NOTE ON (ϕ, ξ, η)-STRUCTURE

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Prof. Sasaki has recently investigated a structure called (ϕ,ξ,η)-structure [1][2)] which is closely related to the almost contact structure studied by W.Gray and W. M. Boothby-H. C. Wang, and may also be considered as an analogue of almost complex structure for odd dimensional manifolds.

Let M^{2n+1} be a $(2n+1)$-dimensional differentiable manifold of class C^∞. M^{2n+1} is said to have a *(ϕ, ξ, η)-structure* if there exists a tensor field $\phi_j^{\ i}$, a contravariant vector field ξ^i and a covariant vector field η_j (each is of class C^∞) over M^{2n+1} such that the following conditions are satisfied:

(0.1) $$\xi^i\eta_i = 1, \qquad (i, j, k = 1,2,\ldots\ldots,2n+1)$$

(0.2) $$\text{rank } |\phi_j^{\ i}| = 2n,$$

(0.3) $$\phi_j^{\ i}\xi^j = 0,$$

(0.4) $$\phi_j^{\ i}\eta_i = 0,$$

(0.5) $$\phi_j^{\ i}\phi_k^{\ j} = -\delta_k^{\ i} + \xi^i\eta_k.$$

In this note we intend to show that starting from a differentiable manifold M^{2n+1} having (ϕ, ξ, η)-structure one can construct by a natural way a manifold with 3-π-structure [2] in the sense of the present author, and by applying the theory of 3-π-structure we can obtain a tensor analogous to the Nijenhuis tensor in the case of almost complex structure. Moreover, canonic connections for the considered structure are also obtained by the use of the results of π-structure.

1. Associated 3-π-structure. From (0.2) and (0.3) it is evident that ξ^i is a proper vector corresponding to the proper value 0 of $\phi_j^{\ i}$ and the proper subspace of the proper value 0 is spanned only by ξ^i.

Let v^j be any proper vector corresponding to a non-zero proper value λ of $\phi_j^{\ i}$, then

(1.1) $$\phi_j^{\ i}v^j = \lambda v^i.$$

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2) Number in bracket refers to the reference at the end of paper.

Multiply η_i and sum with respect to i, we have

$$\phi_j^{\ i}\eta_i v^j = \lambda v^i \eta_i. \tag{1.2}$$

By (0.4) we get $\lambda v^i \eta_i = 0$, therefore

$$v^i \eta_i = 0. \tag{1.3}$$

If we contract (1.1) with $\phi_i^{\ k}$ we have

$$\phi_i^{\ k}\phi_j^{\ i}v^j = \lambda\phi_i^{\ k}v^i.$$

From this we have by (0.5) and (1.1)

$$(-\delta_j^k + \xi^k\eta_j)v^j = \lambda^2 v^k,$$

which gives by (1.3)

$$\lambda^2 v^k = - v^k \text{ and } \lambda^2 = -1.$$

Thus the only non-zero proper values of $\phi_j^{\ i}$ are i and $-i$. As $\phi_j^{\ i}$ is real, it follows that each of the proper values has the multiplicity n, and the corresponding proper subspace T_P and S_P of the tangent space M_P at point P of M^{2n+1} are both of dimension n. So, over the differentiable manifold M^{2n+1} we have three distributions X, T and S which assign each point P of M^{2n+1} three subspaces X_P, T_P, S_P of the complexification M_P^c of the tangent space M_P. Moreover, $M_P^c = X_P^c + T_P + S_P$ (direct sum) for every point P of M^{2n+1}. Thus the differentiable manifold M^{2n+1} is endowed with a 3-π-structure defined by three distributions X, T and S.

Let v^j be any vector field contained in the distribution $T+S$, then by (0.1) and (1.3) we have

$$(\xi^i\eta_j)\xi^j = \xi^i \quad\text{and}\quad (\xi^i\eta_j)v^j = 0. \tag{1.4}$$

So, $\xi^i\eta_j$ is the projection tensor field for the distribution X.

Let $p_j^{\ i}$ and $q_j^{\ i}$ be respectively the projection tensor field for the distributions T and S. Then if $v_+^{\ i}$ and $v_-^{\ i}$ are respectively contained in T and S, we have

$$\begin{cases} p_j^{\ i}v_+^{\ j} = v_+^{\ i}, & p_j^{\ i}v_-^{\ j} = 0; \\ q_j^{\ i}v_+^{\ j} = 0, & q_j^{\ i}v_-^{\ j} = v_-^{\ i}; \\ p_j^{\ i}\xi^j = q_j^{\ i}\xi^j = 0. \end{cases} \tag{1.5}$$

Since, by definition

$$\begin{cases} \phi_j^{\ i}v_+^{\ j} = iv_+^{\ i}, \quad \phi_j^{\ i}v_-^{\ j} = -iv_-^{\ i}, \\ \phi_j^{\ i}\xi^j = 0, \end{cases} \tag{1.6}$$

we have

$$\phi_j^{\ i} = ip_j^{\ i} - iq_j^{\ i}, \tag{1.7}$$

because both sides of the latter formula have the same effect for all vectors of the tangent space at every point of M^{2n+1}.

As M_P^c is the direct sum of X_P^c, T_P and S_P, we also have

$$\xi^i\eta_j + p_j^i + q_j^i = \delta_j^i. \tag{1.8}$$

Now we start to consider the inverse implication: Assume that we have a $(2n+1)$-dimensional differentiable manifold M^{2n+1} endowed with a 3-π-structure defined by the complete system [2] consists of three distributions X, T and S, of which the first one is a one dimensional real distribution and the latter two are n-dimensional conjugate complex distributions which together span the complexification of a $2n$-dimensional real distribution. If ξ^i, $\underset{\alpha}{v_+}^i$, $\underset{\alpha}{v_-}^i (\alpha = 1,\ldots,n)$ respectively span X, T and S, then there exists a covariant vector field η_i such that

$$\begin{cases} \eta_i \underset{\alpha}{v_+}^i = 0, \qquad \eta_i \underset{\alpha}{v_-}^i = 0, \ (\alpha = 1,\ldots,n) \\ \xi^i\eta_i = 1. \end{cases} \tag{1.9}$$

Then it is evident that $\xi^i\eta_j$ is the projection tensor field for the distribution X. Let p_j^i and q_j^i be respectively the projection tensor field for T and S, then (1.5) holds for any vector field v_+^i and v_-^i contained respectively in T and S. It is also evident that (1.8) and the following relation hold:

$$\bar{p}_j^i = q_j^i. \tag{1.10}$$

Next, define ϕ_j^i by (1.7), then we have by $(1.5)_3$ that

$$\phi_j^i\xi^j = 0. \tag{0.3}$$

Moreover, by $(1.5)_1$ and $(1.9)_1$ we have

$$\eta_i p_j^i v_+^i = v_+^i\eta_i = 0.$$

Similarly, from $(1.5)_1$ and (0.3), we have

$$\eta_i p_j^i v_-^j = 0 \text{ and } \eta_i p_j^i \xi^j = 0.$$

So $\eta_i p_j^i u^j = 0$ holds for any vector u^j of the tangent space. Thus we have

$$\eta_i p_j^i = 0, \text{ and similarly, } \eta_i q_j^i = 0. \tag{1.11}$$

Consequently, we have (1.7)

$$\phi_j^i\eta_i = 0. \tag{0.4}$$

From (1.7) and (1.8) we have moreover

$$\begin{aligned} \phi_j^i\phi_k^j &= (ip_j^i - iq_j^i)(ip_k^j - iq_k^j) \\ &= -(p_k^i + q_k^i) = -\delta_k^i + \xi^i\eta_k. \end{aligned} \tag{0.5}$$

From (1. 10) it follows that ϕ_j^i is real. And, from (1. 7) and (1. 5) it follows that ϕ_j^i satisfies (1. 6), so the rank of $|\phi_j^i|$ is $2n$.

Thus we have the following:

THEOREM 1.1. *A necessary and sufficient condition for a $(2n+1)$-dimensional manifold M^{2n+1} to have a (ϕ, ξ, η)-structure is that the manifold be endowed with a 3-π-structure defined by a complete system consists of three distributions X, T and S, of which the first one is a one dimensional real distribution, and the latter two are n-dimensional conjugate complex distributions which together span the complexification of a $2n$-dimensional real distribution.*

2. Fundamental tensors and torsion tensors. From (1.7) and (1.8) we have

$$(2.1)\qquad \begin{cases} p_j^i = \dfrac{1}{2}(\delta_j^i - \xi^i\eta_j - i\phi_j^i), \\[2mm] q_j^i = \dfrac{1}{2}(\delta_j^i - \xi^i\eta_j + i\phi_j^i), \end{cases}$$

so p_j^i and q_j^i are conjugate complex to each other.

Now the fundamental tensor for the associated 3-π-structure is defined as follows:

$$(2.2)\qquad F_j^i = \lambda(\xi^i\eta_j + \omega_1^2 p_j^i + \omega_1 q_j^i),$$

where ω_1 is a cubic root $(\neq 1)$ of unity and λ is any non-zero complex number. Substitute (2. 1) in (2. 2) we have

$$(2.3)\qquad \overset{1}{F}{}_j^i \equiv F_j^i = \frac{\lambda}{2}\{-\delta_j^i + 3\xi^i\eta_j + i\omega_1(1-\omega_1)\phi_j^i\}.$$

From which it follows that

$$(2.4)\qquad \overset{2}{F}{}_k^i \equiv F_j^i F_k^j = \frac{\lambda^2}{2}\{-\delta_k^i + 3\xi^i\eta_k - i\omega_1(1-\omega_1)\phi_k^i\},$$

and

$$(2.5)\qquad \overset{3}{F}{}_l^i \equiv F_j^i F_k^j F_l^k = \lambda^3\delta_l^i.$$

We also can associate M^{2n+1} with three kinds of 2-π-structure as follows:

$$(2.6)\qquad \underset{1}{F}{}_j^i = \lambda_1\{\xi^i\eta_j - (p_j^i + q_j^i)\},$$

$$(2.7)\qquad \underset{2}{F}{}_j^i = \lambda_2\{(\xi^i\eta_j + p_j^i) - q_j^i\},$$

$$(2.8)\qquad \underset{3}{F}{}_j^i = \lambda_3\{(\xi^i\eta_j + q_j^i) - p_j^i\}.$$

These three tensor fields satisfy the following relations:

$$(2.9)\qquad \underset{1}{F}_j{}^i \underset{1}{F}_k{}^j = \lambda_1{}^2\delta_k^i,\ \underset{2}{F}_j{}^i \underset{2}{F}_k{}^j = \lambda_2{}^2\delta_k^i,\ \underset{3}{F}_j{}^i \underset{3}{F}_k{}^j = \lambda_3{}^2\delta_k^i.$$

$$(2.10)\qquad \begin{cases} \underset{1}{F}_j{}^i \underset{2}{F}_k{}^j = \underset{2}{F}_j{}^i \underset{1}{F}_k{}^j = \dfrac{\lambda_1\lambda_2}{\lambda_3}\underset{3}{F}_k{}^i,\ \underset{2}{F}_j{}^i \underset{3}{F}_k{}^t = \underset{3}{F}_j{}^i \underset{2}{F}_k{}^j = \dfrac{\lambda_2\lambda_3}{\lambda_1}\underset{1}{F}_k{}^i, \\ \underset{3}{F}_j{}^i \underset{1}{F}_k{}^j = \underset{1}{F}_j{}^i \underset{3}{F}_k{}^j = \dfrac{\lambda_1\lambda_3}{\lambda_2}\underset{2}{F}_k{}^i. \end{cases}$$

Substitute (2. 1) in (2. 6), (2. 7) and (2. 8) we have

$$(2.11)\qquad \begin{cases} \underset{1}{F}_j{}^i = \lambda_1(2\xi^i\eta_j - \delta_j{}^i), \\ \underset{2}{F}_j{}^i = \lambda_2(\xi^i\eta_j - i\phi_j{}^i), \\ \underset{3}{F}_j{}^i = \lambda_3(\xi^i\eta_j + i\phi_j{}^i). \end{cases}$$

It is known that the torsion tensor (analogue of the Nijenhuis tensor) for a 2-π-structure is as follows [2].

$$(2.12)\qquad \underset{a}{t}_{jk}{}^i = -\frac{1}{4\lambda_a^2}(\delta_j{}^p\underset{a}{F}_k{}^q + \delta_k{}^q\underset{a}{F}_j{}^p)(\partial_p\underset{a}{F}_q{}^i - \partial_q\underset{a}{F}_p{}^i),\ (a = 1, 2, 3).$$

Thus for respective case of (2. 11) we have

$$(2.13)\qquad \underset{1}{t}_{jk}{}^i = -\xi^i(-N_j\eta_k + N_k\eta_j + \eta_{j,k} - \eta_{k,j}),$$

$$(2.14)\qquad \begin{cases} \underset{2}{t}_{jk}{}^i = -\dfrac{1}{4}P_{jk}{}^i - \dfrac{1}{4}iQ_{jk}{}^i, \\ \underset{3}{t}_{jk}{}^i = -\dfrac{1}{4}P_{jk}{}^i + \dfrac{1}{4}iQ_{jk}{}^i, \end{cases}$$

in which

$$(2.15)\qquad \begin{cases} P_{jk}{}^i = N_{jk}{}^i - \underset{1}{t}_{jk}{}^i - \xi^i(\eta_{j,k} - \eta_{k.j}), \\ Q_{jk}{}^i = -N_j{}^i\eta_k + N_k{}^i\eta_j + \xi^iN_{jk}, \end{cases}$$

where $\eta_{j,k} \equiv \dfrac{\partial\eta_j}{\partial x^k} \equiv \partial_k\eta_j$, and the $N_{jk}{}^i$, N_{jk}, $N_j{}^i$, N_j are tensors obtained by Sasaki and Hatakeyama [3] and defined as follows:

$$(2.16)\qquad \begin{cases} N_{jk}{}^i = \phi_k{}^q(\phi_{j,q}^i - \phi_{q,j}^i) - \phi_j{}^p(\phi_{k,p}^i - \phi_{p,k}^i) - \eta_j\xi^i_{,k} + \eta_k\xi^i_{,j}, \\ N_{jk} = \phi_k{}^q(\eta_{q,j} - \eta_{j,q}) - \phi_j{}^p(\eta_{p,k} - \eta_{k,p}), \\ N_j^i = \xi^q(\phi_{j,q}^i - \phi_{q,j}^i) - \phi_j{}^q\xi^i_{,q}, \\ N_j = \xi^p(\eta_{j,p} - \eta_{p,j}). \end{cases}$$

For the torsion tensor of a 3-π-structure we have [2]:

$$(2.17)\qquad t_{jk}{}^i = \frac{1}{9\lambda^3}\left[\left\{-2(\delta_j{}^p\overset{1}{F}_k{}^q + \delta_k{}^q\overset{1}{F}_j{}^p) + \frac{1}{\lambda^3}\overset{2}{F}_j{}^p\overset{2}{F}_k{}^q\right\}(\partial_p\overset{2}{F}_q{}^i - \partial_q\overset{2}{F}_p{}^i)\right.$$

$$+ \{-2(\delta_j^p \overset{2}{F}_k{}^q + \delta_k{}^q \overset{2}{F}_j{}^p) + \overset{1}{F}_j{}^p \overset{1}{F}_k{}^q\}(\partial_p \overset{1}{F}_q{}^i - \partial_q \overset{1}{F}_p{}^i)\Big].$$

Substituting (2. 3) and (2. 4), we have the following expression after some straightforward calculations:

$$(2.18) \qquad t_{jk}{}^i = \frac{1}{4}\{-N_{jk}{}^i - 3\xi^i(\eta_{j,k} - \eta_{k,j}) + 2\xi^i(N_j\eta_k - N_k\eta_j) + \xi^i\phi_j{}^p\phi_k{}^q(\eta_{l,p} - \eta_{p,q}) - N_p{}^i(\phi_j{}^p\eta_k - \phi_k{}^p\eta_j)\}.$$

3. (ϕ, ξ, η)-connections. It is known [2] that if $\gamma_{jk}{}^i$ is any linear connection of the manifold, and we define a connection $l_{jk}{}^i$ by

$$(3.1) \qquad l_{jk}{}^i = \gamma_{jk}{}^i + T_{jk}{}^i$$

with

$$(3.2) \qquad T_{jk}{}^i = \frac{1}{3}\frac{1}{\lambda^3}\{(\nabla_k \overset{1}{F}_j{}^l)\overset{2}{F}_l{}^i + (\nabla_k \overset{2}{F}_j{}^l)\overset{1}{F}_l{}^i\},$$

where ∇ denotes the covariant derivative with respect to $\gamma_{jk}{}^i$, then $l_{jk}{}^i$ is a π-connection of the differentiable manifold with a 3-π-structure whose fundamental tensor is given by (2. 3), that is, $l_{jk}{}^i$ is a connection which leaves the fundamental tensor $F_j{}^i$ covariant constant. Therefore $\overset{2}{F}_j{}^i$ and $\overset{3}{F}_j{}^i = \lambda^3\delta_j{}^i$ are also left covariant constant. Consequently $l_{jk}{}^i$ leaves also $\xi^i\eta_j$ and $\phi_j{}^i$ covariant constant, i.e.,

$$(3.3) \qquad \begin{cases} (\xi^i\eta_j)_{;k} = \xi^i_{;k}\eta_j + \xi^i\eta_{j;k} = 0, \\ \phi^{\;i}_{j;k} = 0, \end{cases}$$

as $\xi^i\eta_j$ and $\phi_j{}^i$ can respectively be expressed as linear combination of $F_j{}^i$, $\overset{2}{F}_j{}^i$ and $\delta_j{}^i$ by (2. 3) and (2. 4). In (3. 3); denotes the covariant derivative with respect to the π-connection $l_{jk}{}^i$ induced by $\gamma_{jk}{}^i$.

If we substitute (2. 3) and (2. 4) into (3.2) we have

$$(3.4) \qquad T_{jk}{}^i = \frac{1}{2}\{-(\nabla_k\xi^i)\eta_j + 3\xi^i\eta_l(\nabla_k\xi^l)\eta_j + 2\xi^i(\nabla_k\eta_j) - (\nabla_k\phi_j{}^l)\phi_l{}^i\}.$$

Now by simple calculation, we have

$$(3.5) \qquad \xi^i_{;k} = \nabla_k\xi^i + T_{jk}{}^i\xi^j = \xi^i\eta_l(\nabla_k\xi^l).$$

Since Ishihara and Obata [4] have shown that there is a symmetric affine connection which leaves ξ^i covariant constant, if we take $\gamma_{jk}{}^i$ in (3. 1) as such a connection $\mathring{\gamma}_{jk}{}^i$, then it follows from (3.5) that the induced π-connection leaves ξ^i covariant constant, and consequently also leaves η_j covariant constant by

$(3.3)_1$. Thus if we define connection $\overset{\circ}{l}_{jk}{}^{i}$ by

$$\overset{\circ}{l}_{jk}{}^{i} = \overset{\circ}{\gamma}_{jk}{}^{i} + \overset{\circ}{T}_{jk}{}^{i} \tag{3.6}$$

with

$$\overset{\circ}{T}_{jk}{}^{i} = \frac{1}{2}\{2\xi^{i}(\overset{\circ}{\nabla}_{k}\eta_{j}) - (\overset{\circ}{\nabla}_{k}\phi_{j}{}^{l})\phi_{l}{}^{i}\}, \tag{3.7}$$

where $\overset{\circ}{\nabla}$ denotes the covariant derivative with respect to $\overset{\circ}{\gamma}_{jk}{}^{i}$, then it leaves $\phi_{j}{}^{i}$, ξ^{i} and η_{j} covariant, that is, $\overset{\circ}{l}_{jk}{}^{i}$ is a (ϕ, ξ, η)-connection in the sense of Sasaki and Hatakeyama. Thus we have

THEOREM 3.1. *On a manifold with a* (ϕ, ξ, η)*-structure we can find an affine* (ϕ, ξ, η)*-connection.*

By the way we note that the torsion tensor of the connection $\overset{\circ}{l}_{jk}{}^{i}$ is as follows.

$$\begin{aligned}\overset{\circ}{S}_{jk}{}^{i} &= \frac{1}{2}(\overset{\circ}{T}_{jk}{}^{i} - \overset{\circ}{T}_{kj}{}^{i}) \\ &= \frac{1}{2}\Big[\{\xi^{i}(\overset{\circ}{\nabla}_{k}\eta_{j}) - \xi^{i}(\overset{\circ}{\nabla}_{j}\eta_{k})\} - \frac{1}{2}\{(\overset{\circ}{\nabla}_{k}\phi_{j}{}^{l})\phi_{l}{}^{i} - (\overset{\circ}{\nabla}_{j}\phi_{k}{}^{l})\phi_{l}{}^{i}\}\Big].\end{aligned} \tag{3.8}$$

Moreover, it is also known that $l_{jk}{}^{i}$ is a π-connection if and only if it can be expressed as

$$\left\{\begin{aligned} l_{jk}{}^{i} &= \overset{\circ}{l}_{jk}{}^{i} + U_{jk}{}^{i}, \text{ where} \\ U_{jk}{}^{i} &= \frac{1}{3}\{\sigma_{jk}{}^{i} + \frac{1}{\lambda^{3}}(\overset{2}{F}_{j}{}^{d}\sigma_{dk}{}^{c}\overset{1}{F}_{c}{}^{i} + \overset{1}{F}_{j}{}^{d}\sigma_{dk}{}^{c}\overset{2}{F}_{c}{}^{j})\}\end{aligned}\right. \tag{3.9}$$

with some tensor $\sigma_{jk}{}^{i}$.

Let $\xi^{i}_{|k}$ and $\xi^{i}_{;k}$ be respectively the covariant derivative of ξ^{i} with respect to $l_{jk}{}^{i}$ and $\overset{\circ}{l}_{jk}{}^{i}$, then

$$\xi^{i}_{|k} = \xi^{i}_{;k} + U_{jk}{}^{i}\xi^{i}. \tag{3.10}$$

So the condition for $l_{jk}{}^{i}$ in (3.9) to be a (ϕ, ξ, η)-connection is that $U_{jk}{}^{i}\xi^{i} = 0$. Substitute (3.9), (2.3) and (2.4) into this relation we have the following condition after some simple calculation:

$$\xi^{j}\sigma_{jk}{}^{i}\eta_{i} = 0. \tag{3.11}$$

Thus we have

THEOREM 3.2. *A connection* $l_{jk}{}^{i}$ *is a* (ϕ, ξ, η)*-connection if and only if it can be expressed as* (3.9) *with some tensor* $\sigma_{jk}{}^{i}$ *satisfying* (3.11).

4. Symmetric (ϕ, ξ, η)-connections. As $\overset{\circ}{l}_{jk}{}^{i}$ in (3.6) is a π-connection induced by a symmetric connection, it follows from the general theory of π-structure [2] that the connection defined by

$$(4.1) \qquad \hat{l}_{jk}{}^{i} = \overset{\circ}{l}_{jk}{}^{i} - \frac{2}{3}\{\overset{\circ}{S}_{jk}{}^{i} + \frac{1}{\lambda^3}(\overset{2}{F}_j{}^d \overset{\circ}{S}_{dk}{}^c \overset{1}{F}_c{}^i + \overset{1}{F}_j{}^d \overset{\circ}{S}_{dk}{}^c \overset{2}{F}_c{}^i)\}$$

is a distinguished π-connection, that is a π-connection having the torsion tensor $t_{jk}{}^{i}$ in (2. 18) of the considered 3-π-structure as its torsion tensor. In (4. 1), $\overset{\circ}{S}_{jk}{}^{i}$ is the torsion tensor of the connection $\overset{\circ}{l}_{jk}{}^{i}$ and can be expressed as in (3. 8).

On the other hand, since $\overset{\circ}{l}_{jk}{}^{i}$ is a (ϕ, ξ, η)-connection, we see by Theorem 3. 2 that $\hat{l}_{jk}{}^{i}$ is also a (ϕ, ξ, η)-connection if and only if the following condition is satisfied:

$$(4.2) \qquad 2\xi^j \overset{\circ}{S}_{jk}{}^{i}\eta_i = 0.$$

Substitute from (3. 8), we have

$$(4.3) \qquad \xi^j\{(\overset{\circ}{\nabla}_k\eta_j) - (\overset{\circ}{\nabla}_j\eta_k)\} = 0.$$

Since $\overset{\circ}{\nabla}$ denote the covariant derivative with respect to a symmetric connection specified above, (4.3) is equivalent to the following

$$(4.4) \qquad N_k \equiv \xi^j(\eta_{k,j} - \eta_{j,k}) = 0.$$

Thus we have

THEOREM 4. 1. *Let M^{2n+1} be a manifold with a (ϕ, ξ, η)-structure. If $N_j = 0$, then we can find a (ϕ, ξ, η)-connection whose torsion tensor is equal to the tensor $t_{jk}{}^{i}$ in* (2. 18).

If η_j is a gradient and $N_j{}^i = 0$, then $t_{jk}{}^{i}$ in (2. 18) turns out to be

$$(4.5) \qquad t_{jk}{}^{i} = -\frac{1}{4}N_{jk}{}^{i}.$$

Thus we have the following theorem of Sasaki and Hatakeyama[3]:

THEOREM 4. 2. *Let M^{2n+1} be a manifold with a (ϕ, ξ, η)-structure. If η_j is a gradient and $N_j{}^i = 0$, then we can find (ϕ, ξ, η)-connection whose torsion tensor is equal to* $-\frac{1}{4}N_{jk}{}^{i}$.

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