\varphi$ is a non-zero functional on $\boldsymbol{A}$, we may suppose that all $\varphi_\alpha$'s are non-zero functionals on $\boldsymbol{A}$ and, since $\boldsymbol{A}$ is a closed ideal in $\boldsymbol{A}_1$, this implies that all $\varphi_\alpha$'s are pure states of $\boldsymbol{A}$ by an argument in the proof of Theorem 2 in Tomiyama-Takesaki [11]. Hence $\varphi | \boldsymbol{A}$, the restriction of $\varphi$ to $\boldsymbol{A}$, belongs to the pure states of $\boldsymbol{A}$, too. On the other hand, it is clear that the $w^*$-closure of the pure states

of $\boldsymbol{A}$ contains zero-functional if $\boldsymbol{A}$ has no unit (cf. Glimm [4; Lemma 9]).

Now let $\varphi$ and $\psi$ be different elements of the $w^*$-closure of the pure states of $\boldsymbol{A}_1$. Then we have $\varphi \neq \psi$ on $\boldsymbol{A}$. Since $\varphi|\boldsymbol{A}$ and $\psi|\boldsymbol{A}$ belong to the $w^*$-closure of the pure states of $\boldsymbol{A}$ as mentioned above, we can find an element $a \in \boldsymbol{B}$ such as $\varphi(a) \neq \psi(a)$. Hence $\boldsymbol{B}_1$ separates the $w^*$-closure of the pure states of $\boldsymbol{A}_1$ and we get $\boldsymbol{A}_1 = \boldsymbol{B}_1$ by Glimm [3: Theorem 1]. Therefore we can deduce the conclusion in each case stated in the theorem.

It is not difficult to see that the last case in Theorem 2. 1. really arises even if $\boldsymbol{A}$ is a CCR algebra. This case corresponds to the case in usual Stone-Weierstrass theorem that $\boldsymbol{B}$ coincides with the algebra of all continuous functions vanishing at a single point. Thus the non-commutative Stone-Weierstrass theorem of *CCR* algebras stated in Kaplansky [7: Theorem 7.2] is generally insufficient if we do not restrict the case to a certain limit.

Using Theorem 2.1 we can prove the following non-commutative Stone-Weierstrass theorem for the cross-section algebra $C_F(X,\mathcal{B})$ defined in section 1.

THEOREM 2.2. *Let $\boldsymbol{C}$ be a self-adjoint subalgebra of $C_F(X,\mathcal{B})$* where $\mathcal{B} = \bigcup_{x\in X} \boldsymbol{A}(x)$. *Suppose that for any distinct points $x, y \in X$, $\boldsymbol{C}$ contains cross-sections taking arbitrary pairs of values in $\boldsymbol{A}(x)$, $\boldsymbol{A}(y)$ at $x,y$. Then $\boldsymbol{C}$ is dense in $C_F(X,\mathcal{B})$.*

PROOF. Let $\varphi$ be an element of the $w^*$-closure of the pure states of $C_F(X,\mathcal{B})$ and $\{\varphi_\alpha\}$ a net of pure states converging weakly to $\varphi$. Put

$P_\alpha = \{a \in C_F(X,\mathcal{B}) | \varphi_\alpha(b^*ac) = 0$ for every $b, c \in C_F(X,\mathcal{B})\}$. Then it is known that $P_\alpha \in \Omega$ for each $\alpha$. Suppose that $\{P_\alpha\}$ is not eventually in any compact set of $\Omega$. Denote by $a(P)$ the homomorphic image of $a \in C_F(X,\mathcal{B})$ in $C_F(X,\mathcal{B})/P$ for an ideal $P$. Since the sets $\{P \in \Omega | \|a(P)\| \geqq \varepsilon\}$ for $\varepsilon$ positive are compact (cf. [7: Lemma 4. 3]), one easily verifies that $\varphi = 0$. Hence if $\varphi \neq 0$, $\{P_\alpha\}$ must be eventually in some compact set in $\Omega$ and in this case we may suppose, without loss of generality, that $P$ converges to some point $P_0$ in $\Omega$.

Now let $\varphi$ and $\psi$ be different elements of the $w^*$-closure of the pure states of $C_F(X,\mathcal{B})$ and $\{\varphi_\alpha\}$, $\{\psi_\beta\}$ nets of pure states converging to $\varphi$ and $\psi$ respectively. Put

$$P_\alpha = \{a \in C_F(X,\mathcal{B}) | \varphi_\alpha(b^*ac) = 0 \text{ for every } b, c \in C_F(X,\mathcal{B})\}$$

and

$$Q_\beta = \{a \in C_F(X,\mathcal{B}) | \psi_\beta(b^*ac) = 0 \text{ for every } b, c \in C_F(X,\mathcal{B})\}.$$

We assume at first that both $\varphi$ and $\psi$ are non-zero functionals on $C_F(X,\mathcal{B})$. Then we may suppose that $\{P_\alpha\}$ and $\{Q_\beta\}$ converge to some points $P_0$ and $Q_0$ in $\Omega$. By Lemma 1.1 for each primitive ideal $P_\alpha$ there exists a point $x_\alpha \in X$ and a primitive ideal $P(x_\alpha)$ in $\boldsymbol{A}(x_\alpha)$ such that

$$P_\alpha = \{a \in C_F(X, \mathcal{B}) \mid a(x_\alpha) \in P(x_\alpha)\}.$$

Similarly $Q_\beta$ may be written as

$$Q_\beta = \{a \in C_F(X, \mathcal{B}) \mid a(y_\beta) \in Q(y_\beta)$$

for some point $y_\beta \in X$ and primitive ideal $Q(y_\beta)$ in $\boldsymbol{A}(y_\beta)$.

Let

$$P_0 = \{a \in C_F(X, \mathcal{B}) \mid a(x_0) \in P(x_0)\}$$

and

$$Q_0 = \{a \in C_F(X, \mathcal{B}) \mid a(y_0) \in Q(y_0)\}$$

where $P(x_0)$ and $Q(y_0)$ mean the primitive ideals in $\boldsymbol{A}(x_0)$ and $\boldsymbol{A}(y_0)$ respectively. Then, by Theorem 1. 3, $x_\alpha$ converges to $x_0$ and $y_\beta$ to $y_0$. Takea cross-section $a \in C_F(X, \mathcal{B})$ with $a(x_0) = 0$, then $\|a(x_\alpha)\|$ converges to $\|a(x_0)\| = 0$ and as

$$|\varphi_\alpha(a)| \leqq \|a(P_\alpha)\| = \|a(x_\alpha)(P(x_\alpha))\| \leqq \|a(x_\alpha)\|$$

we get $\varphi(a) = 0$. Similarly $\psi(a) = 0$ for any cross-section $a \in C_F(X, \boldsymbol{B})$ with $a(y_0) = 0$. Here we have two cases in question.

1. the case $x_0 = y_0$. Let $a$ be an element of $C_F(X, \mathcal{B})$ such as $\varphi(a) \neq \psi(a)$. We can find an element $a'$ in $\boldsymbol{C}$ with $a(x_0) = a'(x_0)$. Then $a(x_0) - a'(x_0) = a(y_0) - a'(y_0) = 0$ and

$$\varphi(a') = \varphi(a) \neq \psi(a) = \psi(a').$$

2. the case $x_0 \neq y_0$. Let

$$P_0' = \{a \in C_F(X, \mathcal{B}) \mid \varphi(b^*ac) = 0 \text{ for every } b, c \in C_F(X, \mathcal{B})\}$$

and

$$Q_0' = \{a \in C_F(X, \mathcal{B}) \mid \psi(b^*ac) = 0 \text{ for every } b, c \in C_F(X, \mathcal{B})\}.$$

$P_0$ and $Q_0'$ are not contained in each other, for $P_0'$ contains the ideal $\{a \in C_F(X, \mathcal{B}) \mid a(x_0) = 0\}$ and $Q_0'$ the ideal $\{a \in C_F(X, \mathcal{B}) \mid a(y_0) = 0\}$. Hence there exists an element $a \in C_F(X, \mathcal{B})$ such as $a \in P_0'$ and $a \notin Q_0'$, so that we get some elements $b, c$ in $C_F(X, \mathcal{B})$ such as $\varphi(b^*ac) = 0$ and $\psi(b^*ac) = 0$. Take an element $a' \in \boldsymbol{C}$ with $a'(x_0) = b^*ac(x_0)$ and $a'(y_0) = b^*ac(y_0)$. We have,

$$\varphi(a') = \varphi(b^*ac) = 0, \text{ and } \psi(a') = \psi(b^*ac) \neq 0.$$

On the other hand, if one of $\varphi$ and $\psi$ is zero, say $\varphi$, then $\psi$ determines a point $x_0 \in X$ and $\psi(a) = 0$ whenever $a(x_0) = 0$. Hence one verifies easily that the restriction of $\psi$ to $\boldsymbol{C}$ is a nonzero functional, too.

Now let $\widetilde{\boldsymbol{C}}$ be the closure of $\boldsymbol{C}$ in $C_F(X, \mathcal{B})$. We must show that $\widetilde{C} = \boldsymbol{C}_F(X, \mathcal{B})$. Clearly $\widetilde{\boldsymbol{C}}$ is a $C^*$-subalgebra of $C_F(X, \mathcal{B})$ and the above discussion shows that

$\widetilde{C}$ separates the $w^*$-closure of the pure states of $C_F(X, \mathcal{B})$. Hence if $C_F(X, \mathcal{B})$ has no unit we get directly $\widetilde{C} = C_F(X, \mathcal{B})$ by Theorem 2.1. In the case that $C_F(X, \mathcal{B})$ has a unit, it is sufficient to show that $\boldsymbol{C}$ has a unit, too. Otherwise, $\boldsymbol{C}$ is a maximal ideal in $C_F(X, \mathcal{B})$ whose quotient algebra is one-dimensional but this is a contradiction as it is easily seen from [7: Theorem 3.1] and the condition for $\boldsymbol{C}$. Therefore in any case $\widetilde{\boldsymbol{C}} = C_F(X, \mathcal{B})$. This completes the proof.

Theorem 2.2. offers the affirmative answer to the question in Kaplansky [7], that is, Theorem 3.3 and 3.4 in [7] can be proved without any restriction on the fibre $\boldsymbol{A}(x)$. Both Corollary 1 and 2 are readily deduced from Theorem 2.2.

COROLLARY 2.1.1. *Let $X$ be a locally compact Hausdorff space at each point of which a $C^*$-algebra $\boldsymbol{A}(x)$ is given. Let $\boldsymbol{A}$ be a $C^*$-algebra of cross-sections $a(x)$ of $\mathcal{B}(=\bigcup_{x\in X} \boldsymbol{A}(x))$ satisfying the postulate that $\|a(x)\|$ is continuous and vanishing at infinity. Suppose further that for any distinct points $x, y \in X$, $\boldsymbol{A}$ contains functions taking arbitrary pairs of values in $\boldsymbol{A}(x)$, $\boldsymbol{A}(y)$ at $x, y$. Then $\boldsymbol{A}$ is closed under multiplication by $C(X)$, the algebra of all bounded continuous functions on $X$.*

COROLLARY 2.2.2. *Let $X$ be a locally compact Hausdorff space, $\boldsymbol{D}$ a $C^*$-algebra and $\boldsymbol{A}$ the $C^*$-algebra of all continuous functions vanishing at infinity from $X$ to $\boldsymbol{D}$. Let $\boldsymbol{B}$ be a $C^*$-subalgebra of $\boldsymbol{A}$, which contains functions taking arbitrary prescribed pairs of values in $\boldsymbol{D}$ at every distinct points $x, y \in X$. Then $\boldsymbol{A} = \boldsymbol{B}$.*

Let $\boldsymbol{C}$ be a self-adjoint subalgebra of $C_F(X, \mathcal{B})$. As in the case of commutative function algebras the weakest topology in $X$ for which each $a(x) \in \boldsymbol{C}$ is norm continuous (that is, the function $\|a(x)\|$ is continuous) is called the $\boldsymbol{C}$-topology in $X$.

THEOREM 2.3. *If $\boldsymbol{C}$ is a self-adjoint subalgebra of $C_F(X, \mathcal{B})$ which contains cross-sections taking arbitrary pairs of values in $\boldsymbol{A}(x)$, $\boldsymbol{A}(y)$ at any distinct points $x, y$ in $X$, then the given topology in $X$ is equivalent to the $\boldsymbol{C}$-topology.*

PROOF. Since the function $\|a(x)\|$ is continuous in the original topology in $X$ for any cross-section $a(x) \in \boldsymbol{C}$, it is clear that the original topology is stronger than the $\boldsymbol{C}$-topology. Hence any closed set in $\boldsymbol{C}$-topology is closed in the original topology, too. Conversely, let $G$ be a closed set $X$ in the original topology. We assert that

$$G = \{x \in X \mid I_x \supset \bigcap_{y\in G} I_y\}.$$

In fact, it is clear that $G \subseteqq \{x \in X \mid I_x \supset \bigcap_{y\in G} I_y\}$. Take a point $x_0$ in the right member. If $x_0$ does not belong to $G$, then we can find a cross-section $a(x)$ in $C_F(X, \mathcal{B})$

such that $a(x) = \mathfrak{0}$ on $G$ and $a(x_0) \neq 0$, a contradiction. Let $x_0$ be a point in the closure of $G$ in the $\boldsymbol{C}$-topology and take a cross-section $a \in \bigcap_{x \in G} I_x$. Clearly $a(x) = 0$ for every $x \in G$. Since $\boldsymbol{C}$ is dense in $C_F(X, \mathfrak{B})$ by Theorem 2.2, all cross-sections in $C_F(X, \boldsymbol{B})$ are norm continuous in the $\boldsymbol{C}$-topology. Therefore $a(x_0) = 0$, hence $I_{x_0} \supset \bigcap_{x \in G} I_x$. We have $x_0 \in G$ and $G$ is closed in the $\boldsymbol{C}$-topology. This completes the proof.

THEOREM 2.3. *Let $G$ be closed set in $X$. Then any $\mathfrak{T}_F$- continuous cross-section $a(x)$ defined on $G$ and vanishing at infinity can always be extended to the whole space $X$.*

PROOF. Let

$$I = \{a \in C_F(X, \mathfrak{B}) \mid a(x) = 0 \text{ for } x \in G\}$$

and $C_0$ the algebra of all $\mathfrak{T}_F$-continuous cross-section on $G$ vanishing at infinity. Consider the factor algebra $C_F(X, \mathfrak{B}/I$, then the mapping $[a] \to a(x) \mid G$ is the natural embedding of $C_F(X, \mathfrak{B})/I$ into $C_0$ where $[a]$ means the class to which $a(x)$ belongs and $a(x) \mid G$ the restriction of $a(x)$ to $G$. By Theorem 2.2 this embedding is onto. Hence any $\mathfrak{T}_F$-continuous cross-section on $G$ vanishing at infinity is the restriction of an element in $C_F(X, \mathfrak{B})$.

**3. Topological representation of $C^*$-algebras as algebras of cross-sections.**

Let $\boldsymbol{A}$ be a $C^*$-algebra and $\Omega$ the structure of $\boldsymbol{A}$, that is, the space of all primitive ideals in $\boldsymbol{A}$ with hull-kernel topology. We denote by $a(P)$ the homomorphic image of $a \in \boldsymbol{A}$ in the quotient algebra $\boldsymbol{A}/P$ by an ideal $P$ in $\boldsymbol{A}$. Let $\Omega = \bigcup_{\alpha \in \Gamma} \Omega_\alpha$ be a decomposition into closed sets of $\Omega$ and put $x_\alpha = k(\Omega_\alpha)$ (kernel of $\Omega_\alpha$). Then there is a one-to-one correspondence between the set of ideals $X = \{x_\alpha \mid \alpha \in \Gamma\}$ and the quotient space of $\Omega$ with respect to this decomposition, so that we can consider on $X$ the quotient topology of this decomposition.

DEFINITION. *Let $\Omega = \bigcup_{\alpha \in \Gamma} \Omega_\alpha$ be a Hausdorff decomposition*[4] *of $\Omega$ and put $X = \{x_\alpha \mid \alpha \in \Gamma\}$ where $x_\alpha = \mathrm{k}(\Omega_\alpha)$. We call $X$ the decomposition space of $\Omega$. If we have*

$$\widetilde{S} = \{x \in X \mid x \supset \bigcap_{y \in S} y\}$$

*for any subset $S$ in $X$ where $\widetilde{S}$ means the closure of $S$ in the quotient topology, this decomposition is called a continuous decomposition of $\Omega$.*

(4) A decomposition is called a Hausdorff decomposition if the quotient space of the decomposition is a Hausdorff space.

With this definition we get the following topological representation theorem of $C^*$-algebras.

THEOREM 3.1. *Let $\boldsymbol{A}$ be a $C^*$-algebra and $\Omega = \bigcup_{\alpha\in\Gamma} \Omega_\alpha$ a continuous decomposition of the structure space $\Omega$ of $\boldsymbol{A}$. Then the decomposition space $X = \{x_\alpha | \alpha \in \Gamma\}$ with quotient topology is a locally compact Hausdorff space on which each element $a \in \boldsymbol{A}$ is represented as the cross-section $a(x)$ satisfying the postulate that $\|a(x)\|$ is continuous on $X$ and vanishing at infinity. Put $\mathcal{B} = \bigcup_{x\in X} \boldsymbol{A}(x)$. Then $\boldsymbol{A}$ is represented as $\boldsymbol{C}_A(X, \mathcal{B})$ the algebra of all cross-sections of $\mathcal{B}$ continuous in $\mathcal{T}_A$-topology and vanishing at infinity of $X$.*

PROOF. From the definition of $X$, $X$ is a Hausdorff space. Let $a$ be an element of $\boldsymbol{A}$ and $\varepsilon$ a positive number. Put $K = \{x \in X | \|a(x)\| \geqq \varepsilon\}$. Then $K$ is an image of the set $\{P \in \Omega | \|a(P)\| \geqq \varepsilon\}$ in $\Omega$ by the quotient map, for it is clear that the latter is mappped into $K$ and moreover for any point $x \in X$ there exists a primitive ideal $P$ which contains $x$ and $\|a(P)\| = \|a(x)\|$ (cf. Kaplansky [7 : p. 234]). Since the set $\{P \in \Omega | \|a(P)\| \geqq \varepsilon\}$ is compact by [7 : Lemma 4.3], $K$ is a compact subset of $X$. Hence $K$ is closed in $X$ because $X$ is a Hausdorff space. Therefore the function $\|a(x)\|$ is upper semi-continuous in $X$.

In order to prove the lower semi-continuity of $\|a(x)\|$ we must show that the sets $\{x \in X | \|a(x)\| \leqq \varepsilon\}$ for $\varepsilon$ positive are closed in $X$. Because of the identity $\|a^*a\| = \|a\|^2$, we need consider only the case where $a$ is self-adjoint. Suppose that $x_0$ is in the closure of the set $S = \{x \in X | \|a(x)\| \leqq \varepsilon\}$ and $\|a(x_0)\| = \rho > \varepsilon$. Let $\gamma(x)$ be a real-valued continuous function defined as follows: $\gamma((-\infty, \varepsilon]) = 0$, $\gamma([\rho, +\infty]) = 1$ and $\gamma(x)$ is linear on $[\varepsilon, \rho]$. Then $\gamma(a)(x) = \gamma(a(x)) = 0$ for every $x \in S$ hence $\gamma(a) \in k(S)$, the kernel of $S$ and $\gamma(a)(x_0) \neq 0$, that is, $\gamma(a) \notin x_0$. However this contradicts the definition of a continuous decomposition. Hence $x_0 \in S$.

Therefore, $\|a(x)\|$ is a continuous function on $X$ and $X$ is a locally compact space.

Now put $\mathcal{B} = \bigcup_{x\in X} \boldsymbol{A}(x)$, then the above argument shows that we can associate with any $a \in \boldsymbol{A}$ the cross-section $a(x)$ of $\mathcal{B}$ such as $\|a(x)\|$ is continuous and vanishing at infinity. Moreover one easily see that $\|a\| = \sup_{x\in X} \|a(x)\|$. Hence we may identify $\boldsymbol{A}$ with the represented algebra of cross-sections of $\mathcal{B}$. Consider the topology $\mathcal{T}_A$ in $\mathcal{B}$ and let $C_A(X, \mathcal{B})$ be the algebra of all cross-sections of $\mathcal{B}$ continuous in $\mathcal{T}_A$-topology and vanishing at infinity of $X$. We assert that $\boldsymbol{A}$ contains cross-sections taking arbitrary pairs of values in $\boldsymbol{A}(x)$, $\boldsymbol{A}(y)$ at distinct points $x, y \in X$. In fact, consider the ideal

$$x + y = \{a + b | a \in x, b \in y\}.$$

Then $x + y$ is dense in $\boldsymbol{A}$, for otherwise there exists a primitive ideal $P$ in $\boldsymbol{A}$ containing $x + y$, that is, $P \in h(x) \cap h(y)$. However, since the decomposition $\Omega = \bigcup_{\alpha \in \Gamma} \Omega_\alpha$ is a Hausdorff decomposition, each class $\Omega_\alpha$ is closed in $\Omega$ and $\Omega_\alpha = hk(\Omega_\alpha)$ which implies that $h(x) \cap h(y) = \phi$ whenever $x \neq y$, a contradiction. Thus $x + y$ is dense in $\boldsymbol{A}$ and by Lemma 8.1 in [5] we have $\boldsymbol{A} = x + y$. Let $a_1(x)$, $a_2(y)$ be an arbitrary pair of values in $\boldsymbol{A}(x)$, $\boldsymbol{A}(y)$ at distinct points $x$, $y \in X$. We can find an element $a_1' \in x$ and an element $a_2' \in y$ such that $a_1 - a_2 = a_1' - a_2'$. Let

$$a_0 = a_1 - a_1' = a_2 - a_2'.$$

Then clearly $a_0(x) = a_1(x)$ and $a_0(y) = a_2(y)$.

Therefore, by Theorem 2. 2, the represented algebra $\boldsymbol{A}$ coincides with $C_A(X, \mathcal{B})$. This completes the proof.

REMARK. It is to be noticed that the decomposition in Theorem 1. 3 is a continuous decomposition of the structure space of $C_F(X, \mathcal{B})$. Thus Theorem 3. 1 is considered as the converse of Theorem 1. 3.

As a direct consequence of this theorem we get the following representation theorem of $C^*$-algebras whose structure spaces are Hausdorff.

COROLLARY 3. 1. 1. *Let $\boldsymbol{A}$ be a $C^*$-algebra and $\Omega$ the structure space of $\boldsymbol{A}$. Suppose that $\Omega$ is a Hausdorff space and put $\mathcal{B} = \bigcup_{P \in \Omega} \boldsymbol{A}(P)$. Then $\boldsymbol{A}$ is represented as $C_A(\Omega, \mathcal{B})$, the algebra of all cross-sections of $\mathcal{B}$ continuous in $\mathcal{T}_A$-topology and vanishing at infinity of $\Omega$.*

Though the above defined topology is slightly different from the bundle space topology defined in Tomiyama-Takesaki [11] in the case that $\boldsymbol{A}$ is an $n$-homogeneous $C^*$-algebra, one may easily see that they are equivalent. Therefore Corollary 3.1.1 is a natural generalization of Theorem 5 in [11].

Now the above result shows that the commutative method is always applicable to the class of $C^*$-algebras whose structure spaces are Hausdorff. Hence there is no reason to distinguish the central $C^*$-algebras from the $C^*$-algebras whose structure spaces are Hausdorff spaces and we get naturally the following

COROLLARY 3.1.2. *Let $\boldsymbol{A}$ be a $C^*$-algebra and $\Omega$ the structure space of $\boldsymbol{A}$. Suppose that any $P \in \Omega$ does not contain the center $\boldsymbol{Z}$ of $\boldsymbol{A}$. Then $\boldsymbol{A}$ is central if and only if $\Omega$ is a Hausdorff space.*

PROOF. It is sufficient to prove the "if" part of this corollary. Suppose that $\Omega$ is a Hausdorff space. Let $P$ and $Q$ be different primitive ideals in $\boldsymbol{A}$ and take an element $z$ in $\boldsymbol{Z}$ such as $z(P) \neq 0$. Let $f$ be a bounded complex-valued continuous function on $\Omega$ such as $f(P) = 1$ and $f(Q) = 0$, then by Corollary 3.1.1 we

have $f\cdot z \in \boldsymbol{A}$. Since $f\cdot z(P) = z(P) \neq 0$ and $f\cdot z(Q) = 0$, one sees that $f\cdot z \notin P \cap \boldsymbol{Z}$ and $f\cdot z \in Q \cap \boldsymbol{Z}$. Thus $P \cap \boldsymbol{Z} \neq Q \cap \boldsymbol{Z}$, hence $\boldsymbol{A}$ is a central $C^*$-algebra.

REMARK. A Hausdorff decomposition is not necessarily a continuous decomposition though, in the whole space $\Omega$, the Hausdorff property is equivalent to the continuity property of $\|a(P)\|$.

**4. Topological representation of $W^*$-algebras and their pure state spaces.** In this section we prove that there exists always the finest continuous decomposition in the structure space of a $W^*$-algebra $\boldsymbol{A}$. As we see below, this is an another interpretation of the decomposition considered by Glimm [3]. We shall make clear the situation of Glimm's theorems by [3] on the pure state spaces of $W^*$-algebras and give more sharpened results for them.

Let $\boldsymbol{A}$ be a $C^*$-algebra and $\Omega$ the structure space of $\boldsymbol{A}$. A decomposition $\Omega = \bigcup_{\alpha\in\Gamma} \Omega_\alpha$ is called finer than the decomposition $\Omega = \bigcup_{\lambda\in\Lambda} \Omega'_\lambda$ if each $\Omega_\alpha$ is contained in some class $\Omega'_\lambda$.

THEOREM 4.1. *Let $\boldsymbol{A}$ be a $W^*$-algebra, $\Omega$ the structure space of $\boldsymbol{A}$ and $\Omega_0$ the structure space of the center $\boldsymbol{Z}$ of $\boldsymbol{A}$, Then $\Omega = \bigcup_{\zeta\in\Omega_0} h(\zeta)$ is the finest continuous decomposition of $\Omega$ whose decomposition space $X$ with quotient topology is homeomorphic with $\Omega_0$.*

Thus, setting $\mathcal{B} = \bigcup_{x\in X} \boldsymbol{A}(x)$, $\boldsymbol{A}$ is represented as $C_A(X, \mathcal{B})$, the algebra of all bounded $\mathcal{T}_A$-continuous cross-sections of $\mathcal{B}$. Notice that in this case a continuous function $f$ on $X$ is considered to be a continuous function on $\Omega_0$, hence an element in $\boldsymbol{Z}$ and $f\cdot a (a \in \boldsymbol{A})$ coincides with the usual product of the central element $f$ and $a$ in $\boldsymbol{A}$.

PROOF OF THEOREM 4.1. Since the map : $P \in \Omega \to P \cap \boldsymbol{Z} \in \Omega_0$ is a continuous map from $\Omega$ to $\Omega_0$, it is not difficult to see that the decomposition $\Omega = \bigcup_{\zeta\in\Omega_0} h(\zeta)$ is a Hausdorff decomposition. Let $\tilde{O} = \bigcup_{\zeta\in O} h(\zeta)$ be an open set in $\Omega$. We assert that $O$ is an open set of $\Omega_0$, so let $\zeta_0$ be a point of $O$ and $P_0$ a primitive ideal in $h(\zeta_0)$. Since $\tilde{O}^c$, the complement of $\tilde{O}$, is closed in $\Omega$ we can find an element $a \in \boldsymbol{A}$ such as $a(P_0) \neq 0$ and $a(P) = 0$ for every $P \in \tilde{O}^c$. Let $X$ be the decomposition space of $\Omega = \bigcup_{\zeta\in\Omega_0} h(\zeta)$, that is, $X = \{x(\zeta) = kh(\zeta) \mid \zeta \in \Omega_0\}$, then one easily see that $a(x(\zeta_0)) \neq 0$ and $a(x(\zeta)) = 0$ for every $\zeta \in O^c$, the compplement of $O$ in $\Omega_0$. Hence, by Lemma 10 in Glimm [3], there exists a neighborhood $U$ of $\zeta_0$ contained in $O$ and this implies that $O$ is an open set in $\Omega_0$,

Now it is clear that there is a one-to-one correspondence between $X$ and $\Omega_0$, and the above discussion shows that this correspondence is bicontinuous where the set $X$ is endowed with the quotient topology with respect to the decomposition $\Omega = \bigcup_{\zeta \in \Omega_0} h(\zeta)$.

Next, let $S$ be an arbitrary subset of $X$ and $\overline{S}$ the closure of $S$ in $X$. Put

$$\widetilde{S} = \{x \in X | x \supset k(S)\}.$$

Then, by the definition of quotient topology, it is not difficult to see that $\widetilde{S} \subset \overline{S}$. Conversely suppose that $a$ is in $k(S)$, the kernel of $S$. Then $a(x) = 0$ on $S$ and by [3 : Lemma 10] $a(x) = 0$ on $\overline{S}$, hence $a \in x$ for every $x \in \overline{S}$. That is, $\overline{S} \subset \widetilde{S}$ and we get, $\overline{S} = \widetilde{S}$. Therefore the above decomposition is a continuous one.

We shall show that the above decomposition is the finest continuous decomposition of $\Omega$. Suppose on the contrary that there exists a continuous decomposition $\Omega = \bigcup_{\alpha \in \Gamma} \Omega_\alpha$ exactly finer than the decomposition $\Omega = \bigcup_{\zeta \in \Omega_0} h(\zeta)$. Then we get at least two distinct class $\Omega_\alpha$ and $\Omega_\beta$ in some class $h(\zeta)$. Let $x = k(\Omega_\alpha)$ and $y = k(\Omega_\beta)$. As $x \neq \boldsymbol{Z}$, there exists an element $z \in \boldsymbol{Z}$ such as $z(x) \neq 0$ hence taking a bounded continuous function $f$ on the decomposition space of the decomposition $\Omega = \bigcup_{\alpha \in \Gamma} \Omega_\alpha$ such as $f(x) = 1$ and $f(y) = 0$ we have, by Theorem 3.1, $fz \in \boldsymbol{Z}$ and $fz \notin x$, $fz \in y$. This is a contradiction.

By the pure state space of a $C^*$algebra $\boldsymbol{A}$ with unit, we mean the $w^*$-closure of the pure states of $\boldsymbol{A}$ and denote it by $\mathfrak{P}(\boldsymbol{A})$. $\mathfrak{S}(\boldsymbol{A})$ means the state space of $\boldsymbol{A}$.

We keep the above notations in Theorem 4.1 for the rest of this section. Next lemma concerns with the first half part of Theorem 4 in Glimm [3].

LEMMA 4.1. *If $\boldsymbol{A}(x)$ has a non-zero GCR ideal, then $\boldsymbol{A}(x)$ is a primitive algebra and contains a minimal projection.*

PROOF. Let $I_x$ be a non-zero GCR ideal in $\boldsymbol{A}(x)$, then $I_x$ has no ideal divisors of zero because $\boldsymbol{A}(x)$ has no ideal divisors of zero (cf. [3: Lemma 11]). Hence, by Kaplansky [7 : Lemma 7.4], $I_x$ is primitive and there exists a primitive ideal $P_x$ in $\boldsymbol{A}(x)$ such as $P_x \cap I_x = \{0\}$, which implies $P_x = \{0\}$, Therefore $\boldsymbol{A}(x)$ is a primitive algebra. On the other hand, $I_x$ contains a minimal projection and, as $I_x$ is an ideal in $\boldsymbol{A}(x)$, this is also a minimal projection of $\boldsymbol{A}(x)$.

LEMMA 4.2. *Every projection in $\boldsymbol{A}(x)$ is the image of some projection in $\boldsymbol{A}$.*

PROOF. Let $e_{x_0}$ be a projection in $\boldsymbol{A}(x_0)$. By the proof of Lemma 12 in

Glimm [3], we can find an element $a \in \boldsymbol{A}$ and a neighborhood $U$ of $x_0$ such that $a(x)$ is a non-zero projectionin $\boldsymbol{A}(x)$ for every $x \in U$ and $a(x_0) = e_{x_0}$. Moreover as $X$ is homeomorphic with $\Omega_0$ which is known to be a totally disconnected space, there exists an open and closed neighborhood $V$ of $x_0$ contained in $U$. Let $f$ be the characteristic function of $V$ and put $e = f \cdot a$, then it is not difficult to see that $e$ is a projection of $A$ and $e(x_0) = e_{x_0}$.

Now we get

THEOREM 4.2. *Let $\boldsymbol{A}$ be a $W^*$-algebra. Then the following statements are equivalent*:

(1) *$\boldsymbol{A}$ is of continuous type, that is, $\boldsymbol{A}$ has no type I portion,*

(2) *$\boldsymbol{A}$ has no non-zero GCR ideal,*

(3) *$\boldsymbol{A}(x)$ has no non-zero GCR ideal for every $x \in X$,*

(4) $\mathfrak{P}(\boldsymbol{A}(x)) = \mathfrak{S}(\boldsymbol{A}(x))$ *for every* $x \in X$,

(5) $\mathfrak{P}(\boldsymbol{A}) = \{{}^t\psi_x(\varphi) \mid \varphi \in \mathfrak{S}(\boldsymbol{A}(x)),\ x \in X\}$, *where $\psi_x$ means the canonical map from $\boldsymbol{A}$ to $\boldsymbol{A}(x)$.*

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (5) were established in Glimm [3] but we prove here all implications for the completeness.

PROOF. (1) $\Rightarrow$ (3). Suppose that there exists a point $x \in X$ such that $\boldsymbol{A}(x)$ has a non-zero GCR ideal. Then, by Lemma 4.1. $\boldsymbol{A}(x)$ contains a minimal projection $e_x$, which is the image of a projection $e$ in $\boldsymbol{A}$. Since $\boldsymbol{A}$ is of continuous type it is well known that $e$ is the sum of two equivalent orthogonal projections $e_1$, $e_2$ in $\boldsymbol{A}$. Hence, $e_x = e(x) = e_1(x) + e_2(x)$ and both of $e_1(x)$ and $e_2(x)$ are non-zero projections in $\boldsymbol{A}(x)$. This contradicts the minimality of $e_x$. Therefore every $\boldsymbol{A}(x)$'s have no non-zero *GCR* ideals.

(3) $\Rightarrow$ (4). Since $\boldsymbol{A}(x)$ has no ideal divisors of zero, (3) implies (4) by [11: Theorem 2]. The implication (4) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (1). Suppose that $\boldsymbol{A}$ has a non-zero type I portion $\boldsymbol{A}z$ where $z$ is a central projection of $\boldsymbol{A}$. By [3: Theorem 4], we have

$$\mathfrak{P}(\boldsymbol{A}z) = \{{}^t\tilde{\psi}_x(\varphi) \mid \varphi \in \mathfrak{P}(\boldsymbol{A}(x)) \text{ for } x \in X \text{ with } z(x) \neq 0\}$$

and $\mathfrak{P}(\boldsymbol{A}(x)) \neq \mathfrak{S}(\boldsymbol{A}(x))$ for all such $x$'s where $\tilde{\psi}_x$ means the restriction of $\psi_x$ to $\boldsymbol{A}z$. Take a functional $\varphi \in \mathfrak{S}(\boldsymbol{A}(x))$ and $\varphi \notin \mathfrak{P}(\boldsymbol{A}(x))$ for some point $x \in X$ with $z(x) \neq 0$. Then ${}^t\psi_x(\varphi) \in \mathfrak{P}(\boldsymbol{A})$ by the assumption, hence ${}^t\tilde{\psi}_x(\varphi) \in \mathfrak{P}(\boldsymbol{A}z)$, a contradiction. Therefore $\boldsymbol{A}$ has no type I portion.

The implication (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is clear.

It is perhaps worth to notice that though we can not generally conclude that the weak closure of a $C^*$-algebra having no non-zero *GCR* ideal is of continuous type, it is true in the case of a $W^*$-algebra.

THEOREM 4.3. *Let* $\boldsymbol{A}$ *be a* $W^*$*-algebra. Then the following statements are equivalent*;

(1) $\boldsymbol{A}$ *is of type* I,

(2) $\boldsymbol{A}(x)$ *has a non-zero GCR ideal for everx* $x \in X$.

PROOF. The implication (1) $\Rightarrow$ (2) is due to [3]. Roughly speaking, the discussion is as follows: the canonical image of an abelian projection in $\boldsymbol{A}$ by $\psi_x$ is a minimal projection in $\boldsymbol{A}(x)$ or zero and as $\boldsymbol{A}$ has stufficiently many abelian projections this means that each of $\boldsymbol{A}(x)$'s has a minimal projecion, hence a non-zero *GCR* ideal.

The converse is clear from Theorem 4.2.

Combining (4) of Theorem 4.2 and Theorem 4 in [3] we can easily show that the pure state space of a $W^*$-algebra is determined completely by the pure state spaces of its component algebras.

THEOREM 4.4. *Let* $\boldsymbol{A}$ *be a* $W^*$*-algebra. Then*

$$\mathfrak{P}(\boldsymbol{A}) = \{{}^t\psi_x(\varphi) \mid \varphi \in \mathfrak{P}(\boldsymbol{A}(x)) \text{ for } x \in X\}.$$

REMARK. One might suspect that Theorem 4.4 is valid for any cross-section algebras, but this is not the case. Generally speaking, the weak closure of the pure states of an algebra $C_F(X,\mathcal{B})$ is not determined by those of component algebras though an element in the weak closure of the pure states of $C_F(X,\mathcal{B})$ determines a linear functional on some component algebra. We can find a counter example by Glimm [4: Theorem 6] or Tomiyama-Takesaki [11: Theorem 1].

After having prepared the manuscript of this paper, Fell's paper, *"The structure of operator fields, Acta Math.*, 106 (1961), 233-280", has appeared. Although our research has been done quite independently from Fell, there are several similar results; for examples, our Theorem 2.2 and its Corollary 2.2.1 correspond to Theorem 1.4 and its Corollary in Fell's paper. However, in stead of the dual space $\hat{\boldsymbol{A}}$ of a $C^*$-algebra $\boldsymbol{A}$ as in Fell's paper, we employed mainly the ideal dual space $\Omega$ of $\boldsymbol{A}$ through our paper and this makes differences such as we see, for example, in our Theorem 1.3 and Corollary of Theorem 1.2 in Fell's paper.

In our § 3, we have treated the topological representation of $C^*$-algebras; in this case, our method is quite different from Fell, however incidently our Corollary 3.1.1 corresponds to Theoreml 2.3 in Fell's paper.

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