TRANSFORMATIONS OF CONJUGATE FUNCTIONS II

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1. Paley and Wiener proved that *the conjugate functiun $\widetilde{f}(x)$ of $f(x)$ is integrable, if $f(x)$ is integrable, monotone in $(0, \pi)$ and odd.* Concerning with this theorem G. H. Hardy [1] proved that *if*

$$f \in L(-\infty, \infty)$$

and

$$\int_{-\infty}^{\infty} |x df(x)| < \infty,$$

then

$$g(x) = -\frac{1}{\pi|x|}\int_{-x}^{x} f(t)dt + h(x)$$

where

$$h \in L(-\infty, \infty),$$

and $g(x)$ is Hilbert transform of $f(x)$, that is,

$$g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\, dt.$$

Here we prove an analogous theorem, which is a more direct generalization of Paley-Wiener theorem, and consider some of its applications.

2. THEOREM. *If $f(x)$ is integrable in $(-\pi, \pi)$ and*

$$\int_{-\pi}^{\pi} |x df(x)| < \infty,$$

then

$$\widetilde{f}(x) = \frac{1}{\pi|x|}\int_{-x}^{x} f(t)dt + h(x),$$

where $h(x)$ is integrable in $(-\pi, \pi)$ and $\widetilde{f}(x)$ is the conjugate function of $f(x)$.

Throughout this note, A, A' are constants and may be different in each case.

PROOF. From the hypothesis we observe that $f(x)$ is of bounded variation and therefore bounded in $\{[-\pi, \pi] - (-\varepsilon, \varepsilon)\}$ for any $\varepsilon > 0$. It is sufficient to prove the theorem for even and odd $f(x)$. Then if integral is interpreted as Cauchy principal values, the conjugate is written as

$$\widetilde{f}(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)}{2\tan(x-t)/2}\, dt$$

$$= \frac{1}{\pi}\int_0^\pi \left(\frac{f(t)}{2\tan(x-t)/2} \pm \frac{f(t)}{2\tan(x+t)/2}\right)dt,$$

where the plus and minus signs correspond to even and odd $f(x)$ respectively. In any case we suppose that $\pi \geqq x \geqq 0$ and write

$$\tilde{f}(x) = \frac{1}{\pi}\left(\int_0^{x/2} + \int_{x/2}^{\pi}\right)\left(\frac{f(t)}{2\tan(x-t)/2} \pm \frac{f(t)}{2\tan(x+t)/2}\right)dt$$
$$= I_1(x) + I_2(x).$$

Now we show that

$$I'(x) \equiv I_1(x) - \frac{1}{\pi x}\int_{-x}^{x} f(t)dt$$

is integrable in $(0, \pi)$. In fact,

$$|I'(x)| \leqq \frac{1}{\pi}\int_0^{x/2}\left|\frac{1}{2\tan(x-t)/2} - \frac{1}{x}\right| |f(t)|dt$$
$$+ \frac{1}{\pi}\int_0^{x/2}\left|\frac{1}{2\tan(x+t)/2} - \frac{1}{x}\right| |f(t)|dt + \frac{2}{\pi x}\int_{x/2}^{\pi} |f(t)|dt.$$

It is easy to see that the third term is integrable in $(0, \pi)$. For the first two terms we observe that

$$\left|\frac{1}{2\tan(x-t)/2} - \frac{1}{x}\right| \leqq A\frac{t}{x^2} + A'$$

and
$$\left|\frac{1}{2\tan(x+t)/2} - \frac{1}{x}\right| \leqq A\frac{t}{x^2} + A' \text{ for } 0 \leqq t \leqq x/2 \leqq \pi/2.$$

Then, it is sufficient to verify that $\int_0^{x/2} \frac{t}{x^2} |f(t)|dt \in L(0, \pi)$.

In order to show $I_2(x) \in L(0, \pi)$, we prove first that $I_2(x) \in L(0, 2\pi/3)$. In this caee, $0 \leqq x \leqq 2\pi/3$ and we have

$$I_2(x) = \frac{1}{\pi}\left(\int_{x/2}^{3x/2} + \int_{3x/2}^{\pi}\right)\frac{f(t)}{2\tan(x-t)/2}dt \pm \int_{x/2}^{\pi}\frac{f(t)}{2\tan(x+t)/2}dt$$
$$= I_2^1(x) + I_2^2(x) + I_2^3(x), \text{ say.}$$

For $I_2^2(x)$, observing that $0 \leqq x/4 \leqq (t-x)/2 \leqq \pi/2$, we have

$$|I_2^2(x)| \leqq A\int_{3x/2}^{\pi}\frac{|f(t)|}{t-x}dt.$$

Hence
$$\int_0^{3\pi/2} |I_2^2(x)|dx \leqq A\int_0^{2\pi/3} dx\int_{3x/2}^{\pi}\frac{|f(t)|}{t-x}dt$$

$$\leqq A \int_0^{\pi} |f(t)| dt \int_0^{2t/3} \frac{dx}{t-x}$$

$$= A \int_0^{\pi} \log 3 |f(t)| dt < \infty.$$

Since $0 \leqq (x+t)/2 \leqq 5\pi/6$ in $I_2^3(x)$, we get

$$|I_2^3(x)| \leqq A \int_{x/2}^{\pi} \frac{|f(t)|}{x+t} dt \leqq A \int_{x/2}^{\pi} \frac{|f(t)|}{t} dt.$$

The last term is integrable in $(0, 2\pi/3)$.

Next we show that $I_2^1(x)$ is $L(0, 2\pi/3)$.

$$I_2^1(x) = \frac{1}{\pi} \int_{x/2}^{3x/2} \frac{f(t)}{2 \tan (x-t)/2} dt$$

$$= \frac{-1}{\pi} \int_0^{x/2} \frac{dt}{2 \tan t/2} \int_{x-t}^{x+t} df(u)$$

and

$$|I_2^1(x)| \leqq \frac{1}{\pi} \int_0^{x/2} \frac{dt}{2 \tan t/2} \int_{x-t}^{x+t} |df(u)|$$

$$= \frac{1}{\pi} \int_{x/2}^{x} |df(u)| \int_{x-u}^{x/2} \frac{dt}{2 \tan t/2} + \frac{1}{\pi} \int_x^{3x/2} |df(u)| \int_{u-x}^{x/2} \frac{dt}{2 \tan t/2}$$

$$= \frac{1}{\pi} \int_{x/2}^{3x/2} \log \left| \frac{\sin x/4}{\sin (x-u)/2} \right| |df(u)|.$$

Hence we have

$$\int_0^{2\pi/3} |I_2^1(x)| dx \leqq \frac{1}{\pi} \int_0^{2\pi/3} dx \int_{x/2}^{3x/2} \log \left| \frac{\sin x/4}{\sin (x-u)/2} \right| |df(u)|$$

$$\leqq \frac{1}{\pi} \int_0^{\pi/3} |df(u)| \int_{2\pi/3}^{2u} \log \left| \frac{\sin x/4}{\sin (x-u)/2} \right| dx$$

$$+ \frac{1}{\pi} \int_{\pi/3}^{\pi} |df(u)| \int_{2u/3}^{2\pi/3} \log \left| \frac{\sin x/4}{\sin (x-u)/2} \right| dx$$

$$= J_1 + J_2, \text{ say.}$$

For $2\pi/3 \leqq u \leqq \pi$,

$$\int_{2u/3}^{2\pi/3} \left| \log \left| \frac{\sin x/4}{\sin (x-u)/2} \right| \right| dx$$

is bounded by a constant not depending on u. Thus we have $J_2 < \infty$.

To treat J_1 we observe that by changing a variable

$$\int_{2u/3}^{2u}\left|\log\left|\frac{\sin x/4}{\sin (x-u)/2}\right|\right|dx = u\int_{2/3}^{2}\log\left|\frac{\sin ux/4}{\sin (x-1)u/2}\right|dx$$
$$\leqq u\int_{2/3}^{2}\log\left|\frac{x}{x-1}\right|dx$$
$$\leqq A'u.$$

Hence $$J_1 \leqq A\int_0^{\pi/3} u\,|df(u)| < \infty.$$

Collecting these inequalities, we have $I_2(x) \in L(0, 2\pi/3)$. It remains to prove that $I_2(x) \in L(2\pi/3, \pi)$. Since $x/2 \leqq 2x - \pi \leqq \pi$ for x, $2\pi/3 \leqq x \leqq \pi$, we put

$$I_2(x) = \frac{1}{\pi}\left(\int_{x/2}^{2x-\pi} + \int_{2x-\pi}^{\pi}\right)\frac{f(t)}{2\tan (x-t)/2}\,dt \pm \frac{1}{\pi}\int_{x/2}^{\pi}\frac{f(t)}{2\tan (x+t)/2}\,dt$$
$$= k_1(x) + k_2(x) + k_3(x), \text{ say.}$$

$$|k_3(x)| \leqq \frac{1}{\pi}\int_{x/2}^{\pi}\frac{|f(t)|}{-2\tan (x+t)/2}\,dt \leqq A\int_{x/2}^{\pi}\frac{1}{-2\tan (x+t)/2}\,dt$$
$$= A\log\left|\frac{\sin 3x/4}{\sin (x+\pi)/2}\right|, \text{ for } 2\pi/3 \leqq x \leqq \pi,$$

and the last term is integrable in $(2\pi/3, \pi)$. Similarly, since $\pi/2 \geqq x - t \geqq 0$, for $x/2 \leqq t \leqq 2x - \pi$ and $2\pi/3 \leqq x \leqq \pi$,

we have

$$|k_1(x)| \leqq \frac{1}{\pi}\int_{x/2}^{2x-\pi}\frac{|f(t)|}{2\tan (x-t)/2}\,dt$$
$$\leqq A\left|\log\left|\frac{\sin (\pi-x)/2}{\sin x/4}\right|\right|.$$

Therefore it remains to prove the integrability of $k_2(x)$.

$$k_2(x) = \frac{1}{\pi}\int_{2x-\pi}^{\pi}\frac{f(t)}{2\tan (x-t)/2}\,dt$$
$$= -\frac{1}{\pi}\int_0^{\pi-x}\frac{f(x+t)-f(x-t)}{2\tan t/2}\,dt$$
$$= -\frac{1}{\pi}\int_0^{\pi-x}\frac{dt}{2\tan t/2}\int_{x-t}^{x+t}df(u).$$

Hence,

$$|k_2(x)| \leqq \frac{1}{\pi}\int_x^\pi |df(u)| \int_{u-x}^{\pi-x} \frac{dt}{2\tan t/2} + \frac{1}{\pi}\int_{2x-\pi}^{x} |df(u)| \int_{x-u}^{\pi-x} \frac{dt}{|2\tan/2t|}$$

$$\leqq \frac{1}{\pi}\int_{\pi/3}^{\pi} \left|\log\left|\frac{\sin(\pi-x)/2}{|\sin(u-x)/2|}\right|\right| |df(u)|.$$

$$\int_{2\pi/3}^{\pi} |k_2(x)|\,dx \leqq \frac{2}{\pi}\int_{2\pi/3}^{\pi} dx \int_{\pi/3}^{\pi} \left|\log\left|\frac{\sin(\pi-x)/2}{\sin(u-x)/2}\right|\right| |df(u)|$$

$$\leqq \frac{2}{\pi}\int_{\pi/3}^{\pi} |df(u)| \int_0^{2\pi} \left|\log\left|\frac{\sin(\pi-x)/2}{\sin(u-x)/2}\right|\right| dx.$$

Since the inner integral does not exceed some constant not depending on u, the last term is finite.

This proves $I_2(x) \in L(2\pi/3, \pi)$ and therefore $I_2(x) \in L(0, \pi)$. Thus our proof is completed.

3. We consider the following transformations of a function $f(x)$. Let $f(x)$ be even and integrable in $(0, \pi)$ and periodic with period 2π. We set

$$F(x) = \int_x^\pi \frac{f(t)}{2\tan t/2}\,dt,$$

$$F^*(x) = \frac{1}{2\tan t/2}\int^x f(t)dt,$$

for $\pi \geqq x \geqq 0$, and denote by $F_c^*(x)$ the function $F(x)$ and $F^*(x)$ which are extended as even functions.

These transformations have been investigated by many authors. Here we have the following results which are somewhat better than Loo's theorem [3].

(i) *If* $f(x)(\log 2\pi/x)$ *is integrable in* $(0, \pi)$, *then* $\tilde{F}_c(x)$ *is also interable in* $(0, \pi)$.

(ii) *If* $f(x)(\log 2\pi/x)^2$ *is integrable in* $(0, \pi)$, *then* $\tilde{F}_c^*(x)$ *is also integrable in* $(0, \pi)$.

To pove (i), we note

$$\int_0^\pi |xdF(x)| = \int^\pi \left| xd\left(\int_x^\pi \frac{f(t)}{2\tan t/2}\,dt\right)\right|$$

$$\leqq A\int_0^\pi |f(t)|\,dt < \infty,$$

and

$$\int_0^\pi \frac{dx}{x}\int_0^x dt \int_t^\pi \frac{|f(u)|}{2\tan u/2}\,du$$

$$= \int_0^\pi dt\left(\int^\pi \frac{|f(u)|}{2\tan u/2}\,du \int^\pi \frac{dx}{x}\right)$$

$$\leqq A \int_0^\pi \log \frac{\pi}{t} dt \int_0^\pi \frac{f(u)}{u} du$$

$$= A \int_0^\pi (1 + \log \pi/t)\, |f(u)|\, du < \infty$$

by changing the order of integrations. Using our theorem we get (i).

A proof of (ii) is similar.

REFERENCES

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[3] C. T. LOO, Note on the properties of Fourier coefficients, Amer. Journ. of Math., 71 (1949), 269-282.

[4] A. ZYGMUND, Trigonometric series, Cambridge (1959).

For further references, see [1] and [2].

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