ON THE $(K, 1, \alpha)$ METHODS OF SUMMABILITY

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1. Zygmund [5] defined the $(K, 1)$ method of summability of a series. This method has the several similar properties to those of the Riemann's $(R, 1)$ method of summability. Concerning the method $(R, 1)$, we have defined, in the paper [1], the Riemann-Cesàro method $(R, 1, \alpha)$ which reduces the method $(R, 1)$ when $\alpha = -1$. In this note, by the analogous method, concerning the method $(K, 1)$, we shall define the new methods of summability and show that the new methods have the similar properties to those of the methods $(R, 1, \alpha)$.

Let α be a real number such that $-1 \leqq \alpha \leqq 0$, and let s_n^α be the Cesàro sum, of order α, of a series $\sum_{n=0}^{\infty} a_n$ with $a_0 = 0$. If the series in

$$\tau(\alpha, t) = t^{\alpha+1}\sum_{n=1}^{\infty} s_n^\alpha \int_t^\pi \frac{\sin nx}{2\tan x/2}dx$$

converges in some interval $0 < t < t_0$, and if

$$\lim_{t\to 0+} \tau(\alpha, t) = B_\alpha s,$$

where

$$B_\alpha = \begin{cases} \pi/2 & \alpha = -1 \\ (\alpha+1)^{-1}\sin(\alpha+1)\pi/2 & -1 < \alpha < 0 \\ 1 & \alpha = 0, \end{cases}$$

then, we will say that the series $\sum_{n=0}^{\infty} a_n$ is evaluable $(K, 1, \alpha)$ to s. When $\alpha = -1$, the method $(K, 1, \alpha)$ reduces the method $(K, 1)$.

2. The above constant B_α is obtained if we consider the $(K, 1, \alpha)$ transform of the series

$$0 + 1 + 0 + 0 + \cdots\cdots.$$

For $\alpha = -1$, it is obvious that $B_\alpha = \pi/2$. For $-1 < \alpha < 0$, since, A_n^α denoting the Andersen notation,

$$\tau(\alpha, t) = t^{\alpha+1}\sum_{n=1}^{\infty} A_{n-1}^\alpha \int_t^\pi \frac{\sin nx}{2\tan x/2}dx$$

$$= t^{\alpha+1}\int_t^\pi \frac{1}{2\tan x/2}\,\Im\Big(\sum_{n=1}^\infty A_{n-1}^\alpha e^{inx}\Big)dx$$

$$= t^{\alpha+1}\int_t^\pi \frac{1}{2\tan x/2}\,\Im\,\{e^{ix}(1-e^{ix})^{-(\alpha+1)}\}dx$$

$$= t^{\alpha+1}\int_t^\pi \frac{\sin\{(\alpha+1)(\pi-x)/2+x\}}{(2\sin x/2)^{\alpha+1}(2\tan x/2)}dx,$$

we have

$$\lim_{t\to 0+} \tau(\alpha, t) = (\alpha+1)^{-1}\sin(\alpha+1)\pi/2 = B_\alpha.$$

Further we shall prove that $\lim_{t\to 0+} \tau(0, t) = B_0$. By

$$\sin x + \sin 2x + \ldots\ldots + \sin nx = \frac{1}{2\tan x/2} - \frac{\cos(n+1/2)x}{2\sin x/2},$$

we get, by the Riemann-Lebesgue Theorem,

$$\tau(0, t)/t = \sum_{n=1}^\infty \int_t^\pi \frac{\sin nx}{2\tan x/2}dx$$

$$= \lim_{m\to\infty}\int_t^\pi \Big(\sum_{n=1}^m \frac{\sin nx}{2\tan x/2}\Big)dx$$

$$= \int_t^\pi \frac{dx}{(2\tan x/2)^2}$$

$$= \frac{1}{2\tan t/2} - \frac{1}{2}\Big(\frac{\pi}{2} - \frac{t}{2}\Big).$$

Hence

$$\lim_{t\to 0+} \tau(0, t) = 1 = B_0.$$

3. We shall now consider the regularity of the method $(K, 1, \alpha)$. Then we have the following theorem.

THEOREM 1. *The method $(K, 1, \alpha)$ is not regular when $-1 \leqq \alpha \leqq 0$.*

PROOF. Let $-1 \leqq \alpha \leqq 0$. Then, for any sequence $\{s_n\}$, s_n being the partial sum of the series $\sum_{n=0}^\infty a_n$, converges to zero, we have

$$\tau(\alpha, t) = t^{\alpha+1}\sum_{n=1}^\infty s_n^\alpha \int_t^\pi \frac{\sin nx}{2\tan x/2}dx$$

$$= t^{\alpha+1}\sum_{n=1}^{\infty}\Big(\sum_{\nu=1}^{n}A_{n-\nu}^{\alpha-1}s_\nu\Big)\int_t^{\pi}\frac{\sin nx}{2\tan x/2}\,dx$$

$$= t^{\alpha+1}\sum_{\nu=1}^{\infty}s_\nu\sum_{n=\nu}^{\infty}A_{n-\nu}^{\alpha-1}\int_t^{\pi}\frac{\sin nx}{2\tan x/2}\,dx$$

$$= t^{\alpha+1}\sum_{\nu=1}^{\infty}s_\nu\int_t^{\pi}\frac{1}{2\tan x/2}\Big(\sum_{n=\nu}^{\infty}A_{n-\nu}^{\alpha-1}\sin nx\Big)dx$$

$$= t^{\alpha+1}\sum_{\nu=1}^{\infty}s_\nu\int_t^{\pi}\frac{\sin\{(\nu-\alpha/2)x+\pi\alpha/2\}}{(2\sin x/2)^{\alpha}\,(2\tan x/2)}\,dx.$$

Let us assume that the method $(K, 1, \alpha)$ is regular. Then, by the Toeplitz Theorem, for $0 < t < t_0$,

$$t^{\alpha+1}\sum_{\nu=1}^{\infty}\left|\int_t^{\pi}\frac{\sin\{(\nu-\alpha/2)x+\pi\alpha/2\}}{(2\sin x/2)^{\alpha}\,(2\tan x/2)}\,dx\right| = O(1).$$

Hence the series

$$\sum_{\nu=1}^{\infty}\int_t^{\pi}\frac{\sin\{(\nu-\alpha/2)x+\pi\alpha/2\}}{(\sin x/2)^{\alpha}\tan x/2}\,dx$$

is absolutely convergent for $0 < t < t_0$. Now, in virtue of the integration by parts, for $0 < t < \pi$,

$$\int_t^{\pi}\frac{\sin\{(\nu-\alpha/2)x+\pi\alpha/2\}}{(\sin x/2)^{\alpha}\tan x/2}\,dx = \frac{\cos\{(\nu-\alpha/2)t+\pi\alpha/2\}}{(\nu-\alpha/2)\,(\sin t/2)^{\alpha}\tan t/2}$$

$$-\int_t^{\pi}\frac{(1+\alpha\cos^2 x/2)\cos\{(\nu-\alpha/2)x+\pi\alpha/2\}}{2(\nu-\alpha/2)\,(\sin x/2)^{\alpha+2}}\,dx$$

and then, by the second mean value theorem,

$$\left|\int_t^{\pi}\frac{(1+\alpha\cos^2 x/2)\cos\{(\nu-\alpha/2)x+\pi\alpha/2\}}{2(\nu-\alpha/2)\,(\sin x/2)^{\alpha+2}}\,dx\right|$$

$$= \frac{1}{2(\nu-\alpha/2)(\sin t/2)^{\alpha+2}}\left|\int_\xi^{\eta}\cos\left\{\left(\nu-\frac{\alpha}{2}\right)x+\frac{\pi\alpha}{2}\right\}dx\right|$$

$$\leqq \left(\nu-\frac{\alpha}{2}\right)^{-2}(\sin t/2)^{-\alpha-2},$$

where $0 < \xi < \eta < \pi$. Hence the series

$$\sum_{\nu=1}^{\infty}\int_t^{\pi}\frac{(1+\alpha\cos^2 x/2)\cos\{(\nu-\alpha/2)x+\pi\alpha/2\}}{2(\nu-\alpha/2)\,(\sin x/2)^{\alpha+2}}\,dx$$

is absolutely convergent for $0 < t < \pi$. Therefore, by

$$\frac{1}{(\sin t/2)^{\alpha}\tan t/2}\sum_{\nu=1}^{\infty}\frac{\cos\{(\nu-\alpha/2)t+\pi\alpha/2\}}{\nu-\alpha/2}$$
$$=\sum_{\nu=1}^{\infty}\int_{t}^{\pi}\frac{\sin\{(\nu-\alpha/2)x+\pi\alpha/2\}}{(\sin x/2)^{\alpha}\tan x/2}dx$$
$$+\sum_{\nu=1}^{\infty}\int_{t}^{\pi}\frac{(1+\alpha\cos^{2}x/2)\cos\{(\nu-\alpha/2)x+\pi\alpha/2\}}{2(\nu-\alpha/2)(\sin x/2)^{\alpha+2}}dx,$$

where the two series in the right hand are absolutely convergent for $0 < t < t_0$, the series

$$\sum_{\nu=1}^{\infty}\frac{\cos\left\{\left(\nu-\frac{\alpha}{2}\right)t+\frac{\pi\alpha}{2}\right\}}{\nu-\frac{\alpha}{2}}$$

is absolutely convergent for $0 < t < t_0$. But, by the Lusin-Denjoy Theorem [6: p. 131], this series is not absolutely convergent on any set of positive measure. From this contradiction, Theorem is proved.

REMARK. A similar argument give an another proof of Theorem 2, in the paper [1], which asserts that the method $(R, 1, \alpha)$ is not regular when $-1 \leqq \alpha \leqq 0$.

Concluding this paragraph, I take this opportunity of expressing my heartfelt thanks to Dr. T. Tsuchikura for his valuable suggestions.

4. We shall now consider the sufficient conditions for the $(K, 1, \alpha)$ summability of the series $\sum_{n=0}^{\infty} a_n$. The following results are stated without the proofs since, though not immediate, these are similar to the proofs of the corresponding results for the methods $(R, 1, \alpha)$. It is remarkable that these conditions are the same to those for the methods $(R, 1, \alpha)$.

THEOREM 2. *Suppose that the series* $\sum_{n=0}^{\infty} a_n$ *is evaluable Abel to* s *and, for some* r, $-1 < r < 0$,

$$\sum_{\nu=1}^{n}|s_{\nu}^{r}| = O(n^{r+1}).$$

Then the series $\sum_{n=0}^{\infty} a_n$ *is evaluable* $(K, 1, \alpha)$ *to* s *when* $-1 \leqq \alpha \leqq 0$.

$(R, 1, \alpha)$ analogue of this theorem is Theorem 3 in the paper [3] when $-1 \leqq \alpha < 0$ and Corollary 5 in the paper [4] when $\alpha = 0$.

COROLLARY. *The series evaluable* (C, r), $-1 < r < 0$, *to* s *is also evaluable* $(K, 1, \alpha)$ *to* s *when* $-1 \leqq \alpha \leqq 0$.

THEOREM 3. *Suppose that*

$$\sum_{\nu=n}^{2n}(|a_\nu| - a_\nu) = O(1)$$

and the series $\sum_{n=0}^{\infty} a_n$ *is evaluable Abel to* s. *Then the series* $\sum_{n=0}^{\infty} a_n$ *is evaluable* $(K, 1, \alpha)$ *to* s *when* $-1 \leqq \alpha \leqq 0$.

$(R, 1, \alpha)$ analogue is Theorem 4 in the paper [1].

THEOREM 4. *Suppose that*

$$\sum_{\nu=n}^{2n}(|a_\nu| - a_\nu) = O(n^{1-r}), \qquad 0 < r < 1,$$

and

$$\sum_{\nu=1}^{n} |s_\nu - s| = o(n/\log n).$$

Then the series $\sum_{n=0}^{\infty} a_n$ *is evaluable* $(K, 1, \alpha)$ *to* s *when* $-1 \leqq \alpha \leqq 0$.

$(R, 1, \alpha)$ analogue is Theorem 5 in the paper [1].

THEOREM 5. *Suppose that, for* $\delta > 0$,

$$s_n - s = o(1/(\log n)^{1+\delta}).$$

Then the series $\sum_{n=0}^{\infty} a_n$ *is evaluable* $(K, 1, \alpha)$ *to* s *when* $-1 \leqq \alpha \leqq 0$.

$(R, 1, \alpha)$ analogue is Theorem 3 in the paper [2].

THEOREM 6. *The series evaluable* $|C, 1|$ *to* s *is also evaluable* $(K, 1, \alpha)$ *to* s *when* $-1 \leqq \alpha \leqq 0$.

$(R, 1, \alpha)$ analogue is Theorem 3 in the paper [1] when $\alpha = 0$ and Theorem 4 in the paper [3] when $-1 < \alpha < 0$.

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