NOTE ON CHARACTERS OF THE GROUPS OF UNITS OF ALGEBRAIC NUMBER FIELDS

KATSUHIKO MASUDA

(Received November 11, 1960)

Let k be a finite algebraic number field, I the idèle group of k with natural topology, C the idèle class group of k, D the connected component of the neutral element of C. In this short note, we shall study characterization of such characters of certain subgroups of I that are canonically obtained from characters of C/D, and obtain some informations about the role of the totally positive units in the class field theory.

In §§**1-2**, we shall prepare some notations, definitions and easy lemmas. In §**3**, we shall state a theorem of our previous paper [3], as Theorem 1, and obtain some applications, necessary in the following. Then, considering Artin's representatives[1] of E, we shall obtain the aimed results as Theorems 2, 3 and Corollary in §§**4-5**.

The author is greatly indebted to the referees for many suggestions for improving the paper.

1. Let k, I, C, and D be as stated above, throughout the present note. As usual, we identify the multiplicative group k^* of the non-zero elements of k with the principal idèle group P of k. For each prime divisor $\mathfrak{p}$ of k, we identify the multiplicative group $k_\mathfrak{p}^*$ of the non-zero elements of the $\mathfrak{p}$-completion field $k_\mathfrak{p}$ of k with $I_\mathfrak{p}$, respectively, where we denote by $I_\mathfrak{p}$ the subgroup of I that consists of all idèles a with 1 as the $\mathfrak{q}$-component $(a)_\mathfrak{q}$ for each prime divisor $\mathfrak{q}$ of k different from $\mathfrak{p}$. Let Y be an arbitrary, not necessarily closed, subgroup of I. We consider Y as a topological group by the relative topology with reference to I. Y is not necessarily locally compact, but has sufficiently many characters. Each character $\bar{\chi}$ ($\bar{\chi}'$) of C (C/D) gives canonically a character χ of Y, which we call a G- (D-) character of Y, respectively. We shall use the following well known result from the duality of locally compact Abelian groups that we state, without proof, as

LEMMA 1. *Let B be a locally compact Abelian group, and B_1 a closed subgroup. Then, there exists for each character χ of B_1 a character $\bar{\chi}$ of B*

1) Cf. [1] & [5].

such that the restriction of $\bar{\chi}$ to B_1 coincides with χ.

Let χ and $\bar{\chi}$ be as stated in the above lemma. Then we call $\bar{\chi}$ an extention of χ.

COROLLARY[2]. *Let B be a locally compact abelian group, B_1 a compact subgroup, and B_2 a closed subgroup of B. Then, a character χ of B_1 is extensible to a character $\bar{\chi}$ of B, such that the restriction of $\bar{\chi}$ to B_2 is trivial, if and only if the restriction of χ to the intersectin $B_1 \cap B_2$ of B_1 and B_2 is trivial.*

2. Let E_0 denote the subgroup of I that consists of all idèles a that satisfy both the following conditions: (i) For every archimedean prime divisor[3] $\mathfrak{p}$, the $\mathfrak{p}$-component $(a)_\mathfrak{p}$ is 1. (ii) For each non-archimedean prime divisor $\mathfrak{q}$, the $\mathfrak{q}$-component $(a)_\mathfrak{q}$ is a $\mathfrak{q}$-unit. Let T be an arbitrary set of non-archimedean prime divisors of k. We define an endomorphism T^* of I, corresponding to T, such that, for each prime divisor $\mathfrak{p}$ of k, the $\mathfrak{p}$-component $(T^*(a))_\mathfrak{p}$ of the image of a is given by

$$(1) \qquad (T^*(a))_\mathfrak{p} = (a)_\mathfrak{p} \qquad (\mathfrak{p} \in T)$$
$$= 1 \qquad (\mathfrak{p} \notin T).$$

Obviously, $T^{*2} = T^*$ and $(T^*(a))_\mathfrak{p} = 1$ for each archimedean prime divisor $\mathfrak{q}$. Let E_T denote the intersection of E_0 and the image $T^*(I)$ of I by T^*: $E_T = E_0 \cap T^*(I)$. When $T = \{\mathfrak{p}\}$ consits of a single prime divisor $\mathfrak{p}$, we use the notation $E_\mathfrak{p}$, for brevity, in place of $E_{\{\mathfrak{p}\}}$. Let χ be a character of E_0, $\mathfrak{p}$ a non-archimedean prime divisor of k. We say that $\mathfrak{p}$ is ramified by χ, if and only if the restriction of χ to $E_\mathfrak{p}$ is non-trivial. We denote by $V(\chi)$ the set of all non-archimedean prime divisors ramified by χ. As is well known, $V(\chi)$ is always a finite set, and we can define conductor of a character of E_0 as usual. The following proposition and corollary follow trivially from Lemma 1 and Corollary to it.

PROPOSITION 1. *Every character χ of E_0 is a G-character.*

COROLLARY. *Let T be an arbitrary set of non-archimedean divisors and χ be a character of E_T. Then, χ is extensible to a G-character χ' of E_0 such that $V(\chi') \subset T$.*

2) This follows easily from a result in the p. 17 of [4] and the duality theorem of loclally compact Abelian groups.

3) We use the words non-archimedean prime divisors and archimedean prime divisors in place of finite prime divisors and infinite prime divisors.

3. For each idèle a of k, we define $Z(a)$ as the set of all prime divisors $\mathfrak{p}$ (not necessarily non-archimedian) of K with $\mathfrak{p}$-component $(a)_{\mathfrak{p}}$ different from 1, i e.,

$$Z(a) = \{\mathfrak{p}; (a)_{\mathfrak{p}} \neq 1\}. \tag{2}$$

Let J_0 be the subset of I that consists of all idèles a that satisfy both the following conditions; (i) $Z(a)$ does not contain any archimedean prime divisor. (ii) The Kronecker density of $Z(a)$ is 0. Obviously, J_0 is a non-closed subgroup of I. The following theorem was proved in our previous paper ([3]):

THEOREM 1. *The natural map of J_0 into C/D is injective.*

We take an arbitrary one of non-empty sets of non-archimedean prime divisors of k with 0 as its Kronecker density, denote it by T and fix it from now on, throughout the rest of this note. Let A be the maximal Abelian extension of k, G the Galois group of A over k, and σ the canonical homomorphism of I onto G given by the class field theory. Then, as E_T is compact, from Theorem 1 follows cleary the following

PROPOSITION 2. *The restriction σ to E_T is an isomorphism.*

Let $\sigma(E_T)$ denote the image of E_T by σ into G. From Proposition 2, there exists the inverse σ_T^{-1} of the restriction σ_T of σ to E_T. (The defining domain of σ_T^{-1} is $\sigma(E_T)$). Let χ be a character of E_T. We obtain a character χ_1 of $\sigma(E_T)$ from χ by σ_T^{-1}. χ_1 is from Lemma 1 extensible to a character χ_1' of G which induces a character of I by σ. Let χ' denote its restriction to E_0. Obviously, χ' is an extension of χ to a D-character of E_0. Thus, we obtain the following proposition.

PROPOSITION 3. *Every character of E_T is a D-character of E_T.*

It is not always possible to extend a character of E_T to a D-character χ' of E_0, such that $V(\chi') \subset T$. In the following, we shall study the condition for the extensibility under this restriction.

4. Let F denote the subgroup of P, that consists of all of the totally positive units of k. Then, we have

THEOREM 2. *Let χ be a character of E_0. Then, χ is a D-character, if and only if F^* is contained in the kernel of χ, where we denote $(V(\chi))^*(F)$ by F^* for brevity.*

This is obviously equivalent with the following theorem, that we shall prove.

THEOREM 2'. *Let S be a finite set of non-archimedean prime divisors of*

k, and χ be a character of E_S. Then χ is extensible into a D-character χ' of E_0, such that $V(\chi') \subset S$, if and only if the kernel of χ contains $S^*(F)$.

PROOF OF THEOREM 2$'$. The only-if-part of the theorem is trivial, and we omit the proof. Let χ be a character of E_S such that the kernel of χ contains $S^*(F)$. We denote by R the set of the non-archimedean prime divisors of k not belonging to S, and by γ and δ the natural maps of E_0 into C and C/D respectively. Let δ_S denote the restriction of δ to E_S. Then there exists, from Proposition 2, the inverse δ_S^{-1}. Let χ^* denote the character of $\sigma(E_S)$, obtained from χ by δ_S^{-1}. From Corollary to Lemma 1, χ^* is extensible to a character of $\delta(E_0) = \delta(E_S)\cdot\delta(E_R)$, such that the restriction to $\delta(E_R)$ is trivial, if (and only if) the kernel of χ^* contains the intersection $\delta(E_S) \cap \delta(E_R)$ of $\delta(E_S)$ and $\delta(E_R)$. Let D_0[4] be as stated in Artin's article [1]. As D/D_0 is a real line and E_0 is compact and totally disconnected, the natural map of E_0 into D/D_0 is trivial, and so, $\gamma(E_0)$ lies in D_0. Therefore, for a pair of idèles $e \in E_S$ and $e' \in E_R$ it holds $\delta(e) = \delta(e'^{-1})$, if and only if ee' can be written as

$$ee' = n\alpha, \tag{3}$$

where n is a number in P and α is one of Artin's representatives. Applying the endomorphisms $(0)^*$ and $(\infty)^*$ to the both sides of (3), we obtain that n is a unit and totally positive, respectively, where we denote by 0 and ∞ the set of all non-archimedean prime divisors and that of all archimedean prime divisors of k. Let $\mathfrak{p}_1^{m_1}\mathfrak{p}_2^{m_2}\ldots\ldots\mathfrak{p}_r^{m_r}$ with $\mathfrak{p}_i \in S$ and non-negative rational integers m_i be the conductor of χ. From the construction of Artin's representatives, there exists an element ε of F, such that, for every one of $i=1,2,\ldots\ldots,r$, it holds

$$(\alpha)_{\mathfrak{p}_i} \equiv (\{\mathfrak{p}_i\}^* (\varepsilon))_{\mathfrak{p}_i} \qquad \mod \mathfrak{p}_i^{m_i}.$$

Then from the assumption that the kernel of χ contains $S^*(F)$ follows $\chi(e) = 1$, accordingly $\chi^*(\delta(e)) = 1$. Therefore χ^* is, from Corollary of Lemma 1, extensible to a character of C/D, such that the restriction to $\delta(E_R)$ is trivial. It certifies obviously the if-part of the theorem, q. e. d..

5. Let M_T be the intermediate field of A/k corresponding to $\sigma(E_T)$. From Theorem 1, $E_T \cong \sigma(E_T)$, and M_T is the intersection of the inertia-fields corresponding to the prime divisors contained in T. Let L be a ray-class-field over k such that the conductor $\mathfrak{f}$ of L/k is divisible by a non-archimedian prime divisor $\mathfrak{p}$, only if $\mathfrak{p}$ is contained in T. The Galois group of A/LM_T is isomorphic with a subgroup of E_T. Let K_T be the union of all of such ray-

4) This is an exception of our terminology in this note to use the suffix o in order to denote concepts concerning the totality of non-archimedean prime divisors of K.

class-fields over k. Then it follows easily from Theorem 2′ that the Galois group of $A/K_T M_T$ is isomorphic with the closure $\overline{T^*(F)}$ of $T^*(F)$ in E_T, which we state as

THEOREM 3[5]. $\mathfrak{G}(A/K_T M_T) \simeq \overline{T^*(F)}$.

From the above theorem and the corollary to Proposition 1, we have

COROLLARY. *The dual group of* $\mathfrak{G}(A/K_T M_T)$ *is canonically isomorphic with the totality of the restrictions to* $(\infty)^*(F)$ *of Grössen-characters*[6] *conductors of which have finite components divisible only by some of prime divisors contained in* T.

REFERENCES

[1] E. ARTIN, Representatives of the connected component of the idèle class group, Proc. Int. Sym., Tokyo & Nikko, 1955.
[2] C. CHEVALLEY, La thèorie du corps de classes, Ann. of Math., 41 (1940).
[3] K. MASUDA, Certain subgroup of the idele group, Proc. Japan Acad., 33 (1957).
[4] A. WEIL, L'integration dans les groupes topologiques et ses applications, Paris, 1953.
[5] ——— Sur la thèorie du corps de classes, J. Math. Soc. Japan, 3 (1951).
[6] F. WHAPLES, Non-analytic class field theory and Grunwald's theorem, Duke Math. J., 9 (1942).

5) Cf. [2], [5], [6], esp. Lemma 3 in [6].
6) Cf. [5].