ON A REPRESENTATION OF A COUNTABLY INFINITE GROUP

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Introduction. A study of crossed products of rings of operators [4] shows that *any countably infinite group is isomorphic to a group of outer automorphisms of an approximately finite factor on a separable Hilbert space*. Combining the discussions in [4] with those in [3] we obtain the following result.

THEOREM. *Any countably infinite group is isomorphic to a group of outer automorphisms of a certain factor of type* III *on a separable Hilbert space.*

The purpose of this paper is to prove this theorem together with the theorem of N. Suzuki [4]. In this paper, an automorphism of a factor means a *-automorphism, and a group of outer automorphisms of a factor is a group of automorphisms all of which are outer automorphisms except the unit.

1. The construction of a factor of type III which contains a singular maximal abelian subring. Let G be a given countably infinite group. Let Δ be the set of all functions α on G such that $\alpha(g) = 1$ on a finite subset of G, and $= 0$ elsewhere. Defining for α, $\beta \in \Delta$ the addition $\alpha + \beta$ by $[\alpha + \beta](g) = \alpha(g) + \beta(g) \pmod 2$, we make Δ into a group with the unit $0 : 0(g) = 0$ for all $g \in G$. Let Δ' be the set of all functions φ on Δ such that $\varphi(\alpha) = 1$ on a finite subset of Δ, and $= 0$ elsewhere. Δ' is a group with addition: $[\varphi + \psi](\alpha) = \varphi(\alpha) + \psi(\alpha) \pmod 2$ for $\varphi, \psi \in \Delta'$. The unit element of Δ' is the function $0 : 0(\alpha) = 0$ for all $\alpha \in \Delta$.

For each $\alpha \in \Delta$, we associate the measure space $(X_\alpha, \mathrm{S}_\alpha, \mu_\alpha)$, where X_α consists of two points 0 and 1, S_α consists of all subsets of X_α, $\mu_\alpha(\{0\}) = p$, $\mu_\alpha(\{1\}) = q$, and $q \geqq p > 0$, $p + q = 1$. We construct the infinite product measure space (X, S, μ) of $(X_\alpha, \mathrm{S}_\alpha, \mu_\alpha)_{\alpha \in \Delta}$. Then a point $x \in X$ is a function on Δ which takes values 0, 1 only, and so Δ' is a subgroup of X when X is considered as an additive group. Let $\mathbf{H} = L_2(X, \mathrm{S}, \mu)$ and the multiplication algebra on $\mathbf{H}$ be denoted by $\mathbf{A}$ and an element of $\mathbf{A}$ corresponding to a function $a \in L_\infty(X, \mathrm{S}, \mu)$ will be written by a.

For $\alpha \in \Delta$, $\varphi \in \Delta'$ and $g \in G$, we define one-to-one mappings $x \to x^\alpha$, $x \to x^\varphi$ and $x \to x^g$ of X onto itself as follows:

$$x^{\alpha}(\gamma) = x(\alpha + \gamma),$$
$$x^{\varphi}(\gamma) = x(\gamma) + \varphi(\gamma) \pmod 2,$$
$$x^{g}(\gamma) = x(\gamma^{g^{-1}}),$$

where γ^g is defined by $\gamma^g(h) = \gamma(gh)$, and g^{-1} is the inverse of g.

Using the above mappings, we define operators $a^{\alpha}, a^{\varphi} \in \mathbf{A}$ for each $a \in \mathbf{A}$, $\alpha \in \Delta$, $\varphi \in \Delta'$ by

$$[a^{\alpha}f](x) = a(x^{\alpha})f(x)$$
$$[a^{\varphi}f](x) = a(x^{\varphi})f(x) \qquad \text{for all } f \in \mathbf{H}.$$

Let (Δ',Δ) be the set of all pairs (φ, α), $\varphi \in \Delta', \alpha \in \Delta$. For each $(\varphi, \alpha) \in (\Delta', \Delta)$ we associate a one-to-one mapping $x \to x^{(\varphi,\alpha)}$ of X onto itself, defined by

$$x^{(\varphi,\alpha)}(\gamma) = x(\alpha + \gamma) + \varphi(\gamma) \qquad (x \in X).$$

For an $\alpha \in \Delta$ and a $\varphi \in \Delta'$ we define an element $\varphi^{\alpha} \in \Delta'$ by $\varphi^{\alpha}(\gamma) = \varphi(\alpha + \gamma)$. Then, as $[x^{(\varphi,\beta)}]^{(\psi,\alpha)} = x^{(\varphi^{\beta}+\psi,\alpha+\beta)}$ for (φ,α), $(\psi,\beta) \in (\Delta', \Delta)$, (Δ', Δ) is a group by the law of composition $(\varphi, \alpha)(\psi, \beta) = (\varphi^{\beta} + \psi, \alpha + \beta)$. In fact,

$$(\varphi, \alpha)(0, 0) = (0, 0)(\varphi, \alpha) = (\varphi, \alpha)$$
$$(\varphi, \alpha)(\varphi^{\alpha}, \alpha) = (\varphi^{\alpha}, \alpha)(\varphi, \alpha) = (0, 0).$$

This group will be denoted by $\mathfrak{G}$.

For $a \in \mathbf{A}$, $\alpha \in \Delta$ and $\varphi \in \Delta'$ we define operators A, U_{α} and U_{φ} on the Hilbert space $\mathbf{H} \otimes l_2(\mathfrak{G})$ as follows: For $F(x, (\psi, \beta)) \in \mathbf{H}\otimes l_2(\mathfrak{G})$

$$[AF](x, (\psi, \beta)) = a(x)F(x, (\psi, \beta)),$$
$$[U_{\alpha}F](x, (\psi, \beta)) = F(x^{\alpha}, (\psi, \beta)(0, \alpha)),$$
$$[U_{\varphi}F](x, (\psi, \beta)) = \left[\frac{d\mu_{\varphi}}{d\mu}(x)\right]^{\frac{1}{2}\,*)} F(x^{\varphi}, (\psi, \beta)(\varphi, 0)),$$

where $\mu_{\varphi}(E) = \mu(E\varphi)$ and $E\varphi = \{x^{\varphi} \mid x \in E\}$ for any $E \in \mathbf{S}$. Here we note the following fact: For any $\varphi \in \Delta'$,

$$\frac{d\mu_{\varphi}}{d\mu}(x) = \prod_{\alpha\in\Delta}\left(\frac{p}{q}\right)^{(x(\alpha)-1/2)\varphi(\alpha)}$$

and thus if $p = q = \dfrac{1}{2}$, $\dfrac{d\mu_{\varphi}}{d\mu}(x) = 1$.

It is easily seen that A is a bounded operator for each $a \in \mathbf{A}$ and U_{α},

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*) By Corollary of Lemma 3 in [4] μ is quasi-invariant under Δ', that is, $\mu(E) = 0$ for $E \in \mathbf{S}$ implies $\mu(E\varphi) = 0$ for all $\varphi \in \Delta'$. Hence μ_{φ} is absolutely continuous with respect to μ.

U_φ are unitary operators for all $\alpha \in \Delta, \varphi \in \Delta'$.

LEMMA 1. *For each* $a \in \mathbf{A}$, $\alpha \in \Delta$ *and* $\varphi \in \Delta'$, *we have*

$$U_\alpha A U_\alpha = A^\alpha,\ U_\varphi A U_\varphi = A^\varphi,\ U_\alpha U_\varphi U_\alpha = U_{\varphi^\alpha},$$

where A^α(*resp.* A^φ) *is an operator on* $\mathbf{H} \otimes l_2(\mathfrak{G})$ *corresponding to* a^α (*resp.* a^φ) *on* $\mathbf{H}$.

PROOF. For $F \in \mathbf{H} \otimes l_2(\mathfrak{G})$, we have

$$\begin{aligned}[][U_\alpha A U_\alpha F](x,(\psi,\beta)) &= [AU_\alpha F](x^\alpha,(\psi,\beta)(0,\alpha)) \\ &= a(x^\alpha)[U_\alpha F](x^\alpha,(\psi,\beta)(0,\alpha)) \\ &= a(x^\alpha)F(x,(\psi,\beta)) = [A^\alpha F](x,(\psi,\beta)),\end{aligned}$$

$$\begin{aligned}[][U_\varphi A U_\varphi F](x,(\psi,\beta)) &= \left[\frac{d\mu_\varphi}{d\mu}(x)\right]^{\frac{1}{2}}[AU_\varphi F](x^\varphi,(\psi,\beta)(\varphi,0)) \\ &= \left[\frac{d\mu_\varphi}{d\mu}(x)\frac{d\mu_\varphi}{d\mu}(x^\varphi)\right]^{\frac{1}{2}} a(x^\varphi)F(x,(\psi,\beta)) \\ &= [A^\varphi F](x,(\psi,\beta)),\end{aligned}$$

because $\frac{d\mu_\varphi}{d\mu}(x)\frac{d\mu_\varphi}{d\mu}(x^\varphi) = 1$ except for a set of μ-measure 0, and

$$\begin{aligned}[][U_\alpha U_\varphi U_\alpha F](x,(\psi,\beta)) &= \left[\frac{d\mu_\varphi}{d\mu}(x^\alpha)\right]^{\frac{1}{2}} F(x^{\varphi^\alpha},(\psi,\beta)(\varphi^\alpha,0)) \\ &= \left[\frac{d\mu_{\varphi^\alpha}}{d\mu}(x)\right]^{\frac{1}{2}} F(x^{\varphi^\alpha},(\psi,\beta)(\varphi^\alpha,0)) \\ &= [U_{\varphi^\alpha}F](x,(\psi,\beta)).\end{aligned}$$

Put $U_{(\varphi,\alpha)} = U_\alpha U_\varphi$ for each $(\varphi,\alpha) \in \mathfrak{G}$. Then we get the following lemma.

LEMMA 2. *For each* $(\varphi,\alpha), (\psi,\beta) \in \mathfrak{G}$,

$$U_{(\varphi,\alpha)}U_{(\psi,\beta)} = U_{(\varphi,\alpha)(\psi,\beta)},\ [U_{(\varphi,\alpha)}]^{-1} = U_{(\varphi,\alpha)^{-1}}.$$

PROOF. $$\begin{aligned} U_{(\varphi,\alpha)}U_{(\psi,\beta)} &= U_\alpha U_\varphi U_\beta U_\psi = U_\alpha U_\beta U_\beta U_\varphi U_\beta U_\psi \\ &= U_{\alpha+\beta}U_{\varphi^\beta}U_\psi = U_{\alpha+\beta}U_{\varphi^\beta+\psi} = U_{(\varphi^\beta+\psi,\alpha+\beta)} \\ &= U_{(\varphi,\alpha)(\psi,\beta)},\end{aligned}$$

and $[U_{(\varphi,\alpha)}]^{-1} = U_\varphi U_\alpha = U_\alpha U_\alpha U_\varphi U_\alpha = U_\alpha U_{\varphi^\alpha} = U_{(\varphi^\alpha,\alpha)} = U_{(\varphi,\alpha)^{-1}}$.

If $q > p > 0$, $\mathfrak{G}$ is *free, ergodic* and *non-measurable* in the sense of [2] (cf. [3: Lemma 9]), and if $p = q = \frac{1}{2}$, $\mathfrak{G}$ is *free, ergodic* and *measurable* since, in this case μ is Lebesgue measure on [0. 1]. Hence, employing [1: Lemma 5. 2. 2] and [4: Lemma 1] we have

LEMMA 3. *Let* $\mathbf{M}$ *be the ring of operators generated by* A $(a \in \mathbf{A})$ *and* $U_{(\varphi,\alpha)}$ $((\varphi, \alpha) \in \mathfrak{G})$. *Then* $\mathbf{M}$ *is either a factor of type* **III** *or a factor of type* $\mathbf{II}_1$ *according as either* $q > p > 0$ *or* $p = q = \frac{1}{2}$. *In the latter case,* $\mathbf{M}$ *is an approximately finite factor since* $\mathfrak{G}$ *is locally finite.*

2. The proof of the theorem. First we observe that the subalgebra $\mathbf{P} \subseteqq \mathbf{M}$ generated by $U_{(0,\alpha)}$ $(\alpha \in \Delta)$ is a singular maximal abelian subalgebra of $\mathbf{M}$, i.e. the subalgebra of $\mathbf{M}$ generated by the unitary operators $U \in \mathbf{M}$ satisfying $U\mathbf{P}U^* \subseteqq \mathbf{P}$ coincides with $\mathbf{P}$. In fact, Δ is an abelian group and hence the discussion in the proof of [3: Theorem 2] is directly applicable, and the conclusion for $\mathbf{P}$ is obtained.

For each $g \in G$ we define a one-to-one mapping of Δ' onto itself $\varphi \to \varphi^g$ by $\varphi^g(\gamma) = \varphi(\gamma^{g^{-1}})$. The element (φ^g, γ^g) $((\varphi, \alpha) \in \mathfrak{G}, g \in G)$ will be denoted by $(\varphi, \alpha)^g$ shortly. Using these we define the unitary operator U_g on $\mathbf{H} \otimes l_2(\mathfrak{G})$ for each $g \in G$ by

$$[U_gF](x, (\psi, \beta)) = F(x^g, (\psi, \beta)^g).$$

Then the following lemma is obtained.

LEMMA 4. *For* $(\varphi, \alpha) \in \mathfrak{G}$, $g \in G$ *and* $a \in \mathbf{A}$ *we have*

$$U_{g^{-1}}U_{(\varphi,\alpha)}U_g = U_{(\varphi,\alpha)^g}, \quad U_{g^{-1}}AU_g = A^g$$

where A^g *is defined by* $[A^gF](x, (\psi, \beta)) = a(x^{g^{-1}})F(x, (\psi, \beta))$ *for* $F \in \mathbf{H} \otimes l_2(\mathfrak{G})$.

PROOF. For $F \in \mathbf{H} \otimes l_2(\mathfrak{G})$, we have

$$\begin{aligned}[U_{g^{-1}}AU_gF](x, (\psi, \beta)) &= [AU_gF](x^{g^{-1}}, (\psi, \beta)^{g^{-1}}) \\ &= a(x^{g^{-1}})[U_gF](x^{g^{-1}}, (\psi, \beta)^{g^{-1}}) \\ &= a(x^{g^{-1}})F(x, (\psi, \beta)) = [A^gF](x, (\psi, \beta)),\end{aligned}$$

and the second equality is proved.

To prove the first equality, we show at first that for each $x \in X$, $g \in G$, $\alpha \in \Delta$, $\varphi \in \Delta'$ and $(\psi, \beta) \in \mathfrak{G}$,

$$(((x^{g^{-1}})^{\alpha})^{\varphi})^g = x^{(\varphi,\alpha)^g},$$
$$((\psi, \beta)^{g^{-1}}(\varphi, \alpha))^g = (\psi, \beta)(\varphi, \alpha)^g.$$

In fact,

$$\begin{aligned}(((x^{g^{-1}})^{\alpha})^{\varphi})^g(\gamma) &= ((x^{g^{-1}})^{\alpha})^{\varphi}(\gamma^{g^{-1}}) = (x^{g^{-1}})^{\alpha}(\gamma^{g^{-1}}) + \varphi(\gamma^{g^{-1}}) \\ &= x^{g^{-1}}(\alpha + \gamma^{g^{-1}}) + \varphi^g(\gamma) = x(\alpha^g + \gamma) + \varphi^g(\gamma)\end{aligned}$$

$$= x^{(\varphi,\alpha)^g}(\gamma),$$

and

$$\begin{aligned}((\psi,\beta)^{g^{-1}}(\varphi,\alpha))^g &= ((\psi^{g^{-1}})^\alpha + \varphi, \beta^{g^{-1}} + \alpha)^g = (((\psi^{g^{-1}})^\alpha)^g + \varphi^g, \beta + \alpha^g) \\ &= (\psi^{\alpha g} + \varphi^g, \beta + \alpha^g) = (\psi,\beta)(\varphi^g,\alpha^g) \\ &= (\psi,\beta)(\varphi,\alpha)^g.\end{aligned}$$

Using these relations we obtain

$$\begin{aligned}&[U_{g^{-1}}U_{(\varphi,\alpha)}U_gF](x,(\psi,\beta)) \\ &= \left[\frac{d\mu_\varphi}{d\mu}((x^{g^{-1}})^\alpha)\right]^{\frac{1}{2}} F((((x^{g^{-1}})^\alpha)^\varphi)^g, ((\psi,\beta)^{g^{-1}}(\varphi,\alpha))^g) \\ &= \left[\frac{d\mu_{\varphi^g}}{d\mu}(x^{\alpha g})\right]^{\frac{1}{2}} F(x^{(\varphi,\alpha)^g}, (\psi,\beta)(\varphi,\alpha)^g) \\ &= [U_{(\varphi,\alpha)^g}F](x,(\psi,\beta)).\end{aligned}$$

for $F \in \mathbf{H} \otimes l_2(\mathfrak{G})$, and thus we get

$$U_{g^{-1}}U_{(\varphi,\alpha)}U_g = U_{(\varphi,\alpha)^g}.$$

From Lemma 4 we have

LEMMA 5. *The mapping $g \to U_g$ $(g \in G)$ is a faithful unitary representation of G on $\mathbf{H} \otimes l_2(\mathfrak{G})$ and the mapping $T \to U_{g^{-1}}TU_g$ $(T \in \mathbf{M})$ defines an automorphism of* $\mathbf{M}$.

PROOF OF THE THEOREM. It is sufficient to show that for each $g \in G$, $g \neq e$, the unit of G, the mapping $T \to U_{g^{-1}}TU_g$ $(T \in \mathbf{M})$ defines an outer automorphism of $\mathbf{M}$. Suppose that there is a $g \in G$ such that the mapping $T \to U_{g^{-1}}TU_g$ defines an inner automorphism of $\mathbf{M}$. Then there exists a unitary operator $U \in \mathbf{M}$, and

$$U^{-1}TU = U_{g^{-1}}TU_g \qquad \text{for all } T \in \mathbf{M}.$$

In particular we get

$$U^{-1}U_{(0,\alpha)}U = U_{g^{-1}}U_{(0,\alpha)}U_g = U_{(0,\alpha^g)} \in \mathbf{P} \text{ for all } \alpha \in \Delta.$$

Hence, by the singularity of $\mathbf{P}$, $U \in \mathbf{P}$ and $(0,\alpha) = (0,\alpha^g)$ for all $\alpha \in \Delta$. Thus $\alpha = \alpha^g$ for all $\alpha \in \Delta$ and $g = e$. Therefore $T \to U_{g^{-1}}TU_g(T \in \mathbf{M})$ defines an outer automorphism of $\mathbf{M}$ for each $g \in G$, $g \neq e$, and the proof is completed.

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