CONFORMAL MAPS OF ALMOST KAEHLERIAN MANIFOLDS

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1. Consider a campact Riemannian manifold M[2] and let Γ be a connection in the bundle of orthogonal frames over M whose homogeneous holonomy group is irreducible. Then, with respect to Γ every affine transformation is homothetic and hence an affine transformation with respect to the Riemannian connection γ. Since the largest connected group of affine transformations $A_0^\gamma(M)$ with respect to γ coincides with the largest connected group $I_0(M)$ of isometries [3] it follows that the largest connected group $A_0^\Gamma(M)$ of affine transformations with respect to Γ consists of isometries. In particular, if M is an almost Kaehlerian manifold, $A_0^\Gamma(M)$ coincides with the largest connected group of automorphisms of the almost Kaehlerian structure provided the homogeneous holonomy group associated with Γ is irreducible. Now, an infinitesimal conformal transformation of a compact flat space is homothetic (hence isometric), and so it is an infinitesimal automorphism of the almost Kaehlerian structure. On the other hand, if the Ricci scalar curvature is a non-positive constant an infinitesimal conformal transformation is isometric and the same conclusion prevails.

For compact Kaehlerian manifolds M it is known that the largest connected Lie group $C_0(M)$ of conformal transformations coincides with the largest connected group $\widetilde{A}_0(M)$ of automorphisms of the Kaehlerian structure provided the topological dimension of M is greater than 2. If dim $M = 2$ it coincides with the largest connected group of analytic homeomorphisms [6]. The following theorem was recently proved [4]:

If M^{4k} is a $4k$-dimensional compact almost Kaehlerian manifold then $C_0(M^{4k}) = \widetilde{A}_0(M^{4k})$.

It is one of the main purposes of this paper to extend this result so that it holds for all dimensions. We shall prove, in particular,

THEOREM 1. *The largest connected Lie group of conformal transformations of a compact almost Kaehlerian manifold $M^{2n} (n > 1)$ coincides with the largest connected group of automorphisms of the almost Kaehlerian structure, that is $C_0(M^{2n}) = \widetilde{A}_0(M^{2n})$.*

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2) The manifolds, differential forms and tensorfields considered are assumed to be of class C^∞.

If the almost complex structure is completely integrable and comes from a complex analytic structure we obtain as an immediate consequence the theorem of Lichnerowicz [6]. The proof given in [5] was recently extended to include the almost Kaehlerian manifolds as well [9][3]. Indeed, an integral formula is established relating a certain tensorfield (whose vanishing, in the Kaehlerian case, gives a necessary and sufficient condition for an infinitesimal transformation to be analytic) with the Ricci curvature of the manifold. The derivation of this formula is rather complicated as it involves several lengthy non-trivial computations.

Our proof does not differ essentially from that given in [4], that is it stems from the same general formula (cf. equation 4 below) upon which the results of this paper depend. For the sake of completeness and because of its intrinsic interest we shall reproduce this result thereby giving an alternate proof for the dimensions $4k$

In §8, Theorem 1 is generalized to include those compact orientable Riemannian maniforlds carrying an harmonic form of a given degree whose coefficients satisfy a certain relation. Theorem 7 seems to be the best possible result that can be obtained by our methods.

2. In the non-compact case if we consider infinitesimal maps whose covariant forms are closed a wider class of manifolds may be considered. Let X be a vector field on a pseudo Kaehlerian manifold whose image by the almost complex structure operator J is an infinitesimal conformal map preserving the pseudo-Kaehlerian structure. The vector field JX is then "closed", that is its covariant form (by the duality defined by the metric) is closed. We shall show that, in general, a "closed conformal map" is a homothetic transformation. In fact, the following theorem is proved:

THEOREM 2. *If M^{2n} $(n > 1)$ is a complete pseudo-Kaehlerian manifold which is not locally flat a closed infinitesimal conformal map is an automorphism of the pseudo Kaehlerian structure.*

In the locally flat case we may prove

THEOREM 3. *Let M^{2n} $(n > 1)$ be a complete locally flat pseudo-Kaehlerian manifold. Then a closed infinitesimal conformal transformation of bounded length is an automorphism of the pseudo Kaehlerian structure.*

REMARK. The linear space A of all closed infinitesimal conformal maps of a compact Kaehlerian manifold M^{2n} is an abelian subalgebra of the Lie algebra L of infinitesimal conformal transformations of $M^{2n}(n > 1)$. An alternate way of expressing this is to say that any two elements of A are in involution. If the first

3) Theorem 1 was established by Tachibana [9] and indepently by the author since he was unaware of his result.

Betti number vanishes $A = (0)$. To see this, we observe first that the image by J of an element X of A is isometric (cf. Proofs of Theorems 2 and 3). Hence, $\delta C\xi = 0$, where ξ is the covariant form of X and C the complex structure operator applied toforms. By Theorems 2 and 3, $X \in \widetilde{A}(M)$ that is $dC\xi$ vanishes. It follows that $C\xi$ is harmonic and therefore since M is a Kaehlerian manifold ξ is also harmonic. The condition $\delta\xi = 0$ implies that X is an infinitesimal isometry. But a harmonic vector field which is at the same time a Killing field must be a parallel field of vectors. For $X, Y \in A$, $X = \xi^i \dfrac{\partial}{\partial x^i}$, $Y = \eta^i \dfrac{\partial}{\partial x^i}$

$$[X, Y]^i = \xi^k \frac{\partial \eta^i}{\partial x^k} - \eta^k \frac{\partial \xi^i}{\partial x^k} = \xi^k D_k \eta^i - \eta^k D_k \xi^i = 0$$

where D_k denotes the covariant derivative with respect to the canonical connection. (The summation convention is used throughout.)

If we consider the subalgebra of closed bounded conformal maps the compactness condition may be replaced by completeness.

3. A real manifold M^{2n} of even dimension $2n$ is said to be *almost complex* if there is a linear transformation J defined on the tangent space at each point which varies differentiably with respect to local coordinates and whose square is minus the identity, that is if there is a real tensorfield $F^i{}_j$ on M^{2n} satisfying

$$F^i{}_j F^j{}_k = - \delta^i_k, \qquad (i, j, k = 1, \cdots\cdots, 2n).$$

In a coordinate neighborhood of an even dimensional real manifold with coordinates $x^1, \cdots\cdots, x^{2n}$ complex coordinates may be introduced by setting $z^j = x^j + ix^{j+n}$, $j = 1, \cdots\cdots, n$. The almost complex structure given by J is called *completely integrable* if the manifold can be made into a complex manifold with local coordinates $z^1, \cdots\cdots, z^n$, so that operating with J is equivalent to transforming dz^j and $d\bar{z}^j$ into idz^j and $- id\bar{z}^j$. In this case multiplication by i in the tangent space has an invariant meaning.

Consider a manifold M^{2n} admitting a 2-form

$$\omega = F_{ij} dx^i \wedge dx^j$$

of rank $2n$ everywhere. If ω is closed the manifold is said to be *symplectic.* Let g be a metric on M^{2n} with the property

$$F_{ik} F_{jl} g^{kl} = g_{ij} \qquad (i, j, k, l = 1, \cdots\cdots, 2n).$$

Such a metric always exists. The operator J acting in the tangent space at each point

$$J: \ \xi^k \rightarrow (i(X)\omega)^k$$

(where $i(X)$ is the interior product by X operator) defines on M^{2n} an almost complex structure and together with g an *almost Hermitian* structure. If the manifold is symplectic with respect to the fundamental form ω the almost Hermitian structure is called *almost Kaehlerian*. In this case, M^{2n} is said to be an *almost Kaehlerian manifold*. In such a manifold the fundamental 2-form ω is both closed and co-closed, that is harmonic. If J defines a completely integrable almost complex structure M^{2n} is said to be *pseudo-Kaehlerian*. If the almost complex structure comes from a complex structure M^{2n} is said to be a *Kaehlerian manifold*. A Kaehlerian manifold is thus an Hermitian manifold which is symplectic for the fundamental 2-form of the Hermitian structure.

4. A transformation f of a complex manifold is said to be *analytic* if it preserves the complex analytic structure. The corresponding almost complex structure J is therefore invariant by f. If two complex analytic structures induce the same almost complex structure they must coincide. Hence, in order that f be analytic it is necessary and sufficient that J be preserved. If the manifold is compact it is well known that the largest group of analytic transformations is a complex Lie group [2].

Let G denote a connected Lie group of analytic transformations of the complex manifold M. To each element A of the Lie algebra of G is associated the 1-parameter subgroup a_t of G generated by A. The corresponding 1-parameter group of transformations R_{a_t} on $M(R_{a_t}\cdot P = Pa_t,\ P \in M)$ induces a (right invariant) vector field X on M. From the action on the almost complex structure J it follows that $\theta(X)J$ vanishes where $\theta(X)$ is the Lie derivative operator applied to J. An *infinitesimal analytic transformation* is an infinitesimal transformation defined by a vector field X satisfying $\theta(X)J = 0$. On the other hand, a vector field on M satisfying this equation generates a local 1-parameter group of local transformations of M. In order that a connected Lie group G of transformations be a group of analytic transformations it is necessary and sufficient that the vector fields on M induced by the 1-parameter subgroups of G define infinitesimal analytic transformations. If M is complete, E. Cartan has shown that not every infinitesimal analytic transformation generates a 1-parameter global group of analytic transformations of M. (cf. following paragraph).

Consider a Riemannian manifold M with metric g. By a *conformal transformation* of M is meant a differentiable homeomorphism f of M onto itself with the property that

$$f^*g = a^2 g$$

where f^* is the induced map in the bundle of frames over M and a is a real differentiable function on M. If a is a constant f is said to be *homothetic*. If the metric is preserved $(a = 0)$ f is an *isometry*. The group of all the isometries of M is a Lie group with respect to the natural topology. Let G denote a connected Lie group of conformal transformations of M and L its Lie algebra. To each element A of L is associated the 1-parameter subgroup a_t of G generated by A. The corresponding 1-parameter group of transformations of M induces a differentiable vector field X on M. From the action on the metric tensor g we conclude that

$$\theta(X)g = \lambda g$$

where λ is a real differentiable function depending on X. A vector field satisfying this equation is called an *infinitesimal conformal transformation*. If M is complete every solution of $\theta(X)g = \lambda g$ generates a 1-parameter global group of conformal transformations of M.

In terms of a system of local coordinates the (symmetric) tensor $\theta(X)g$ has the components

$$(\theta(X)g)_{ij} = D_j\xi_i + D_i\xi_j.$$

Hence, if dim $M = m$ an infinitesimal conformal map is a solution of the equation

$$\theta(X)g + \frac{2}{m}\delta\xi.\ g = 0.$$

5. Let α and β be any two p-forms on a compact and orientable Riemannian manifold M^m. Then, for any vector field X on M^m it follows from Stokes' theorem and the identity

$$\theta(X) = di(X) + i(X)d$$

that

$$\int_{M^m} \theta(X)(\alpha\wedge *\beta) = \int_{M^m} di(X)(\alpha\wedge *\beta) = 0$$

where $\theta(X)$, $i(X)$ and d denote the operations of Lie derivation, contraction (interior product) and exterior derivation, resp. and $*$ denotes the duality (star) operator of Hodge. We employ the notation (,) for the global scalar product

$$(\alpha, \beta) = \int_{M^m} \alpha \wedge *\beta.$$

Since $\theta(X)$ is a derivation,

$$(\theta(X)\alpha, \beta) = -\int_{M^m} \alpha \wedge \theta(X)*\beta$$

If, therefore, we put

$$*\bar{\theta}(X)\beta = -\theta(X)*\beta,$$

that is

$$(1) \qquad \bar{\theta}(X) = (-1)^{mp+p+1} * \theta(X) *,$$

we have

$$(2) \qquad (\theta(X)\alpha, \beta) = (\alpha, \bar{\theta}(X)\beta).$$

It follows that the operator $\bar{\theta}(X)$ is the dual of $\theta(X)$. One therefore obtains

$$(3) \qquad \bar{\theta}(X) = \delta\varepsilon(\xi) + \varepsilon(\xi)\delta$$

where $\xi = X_i dx^i$ is the covariant form for X and $\varepsilon(\xi)$ is the dual of $i(X)$:

$$i(X) = (-1)^{m(p-1)} * \varepsilon(\xi) *,$$

that is

$$\varepsilon(\xi)\alpha = \xi \wedge \alpha$$

for any p-form α. The operators $\theta(X)$ and d commute and, clearly, so do their duals as one may also infer from (3). Moreover, if g denotes the metric tensor of M^m

$$(4) \qquad (\theta(X) + \bar{\theta}(X))\alpha = \alpha \cdot \delta\xi + \sum_{r=1}^{p} g^{jk}(\theta(X)g)_{ki_r} \alpha_{i_1 \cdots i_{r-1} j i_{r+1} \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

LEMMA 1. *The harmonic forms on a compact and orientable Riemannian manifold M are invariant under the Lie algebra K of the largest connected group of isometies on M.*

The proof depends on the fact that $\theta(X) + \bar{\theta}(X)$, $X \in K$ annihilates harmonic forms [3]. Since X is an infinitesimal motion, $\theta(X)g = 0$ from which it follows that $\delta\xi$ vanishes also. If α is a harmonic form $d\theta(X)\alpha = \theta(X)d\alpha = 0$ and $\delta\theta(X)\alpha = -\delta\bar{\theta}(X)\alpha = -\bar{\theta}(X)\delta\alpha = 0$. Hence $\theta(X)\alpha$ is a harmonic form. But $\theta(X)\alpha = di(X)\alpha$ from which by the Hodge decomposition of a form $\theta(X)\alpha$ vanishes.

LEMMA 2. *The harmonic forms of degree $p = \frac{m}{2}$ of a compact, orientable Riemannian manifold M of even dimension m are invariant under the Lie algebra L of the largest connected group of conformal transformations of M.*

Indeed, let X be an element of L. Then,

$$(5) \qquad \theta(X)g = -\frac{2}{m}\delta\xi \cdot g$$

and, from (4)

$$\theta(X)\alpha + \bar{\theta}(X)\alpha = \left(1 - \frac{2p}{m}\right)\delta\xi\cdot\alpha. \tag{6}$$

LEMMA 3. *If* dim $M = 2$, *the inner product of a harmonic vector field and a vector field defining an infinitesimal conformal transformation is a constant on* M.

This is clearly the case if M is a Riemann surface.

6. In a compact almost Kaehlerian manifold M the fundamental 2-form which is canonically defined by the almost Hermitian metric is closed and co-closed. The Laplace-Beltrami operator $\Delta = d\delta + \delta d$ and the operator $L: \alpha \to \alpha \wedge \omega$ (α: a p-form) do not commute in general. In fact,

$$L\Delta - \Delta L = dCdC + CdCd.$$

However, since $C\omega = \omega$, $\Delta L\omega = L\Delta\omega = 0$ from which we may conclude that ω^k is a harmonic $2k$-form. Hence, if dim $M = 4k$ and X is an infinitesimal conformal map it follows from Lemma 2 and compactness that $\theta(X)\omega^k$ vanishes. Now, $\theta(X)\omega^k = k\theta(X)\omega \wedge \omega^{k-1}$, and so since the map L defines an isomorphism between the spaces $\Lambda^p(M)$ of p-forms over M and $\Lambda^{p+2}(M)$ for $p \leqq 2k-2$ we conclude that $\theta(X)\omega$ vanishes, that is X defines an automorphism of the almost Kaehlerian structure. That the automorphisms are isometries is seen as follows: Since ω is closed $\theta(X)\omega = di(X)\omega$ and hence $i(X)\omega$ is closed. Thus, by the Hodge-De Rham decomposition of a form $i(X)\omega = d\phi + H[i(X)\omega]$ for some real C^∞ function ϕ; the operator H is the harmonic projector. Since $i(X)\omega = C\xi$, $\xi = -Cd\phi + CH[C\xi]$, from which we conclude that $\delta\xi$ vanishes.

REMARK. In an almost Kaehlerian manifold the operators C and H do not commute. Nevertheless, if h is a harmonic 1-form its image by C has zero divergence. However, if α is an effective closed p-form ($p \geqq 1$), $\delta C\alpha$ vanishes.

PROOF OF THEOREM 1. This is an application of equation (6) applied to the fundamental 2-form ω:

$$\theta(X)\omega + \bar{\theta}(X)\omega = \left(1 - \frac{2}{n}\right)\delta\xi\cdot\omega.$$

Applying δ to both sides of this relation we derive since $\bar{\theta}(X)$ and δ commute and the fact that $\delta\omega$ vanishes

$$\begin{aligned}\delta\theta(X)\omega &= \left(1 - \frac{2}{n}\right)\delta(\delta\xi\cdot\omega)\\ &= -\left(1 - \frac{2}{n}\right)D_k(\delta\xi\cdot F^k{}_i)dx^i\end{aligned}$$

$$= -\left(1-\frac{2}{n}\right)Cd\delta\xi.$$

Taking the global scalar product with $C\xi$ we have since the manifold is compact

$$(\delta\theta(X)\omega,\ C\xi) = (\theta(X)\omega,\ dC\xi) = (\theta(X)\omega,\ \theta(X)\omega)$$

and

$$(Cd\delta\xi,\ C\xi) = (d\delta\xi,\ \xi) = (\delta\xi,\ \delta\xi).$$

Hence,

$$||\theta(X)\omega||^2 = -\left(1-\frac{2}{n}\right)||\delta\xi||^2$$

where we have employed the notation $||\alpha||^2 = \int_M \alpha\wedge *\alpha$. The l. h. s. being non-negative and the r. h. s. non-positive we conclude for $n > 1$ that $\theta(X)\omega$ vanishes.

For $n > 2$ it is immediate that $\delta\xi = 0$, that is X is an infinitesimal isometry. For $n = 2$ a previous argument gives the same result. Hence, $C_0(M^{2n}) = I_0(M^{2n}) = \tilde{A}_0(M^{2n})$.

7. Throughout this section we assume that M is a pseudo-Kaehlerian manifold. Let X be a vector field on M whose image by the almost complex structure operator J is closed. Then, X is an infinitesimal automorphism of the pseudo-Kaehlerian structure since the fundamental form ω of M is closed. Denote by $t(X)$ the tensorfield $\theta(X)J$ modulo $i(X)D\omega$ where D is the covariant differential operator. For pseudo-Kaehlerian manifolds $D\omega$ vanishes, that is $t(X)$ and $\theta(X)J$ coincide. If J is induced by a complex structure the vanishing of $t(X)$ characterizes the infinitesimal analytic maps. Let t be a covariant real tensor of order 2 and denote by J again the operator

$$J:\ t_{ij} \rightarrow t_{ir}F_j^{\ r}.$$

Since $F_{ik}F_{jl}g^{kl} = g_{ij}$, $J\omega = g$, where we denote by J once again the induced map on 2-forms. The tensorfield $t(X)$ associated with a given tangent vectorfield X is given by

$$(t(X))^i{}_j = F^i{}_k D_j\xi^k - F^k{}_j D_k\xi^i.$$

It is easily checked that $t(JX) = Jt(X)$. Therefore, if X is an analytic map so is JX and the dimension of the group of analytic homeomorphisms of any complex manifold is even.

LEMMA 4. *For any vector field X*

$$t(X) = \theta(X)\omega + J\theta(X)g.$$

Indeed,

$$\begin{aligned}-(t(X))_{ij} &= F^k{}_j(D_k\xi_i - D_i\xi_k + D_i\xi_k) + F^k{}_iD_j\xi_k\\ &= F^k{}_j(\theta(X)g)_{ik} + D_j(C\xi)_i - D_i(C\xi)_j\\ &= -(J\theta(X)g)_{ij} - (\theta(X)\omega)_{ij}.\end{aligned}$$

THEOREM 4. *A vector field X defines an infinitesimal analytic transformation of a Kaehlerian manifold if and only if $J\theta(X)\omega = \theta(X)g$, that is when applied to the fundamental 2-form the operators $\theta(X)$ and J commute.*

Let $\tilde{t}(X)$ denote the 2-form corresponding to the skew-symmetric part of $t(X)$.

LEMMA 5. *For any vector field X on a pseudo-Kaehlerian manifold*

$$\bar{\theta}(X)\omega - \theta(X)\omega = \delta\xi\cdot\omega - 2\tilde{t}(X).$$

This is a straightforward application of equation (4) and Lemma 4.

LEMMA 6. *For any vector field X on a compact pseudo-Kaehlerian manifold*

$$\|\theta(X)\omega\|^2 = \|\delta\xi\|^2 + 2(\tilde{t}(X), \theta(X)\omega).$$

The proof is based on that of Theorem 1.

THEOREM 5. *Let X be an infinitesimal analytic transformation of a compact Kaehlerian manifold. Then*

$$\|\theta(X)\omega\| = \|\delta\xi\|.$$

Hence, a divergence free analytic map is an infinitesimal automorphism of the Kaehlerian structure.

This follows immediately from Lemma 6. An application of Lemma 4 together with Theorem 4 results in

THEOREM 6. *Let M be a Kaehlerian manifold. Then, in order that an infinitesimal analytic transformation be the image by J of an infinitesimal isometry it is necessary and sufficient that it be closed.*

This generalizes to the non-compact case a theorem of Lichnerowicz [7].

PROOF OF THEOREM 2. Since ξ is closed,

$$\begin{aligned}-(t(X))_{ij} &= F^k{}_j(D_k\xi_i - D_i\xi_k + D_i\xi_k) + F^k{}_iD_j\xi_k\\ &= F^k{}_jD_i\xi_k + F^k{}_iD_j\xi_k\\ &= (\theta(C\xi)g)_{ij},\end{aligned}$$

that is $t(X)$ is a symmetric tensorfield. Since $\theta(X)g = -\dfrac{1}{n}\delta\xi\cdot g$ it follows from

Lemma 4 that

$$t(X) = \theta(X)\omega + \frac{1}{n}\,\delta\xi\cdot\omega. \tag{7}$$

Hence, $t(X)$ is also skew-symmetric and must therefore vanish. It follows that $d\delta\xi\wedge\omega = 0$ and for $n > 1$ we may conclude that $d\delta\xi$ vanishes, that is X defines a homothetic transformation. But a homothetic map of a complete connected Riemannian manifold which is not locally flat is isometric [5], hence volume preserving and therefore from equation (7) we conclude that the fundamental form is preserved.

PROOF OF THEOREM 3. Every homothetic transformation of a Riemannian manifold is also an affine transformation corresponding to the Riemannian connection. Moreover, an infinitesimal affine transformation of a complete locally flat Riemannian manifold is an isometry if and only if its length is bounded.

REMARKS: (a) Clearly, if the manifold is compact every vector field has bounded length.

(b) Every conformal map of a complete flat space is homothetic but this is not sufficient to insure that it is an automorphism of the pseudo-Kaehlerian structure.

(c) It is known that every affine transformation of a complete pseudo-Kaehlerian manifold whose Ricci curvature is non-degenerate is an automorphism of the pseudo-Kaehlerian structure.

(d) If X is a homothetic transformation of an almost Kaehlerian manifold M the 2-form $\theta(X)\omega$ is harmonic. Indeed, $\theta(X)\omega$ is closed. Moreover from the proof of Theorem 1 it is also co-closed. If dim $M = 4$ this is so for any infinitesimal conformal transformation.

(e) M. Obata has communicated to us the following result (unpublished): "A closed infinitesimal conformal transformation of a (locally) reducible Riemannian manifold is homothetic." This means that only an absolutely irreducible Riemannian manifold can admit closed non-homothetic maps.

For non-compact Kaehlerian manifolds an infinitesimal isometry X need not be analytic. Indeed, the condition $\theta(X)\omega = di(X)\omega$ where $\theta(X)\omega$ is a harmonic 2-form does not imply that $\theta(X)\omega$ vanishes. However, if the manifold is complete the proof of Theorem 2 shows that a closed infinitesimal conformal transformation X is analytic and, in this case X is isometric. Without the assumption of completeness we may conclude in any case that an infinitesimal isometry which is closed is analytic and, in fact by Lemma 4 preserves the fundamental form.

Let X and Y be infinitesimal automorphisms of an almost Kaehlerian manifold M. Since $\theta([X, Y])\omega$ vanishes where ω is the fundamental 2-form of M, $[X, Y]$

is also an infinitesimal automorphism. Moreover,

$$\begin{aligned} i([X, Y])\omega &= \theta(X)i(Y)\omega - i(Y)\theta(X)\omega \\ &= i(X)di(Y)\omega + di(X)i(Y)\omega \\ &= di(\xi \wedge \eta)\omega. \end{aligned}$$

Put $Z = [X, Y]$. Then, $C\zeta = i(Z)\omega = di(\xi \wedge \eta)\omega$. Hence if the Lie algebra of infinitesimal automorphisms is abelian

$$i(\xi \wedge \eta)\omega = \text{const.}$$

for any infinitesimal automorphisms Z and Y.

REMARK. We have assumed throughout that the manifolds under consideration are of class C^∞. It is known that on an almost complex manifold M^{2n} of class C^{2n+1} with completely integrable almost complex structure defined by a tensorfield of class C^{2n} it is possible to introduce complex analytic coordinates [8]. Under these conditions every pseudo-Kaehlerian manifold is differentiably homeomorphic with a Kaehlerian manifold. Our results are valid if the C^∞ condition is replaced by only C^3. Then, only in the case $n = 1$ is it known that a pseudo-Kaehlerian manifold is Kaehlerian.

8. In this section we defie two classes of spaces each of which includes the almost Kaehlerian manifolds and for which the only infinitesimal conformal transformations (in the compact case) are infinitesimal isometries.

Consider a Riemannian manifold M of dimension m on which there is defined an harmonic p-form

$$\alpha = a_{i_1\dots i_p} dx^{i_1} \wedge \dots\dots \wedge dx^{i_p}$$

with the property

$$(P):\ a_i^{\ i_2\dots i_p} a_{j i_2\dots i_p} = g_{ij}.$$

For $p = 2$, Lichnerowicz has shown that the metric of any Riemannian space with an almost complex structure can be modified so that (P) is satisfied. Let $H(m, p)$ denote the class of Riemannian manifolds carrying an harmonic p-form satisfying the conditon (P). Clearly, an almost Kaehler manifold is an $H(m, 2)$, $m = 2n$. Conversely, an $H(2n, 2)$ is almost Kaehlerian. Moreover, a compact semi-simple Lie group carries an 3-form whose coefficients satisfy the property (P). For this class of spaces, the proof of Theorem 1 yields.

THEOREM 7. *An infinitesimal conformal transformation X of a compact and orientable $H(m, p)$, $m > 2$ is an infinitesimal isometry.*

Indeed, applying the operator δ to both sides of the relation (6) we obtain

$$\begin{aligned}\delta\theta(X)\alpha &= \left(1 - \frac{2p}{m}\right)\delta(\delta\xi\cdot\alpha)\\ &= -\left(1 - \frac{2p}{m}\right)D_i(\delta\xi\cdot a^i_{i_2\ldots i_p})dx^{i_2}\wedge\ldots\ldots\wedge dx^{i_p}\\ &= -\left(1 - \frac{2p}{m}\right)i(d\delta\xi)\alpha.\end{aligned}$$

Taking the global scalar product with $i(X)\alpha$ we obtain since α is closed

$$(\delta\theta(X)\alpha,\ i(X)\alpha) = \|\theta(X)\alpha\|^2$$

and, by applying the property (P)

$$(i(d\delta\xi)\alpha,\ i(X)\alpha) = \|\delta\xi\|^2.$$

Thus, $\|\theta(X)\alpha\|^2 = -\left(1 - \frac{2p}{m}\right)\|\delta\xi\|^2.$

Hence, for $m > 2p$, $\|\delta\xi\| = 0$, that is $\delta\xi$ vanishes.

The case $m < 2p$ follows by considering the form $*\alpha$. For then, from (6)

$$\theta(X)*\alpha + \bar{\theta}(X)*\alpha = \left(\frac{2p}{m} - 1\right)\delta\xi*\alpha.$$

Moreover, it can be shown that $*\alpha$ s atisfies (P).

The case $m = 2p$ is dealt with in the following way: From (6), if the degree of α is $\frac{m}{2}$,

$$\theta(X)\alpha + \bar{\theta}(X)\alpha = 0$$

from which we conclude (since M is compact) that $\theta(X)\alpha$ vanishes. Hence, since $\theta(X)$ is a derivation $\theta(X)\ (\alpha\wedge*\alpha) = 0$, and so, since $\alpha\wedge*\alpha = \langle\alpha,\alpha\rangle *1 = m*1$ (where $\langle\ ,\ \rangle$ denotes the local scalar product),

$$\theta(X)(\alpha\wedge*\alpha) = m\theta(X)*1 = 0.$$

This completes the proof.

COROLLARY. *An infinitesimal conformal transformation of an m-dimensional compact semi-simple Lie group with the canonical (left-invariant) metric is an infinitesimal isometry.*

The above proof for the case $m = 2p$ yields

THEOREM 8. *Let M be a compact and orientable Riemannian manifold of dimension $2n$ carrying an harmonic n-form α whose square length $\langle\alpha,\alpha\rangle$*

is a non-zero constant. Then, an infinitesimal conformal tranformation of M is an infinitesimal isometry.

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