ON HARMONIC TENSORS IN AN ALMOST TACHIBANA SPACE

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1. Introduction. In a compact Kählerian space, a skew-symmetric pure tensor field of type $(0, q)$ is harmonic if and only if it is analytic, [5] [6][1]. But if the space is non-Kählerian, an (almost) analytic tensor is not necessarily harmonic.

For this problem, S.Tachibana [4] proved the following

THEOREM. *In a compact almost Tachibana space, a necessary and sufficient condition that a vector be covariant almost analytic is that* v_j *and* $\tilde{v}_j = \varphi_j{}^l v_l$ *are both harmonic.*

In this paper, we shall generalize this theorem to a tensor of type $(0, q)$

MAIN THEOREM. *In a compact almost Tachibana space, a necessary and sufficient condition that a skew-symmetric pure tensor* $T_{(j)}$ *of type* $(0, q)$ *be almost analytic is that* $T_{(j)}$ *and* $\tilde{T}_{(j)} \equiv \varphi_{j_s}{}^s T_{j_q \ldots s \ldots j_1}$ *are both harmonic.*

This main theorem follows from the following two lemmas.

LEMMA A[2]. *In an almost complex space, if skew-symmetric pure tensors* $T_{(j)}$ *and* $\tilde{T}_{(j)}$ *of type* $(0, q)$ *are both closed, then they are almost analytic.*

LEMMA B. *In a compact almost Tachibana space, if a skew-symmetric pure tensor* $T_{(j)}$ *of type* $(0, q)$ *is almost analytic, then it is harmonic.*

In §2 we shall give some well known lemmas concerning an almost analytic tensor in an almost complex space and prove Lemma A. In §3 we shall deal with an almost Tachibana space (which is called a K-space by some writers) and give two lemmas obtained by S.Sawaki [3]. In the last section, we shall prove Lemma B.

2. Almost analytic tensors. Let X_{2n} be a $2n$-dimensional real differentiable manifold of class C^∞, with local coordinate $\{x^i\}$, admitting an almost complex structure defined by the tensor field $\varphi_i{}^h$ of type (1. 1) satisfying

$$\varphi_i{}^l \varphi_l{}^h = -\delta_i^h, \qquad h,i,\ldots = 1, 2, \ldots\ldots, 2n. \tag{2. 1}$$

1) The numbers in brackets refer to References at the end of the paper.
2) This lemma the author owes to Dr. S.Tachibana. Cf., S.Tachibana [5], p.213.

A manifold with such an almost complex structure $\varphi_i^{\ h}$ is called an *almost complex space.*

A tensor field $T_{(j)} \equiv T_{j_q \dots j_1}$ of type $(0,q)$ is called *pure* in j_s and j_t if it satisfies

$$(2.2) \qquad 2^* O_{j_t j_s}^{t\ s} T_{j_q \dots t \dots s \dots j_1} \equiv T_{j_q \dots j_1} + \varphi_{j_t}^{t} \varphi_{j_s}^{s} T_{j_q \dots t \dots s \dots j_1} = 0,$$

where we denote $T_{j_q \dots t \dots s \dots j_1}$ instead of $T_{j_q \dots j_{t+1} t j_{t-1} \dots j_{s+1} s j_{s-1} \dots j_1}$, etc.. By a *pure tensor* we shall mean that it is pure in every pair of its indices.

Next, we shall say that a pure tensor field $T_{(j)}$ of type $(0, q)$ is *almost analytic* if it satisfies[3]

$$(2.3) \qquad \varphi_h^{\ l} \partial_l T_{(j)} - \partial_h \widetilde{T}_{(j)} + \sum_{s=1}^{q} (\partial_{j_s} \varphi_h^{\ s}) T_{j_q \dots s \dots j_1} = 0,$$

which may be written in the tensor form

$$(2.4) \qquad \varphi_h^{\ l} \nabla_l T_{(j)} - \nabla_h \widetilde{T}_{(j)} + \sum_{s=1}^{q} (\nabla_{j_s} \varphi_h^{\ s}) T_{j_q \dots s \dots j_1} = 0,$$

where $\partial_j = \partial / \partial x^j$ and ∇ denotes the operator of covariant differentiation with respect to the Riemannian connection.

For the almost analytic tensors the following lemmas are well known.

LEMMA 2.1. (S.Tachibana [5]) *In an almost complex space, if a skew-symmetric tensor $T_{(j)}$ of type $(0, q)$ is pure, then $\widetilde{T}_{(j)}$ is also a skew-symmetric pure tensor.*

LEMMA 2.2. (S.Tachibana [5][4]) *In an almost complex space, if a pure tensor field $T_{(j)}$ of type $(0, q)$ is almost analytic, then so is $\widetilde{T}_{(j)}$.*

Lastly, we shall prove Lemma A. Since

$$\partial_{j_s} (\varphi_h^{\ s} T_{j_q \dots s \dots j_1}) = (\partial_{j_s} \varphi_h^{\ s}) T_{j_q \dots s \dots j_1} + \varphi_h^{\ s} \partial_{j_s} T_{j_q \dots s \dots j_1},$$

we have

$$\sum_{s=1}^{q} (\partial_{j_s} \varphi_h^{\ s}) T_{j_q \dots s \dots j_1} = \sum_{s=1}^{q} \partial_{j_s} (\varphi_h^{\ s} T_{j_q \dots s \dots j_1}) - \sum_{s=1}^{q} \varphi_h^{\ s} \partial_{j_s} T_{j_q \dots s \dots j_1}.$$

Therefore if the tensor is skew-symmetric, by virtue of Lemma 2.1, (2.3) may be written as

$$\varphi_{[h}^{l} \partial_{|l|} T_{j_q \dots j_1]} - \partial_{[h} \widetilde{T}_{j_q \dots j_1]} = 0,$$

3) For (p, q)-type tensors, see S. Tachibana [5] or S. Kotō [1].
4) For (p, q)-type tensors, see S. Kotō [2].

where the square brackets denote the alternating part. If $T_{(j)}$ is an almost analytic tensor, then by Lemma 2.2 so is $\widetilde{T}_{(j)}$. Thus Lemma A was proved.

3. Almost Tachibana spaces. An *almost Tachibana space* is first of all an almost complex space and secondly has a Riemannian metric g_{ih} satisfying

$$\varphi_i{}^m \varphi_h{}^l g_{ml} = g_{ih}, \tag{3.1}$$

from which

$$\varphi_{ih} = -\varphi_{hi}, \tag{3.2}$$

where $\varphi_{ih} = \varphi_i{}^l g_{lh}$, and finally has the property that the skew-symmetric tensor φ_{ih} is a Killing tensor

$$\nabla_j \varphi_{ih} + \nabla_i \varphi_{jh} = 0. \tag{3.3}$$

Let $R_{kji}{}^h$ and $R_{ji} = R_{lji}{}^l$ be Riemannian curvature tensor and Ricci tensor, respectively, then by Ricci identity we have

$$\nabla_m \nabla_l \varphi_{jh} - \nabla_l \nabla_m \varphi_{jh} = \varphi_h{}^s R_{mljs} - \varphi_j{}^s R_{mlhs}. \tag{3.4}$$

Recently, S. Sawaki [3] proved the following two lemmas

LEMMA 3.1. *In an almost Tachibana space, a pure tensor $T_{(j)}$ is almost analytic if and only if*

(1) $\nabla_h T_{(j)}$ *is a pure tensor,*

(2) $(\nabla_{j_s} \varphi_h{}^s) T_{j_q \dots s \dots j_1} = 0, \qquad s = 1, 2, \dots\dots, q.$

LEMMA 3.2.[5] *In a compact almost Tachibana space, a necessary and sufficient condition that a pure tensor $T_{(j)}$ of type $(0, q)$ be almost analytic is that it satisfies*

(1) $g^{ml} \nabla_m \nabla_l T_{(j)} - \sum_{s=1}^{q} R_{j_s}{}^s T_{j_q \dots s \dots j_1} = 0,$

(2) $(\nabla_{j_s} \varphi_h{}^s)\, T_{j_q \dots s \dots j_1} = 0, \qquad s = 1, 2, \dots\dots, q.$

4. Proof of Lemma B. Let $T_{j_q \dots j_1}$ be a skew-symmetric almost analytic tensor. Operating $\nabla^{j_t} = g^{j_t l} \nabla_l \ (s \neq t)$ to (2) of Lemma 3.2 and taking account of (3.3) we get

$$(\nabla^{j_t} \nabla_{j_s} \varphi_h{}^s)\, T_{j_q \dots s \dots j_1} = (\nabla_h \varphi_{j_s}{}^s)(\nabla^{j_t} T_{j_q \dots s \dots j_1}). \tag{4.1}$$

On the other hand, transvecting (1) of Lemma 3.1 with g^{hj_t} it follows that

$$\nabla^t T_{j_q \dots t \dots j_1} = 0. \tag{4.2}$$

5) For an almost complex space, see S. Kotō [2].

Substituting (4.2) in the right hand m e mber of (4.1) and using (3.4) we find

$$(R_{j_t j_s ts} - \varphi_{j_t}{}^b \varphi_{j_s}{}^a R_{bats})\, T^{j_q \ldots t \ldots s \ldots j_1} = 0,$$

where $T^{j_q \ldots j_1} = T_{j_q \ldots i_1} g^{i_q j_q} \cdots g^{i_1 j_1}$, so that we have

$$R_{j_t j_s}{}^{ts} T_{j_q \ldots t \ldots s \ldots j_1} T^{j_q \ldots j_1} = \frac{1}{2}(R_{ba}{}^{ts} \delta_{j_t}{}^b \delta_{j_s}{}^a + R_{ba}{}^{ts} \varphi_{j_t}{}^b \varphi_{j_s}{}^a) T_{j_q \ldots t \ldots s \ldots j_1} T^{j_q \ldots j_1}$$
$$= R_{j_t j_s}{}^{ts} T_{j_q \ldots t \ldots s \ldots j_1} \, {}^*O_{b\,a}^{j_t j_s} T^{j_q \ldots b \ldots a \ldots j_1}.$$

Since $T_{(j)}$ is a pure tensor, we find

$$R_{j_t j_s}{}^{ts} T_{j_q \ldots t \ldots s \ldots j_1} T^{j_q \ldots j_1} = 0. \tag{4.3}$$

Consequently, for an almost analytic tensor from (1) of Lemma 3.2 and (4.3), it follows that

$$(\Delta T_{j_q \ldots j_1}) T^{j_q \ldots j_1} \tag{4.4}$$
$$\equiv (g^{ml} \nabla_m \nabla_l T_{(j)} - \sum_{s=1}^{q} R_{j_s}{}^s T_{j_q \ldots s \ldots j_1} - \sum_{t>s} R_{j_t j_s}{}^{ts} T_{j_q \ldots t \ldots s \ldots j_1}) T^{j_q \ldots j_1} = 0.$$

In the next, it is a well known fact [6] that in a compact orientable Riemannian space X, the integral formula

$$\int_X [(\Delta T_{j_q \ldots j_1}) T^{j_q \ldots j_1} + (q+1) \nabla^{[h} T^{j_q \ldots j_1]} \nabla_{[h} T_{j_q \ldots j_1]} \tag{4.5}$$
$$+ q (\nabla_l T^{j_q \ldots j_2 l})(\nabla^m T_{j_q \ldots j_2 m})] d\sigma = 0$$

is valid for any skew-symmetric tensor field $T_{(j)}$ of type $(0, q)$ where $d\sigma$ means the volume element of the X.

Hence substituting (4.4) in (4.5), we see that if a skew-symmetric pure tensor $T_{(j)}$ is almost analytic then we have

$$\nabla_{[h} T_{j_q \ldots j_1]} = 0, \text{ and } \nabla^l T_{j_q \ldots j_2 l} = 0,$$

that is, the tensor becomes harmonic. q. e. d.

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