RIEMANN-CESÀRO METHODS OF SUMMABILITY V

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1. Introduction. Let s_n^α be the Cesàro sum of a series $\sum_{n=0}^{\infty} a_n$ with $a_0 = 0$, that is, A_n^α being Andersen's notation,

$$s_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^\alpha \, a_\nu,$$

and let σ_n^α be the Cesàro mean of the series $\sum_{n=0}^{\infty} a_n$, that is, $\sigma_n^\alpha = s_n^\alpha / A_n^\alpha$. The series $\sum_{n=0}^{\infty} a_n$ is said to be evaluable (C, α), $\alpha > -1$, to s, if $\sigma_n^\alpha \to s$ as $n \to \infty$. Let $k > 0$ and $\lambda_n = \log(n+1)$. If, when $\omega \to \infty$,

$$\sum_{\lambda_n < \omega} \left(1 - \frac{\lambda_n}{\omega}\right)^k a_n \to s,$$

then the series $\sum_{n=0}^{\infty} a_n$ is said to be evaluable $(\log n, k)$ to s. It is well-known that a series evaluable (C, k) is also evaluable $(\log n, k)$ to the same sum. In the following, let p be a positive integer and let α be a real number such that $\alpha \geqq -1$. The series $\sum_{n=0}^{\infty} a_n$ is said to be evaluable by Riemann-Cesàro method of order p and index α, or briefly, to be evaluable (R, p, α) to s, if the series

$$t^{\alpha+1} \sum_{n=1}^{\infty} s_n^\alpha \left(\frac{\sin nt}{nt}\right)^p$$

converges in some interval $0 < t < t_0$ and its sum tends to $C_{p,\alpha}$ s as $t \to 0+$, where

$$C_{p,\alpha} = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty u^{\alpha-p} (\sin u)^p \, du, & -1<\alpha<p-1 \text{ or } \alpha=0, p=1, \\ 1 \quad , & \alpha = -1. \end{cases}$$

Under this definition, the summabilities $(R, p, -1)$ and $(R, p, 0)$ are the well-known summabilities (R, p) and (R_p), respectively. In our earlier papers [2, 3, 4], we have investigated some properties on this summability. In particular, sufficient conditions for the summability (R, p, α) has been considered. For example, that the series is evaluable $(C, p-1-\delta)$, $\delta>0$, is sufficient for the summability (R, p, α). See [2; Corollary 1]. The purpose of this paper is to establish necessary conditions for the summability (R, p, α). In this direction, the following theorems are known.

THEOREM A (B. Kuttner [6]) *If a series is evaluable* $(R, 2)$, *it is also evaluable* $(C, 2+\delta)$, $\delta>0$, *to the same sum. Further the series is evaluable* $(\log n, 2)$ *to the same sum.*

THEOREM B (A. Zygmund [8]) *If a series is evaluable* $(R, 1)$, *it is also evaluable* $(C, 1+\delta)$, $\delta>0$, *to the same sum. Further the series is evaluable* $(\log n, 1)$ *to the same sum.*

For the summability (R, p, α), we shall prove the following theorems.

THEOREM 1. *A series evaluable* $(R, 2, \alpha)$, $-1 \leqq \alpha < 1$, *is evaluable* $(C, 2+\delta)$, $\delta>0$, *to the same sum.*

THEOREM 2. *A series evaluable* $(R, 2, \alpha)$, $-1 \leqq \alpha < 1$, *is evaluable* $(\log n, 2)$ *to the same sum.*

THEOREM 3. *A series evaluable* $(R, 1, \alpha)$, $-1 \leqq \alpha \leqq 0$, *is evaluable* $(C, 1+\delta)$, $\delta>0$, *to the same sum.*

THEOREM 4. *A series evaluable* $(R, 1, \alpha)$, $-1 \leqq \alpha \leqq 0$, *is evaluable* $(\log n, 1)$ *to the same sum.*

THEOREM 5. *A series evaluable* $(R, 1, \alpha)$, $-1 \leqq \alpha \leqq 0$, *is evaluable* $(R, 2, \alpha)$ *to the same sum.*

In theorems 1-4, if we put $\alpha=0$, we obtain (R_p) analogues of Theorems A and B. When $\alpha=-1$, Theorem 5 was proved by B. Kuttner [7]. See also G. H. Hardy [1; p. 365]. Theorems 1-5 are proved in the sections 3-7 of this paper. I take this opportunity of expressing my heartfelt thanks to Professors G. Sunouchi and T. Tsuchikura for their kind encouragements and valuable suggestions during the preparation of this paper.

2. Preliminary Lemmas.

LEMMA 1. *If a series* $\sum_{n=1}^{\infty} b_n \sin^2 nt$ *converges for all* t *in an interval, then the series* $\sum_{n=1}^{\infty} b_n$ *is convergent.*

This Lemma is due to B. Kuttner [6]. See also G. H. Hardy [1; p. 366].

LEMMA 2. (B. Kuttner [6]) *If a series* $\sum_{n=1}^{\infty} b_n \cos nt$ *converges to zero uniformly for* t *in an interval including origin, then the series* $\sum_{n=1}^{\infty} n^2 b_n$ *is evaluable* $(C, 2)$ *to zero.*

LEMMA 3. *Let* $t^2\gamma_2(t) = 1 - \cos t$ *and* $\gamma_2'(t) = d\gamma_2(t)/dt$. *Then, for fixed* $\eta > 0$ *and large* ω *and* n,

$$\int_0^{\eta} \left\{\omega^2\gamma_2'(\omega t) + \frac{2}{\omega\eta^3} - \frac{\sin \omega t}{\eta^2} - \frac{2\cos\omega t}{\omega\eta^3}\right\} \sin nt\, dt$$
$$= -n\, G(n, \omega) + R(n,\omega),$$

where

$$G(n, \omega) = \begin{cases} \frac{\pi}{2}\left(1 - \frac{n}{\omega}\right) & \text{for } n \leqq \omega \\ 0 & \text{for } n > \omega, \end{cases} \tag{2.1}$$

$$R(n, \omega) = \frac{2}{\omega n \eta^3} - \frac{2n}{\omega\eta^3(n^2 - \omega^2)} + O\left(\frac{1}{\omega n^2}\right) + O\left(\frac{1}{(\omega - n)^2}\right) \tag{2.2}$$

and

$$R(n, \omega) = O(1). \tag{2.3}$$

PROOF. It is known (E. W. Hobson [5; p. 567]) that

$$\int_0^{\infty} \gamma_2(t) \cos \frac{n}{\omega} t\, dt = G(n, \omega).$$

Hence

$$\frac{1}{2}\omega \int_{-\infty}^{\infty} \gamma_2\{\omega(t-x)\} \sin nt\, dt = \frac{1}{2}\omega \int_0^{\infty} \gamma_2(\omega t)\{\sin n(x+t) + \sin n(x-t)\}\, dt$$
$$= G(n, \omega) \sin nx.$$

Differentiating this equation with respect to x, and then putting $x=0$, we obtain

$$-\omega^2 \int_0^{\infty} \gamma_2'(\omega t) \sin nt\, dt = n\, G(n, \omega). \tag{2.4}$$

On the one hand

$$\omega^2 \int_{\eta}^{\infty} \gamma_2'(\omega t) \sin nt\, dt = -\frac{2}{\omega}\int_{\eta}^{\infty} \frac{\sin nt}{t^3} dt + \int_{\eta}^{\infty} \frac{\sin \omega t \sin nt}{t^2}\, dt$$

$$+\frac{2}{\omega}\int_{\eta}^{\infty}\frac{\cos\omega t\sin nt}{t^3}\,dt$$

$$(2.5)\qquad =-\frac{2\cos n\eta}{\omega n\eta^3}+\frac{\sin(\omega+n)\eta}{2(\omega+n)\eta^2}-\frac{\sin(\omega-n)\eta}{2(\omega-n)\eta^2}+\frac{\cos(\omega+n)\eta}{\omega(\omega+n)\eta^3}+\frac{\cos(n-\omega)\eta}{\omega(n-\omega)\eta^3}+O\Big(\frac{1}{\omega n^2}\Big)+O\Big(\frac{1}{(\omega-n)^2}\Big),$$

and evidently

$$(2.6)\qquad \omega^2\int_{\eta}^{\infty}\gamma_2'(\omega t)\sin nt\,dt=O\Big(\int_{\eta}^{\infty}t^{-2}\,dt\Big)=O(1)$$

On the other hand

$$\int_0^{\eta}\Big\{\frac{2}{\omega\eta^3}-\frac{\sin\omega t}{\eta^2}-\frac{2\cos\omega t}{\omega\eta^3}\Big\}\sin nt\,dt$$

$$(2.7)\qquad =-\frac{2\cos n\eta}{\omega n\eta^3}+\frac{\sin(\omega+n)\eta}{2(\omega+n)\eta^2}-\frac{\sin(\omega-n)\eta}{2(\omega-n)\eta^2}+\frac{\cos(\omega+n)\eta}{\omega(\omega+n)\eta^3}+\frac{\cos(n-\omega)\eta}{\omega(n-\omega)\eta^3}+\frac{2}{\omega n\eta^3}-\frac{2n}{\omega\eta^3(n^2-\omega^2)}$$

and evidently

$$(2.8)\qquad \int_0^{\eta}\Big\{\frac{2}{\omega\eta^3}-\frac{\sin\omega t}{\eta^2}-\frac{2\cos\omega t}{\omega\eta^3}\Big\}\sin nt\,dt=O(1).$$

From these equations Lemma follows, the result (2. 2) following from the equations (2. 4), (2. 5) and (2. 7), while the result (2. 3) following from the equations (2. 4), (2. 6) and (2. 8).

LEMMA 4. *If a series $\sum_{n=1}^{\infty} b_n \sin nt$ converges to zero uniformly for t in an interval $(-\eta,\eta)$, then the series $\sum_{n=1}^{\infty} nb_n$ is evaluable $(C, 1)$ to zero.*

PROOF. The method of proof is similar to that of Kuttner's Lemma 2. We multiply the equation

$$\sum_{n=1}^{\infty} b_n\sin nt=0$$

by

$$\omega^2\gamma_2'(\omega t)+\frac{2}{\omega\eta^3}-\frac{\sin\omega t}{\eta^2}-\frac{2\cos\omega t}{\omega\eta^3},$$

and integrate with respect to t from zero to η. Since the series is uniformly convergent, we may integrate term by term. Then, using Lemma 3, we obtain

$$\frac{\pi}{2}\sum_{n<\omega} nb_n\left(1-\frac{n}{\omega}\right)=\sum_{n=1}^{\infty} b_n\, R(n,\omega)$$
$$=\left(\sum_{|n-\omega|\leqq 1}+\sum_{|n-\omega|>1}\right) b_n\, R(n,\omega)$$
$$=J_1+J_2,$$

say. Since it is known that if $b_n \sin nt$ is convergent to zero for t in a set of positive measure then $b_n=o(1)$, it follows, using (2. 3),

$$J_1=\sum_{|n-\omega|\leqq 1} b_n\, R(n,\omega)=o(1).$$

On the other hand, using (2. 2),

$$J_2=\frac{2}{\omega\eta^3}\sum_{|n-\omega|>1}\frac{b_n}{n}-\frac{2}{\omega\eta^3}\sum_{|n-\omega|>1}\frac{nb_n}{n^2-\omega^2}+O\left(\frac{1}{\omega}\sum_{n=1}^{\infty}\frac{|b_n|}{n^2}\right)$$
$$+O\left(\sum_{|n-\omega|>1}\frac{|b_n|}{(\omega-n)^2}\right)=J_{21}-J_{22}+O(J_{23})+O(J_{24})$$

say, where, using $b_n=o(1)$,

$$J_{23}=\frac{1}{\omega}\sum_{n=1}^{\infty}\frac{|b_n|}{n^2}=O\left(\frac{1}{\omega}\sum_{n=1}^{\infty}\frac{1}{n^2}\right)=o(1).$$

Let us put $\rho=[\omega]-1$, $[\omega]$ denoting the integral part of ω,

$$J_{24}=\sum_{|n-\omega|>1}\frac{|b_n|}{(\omega-n)^2}\leqq\sum_{n=1}^{\rho}\frac{|b_n|}{(\omega-n)^2}+\sum_{n=\rho+3}^{\infty}\frac{|b_n|}{(\omega-n)^2}$$
$$=\sum_{n=1}^{[\rho/2]}\frac{|b_n|}{(\omega-n)^2}+\sum_{n=[\rho/2]+1}^{\rho}\frac{|b_n|}{(\omega-n)^2}+o\left(\sum_{n=\rho+3}^{\infty}\frac{1}{(\omega-n)^2}\right)$$
$$=O\left(\frac{1}{(\omega-\rho/2)^2}\cdot\frac{\rho}{2}\right)+o\left(\sum_{n=1}^{\rho}\frac{1}{(\omega-n)^2}\right)+o(1)$$
$$=O(1/\omega)+o(1)=o(1).$$

Now, since the series $\sum_{n=1}^{\infty} b_n \sin nt$ is uniformly convergent in $(0,\eta)$, we may integrate this series term by term from zero to t, t being included in $(0,\eta)$. Thus the series

$$\sum_{n=1}^{\infty} \frac{b_n}{n}(1-\cos nt) = 2\sum_{n=1}^{\infty} \frac{b_n}{n} \sin^2 \frac{nt}{2}$$

is convergent for $t, 0 < t \leqq \eta$. Hence, by Lemma 1, the series $\sum_{n=1}^{\infty} \frac{b_n}{n}$ is convergent. Therefore

$$J_{21} = \frac{2}{\omega\eta^3} \sum_{|n-\omega|>1} \frac{b_n}{n} = \frac{2}{\omega\eta^3} \sum_{n=1}^{\infty} \frac{b_n}{n} - \frac{2}{\omega\eta^3} \sum_{|n-\omega|\leqq 1} \frac{b_n}{n}$$
$$= o(1).$$

Finally, putting $r_n = \sum_{\nu=n}^{\infty} \frac{b_\nu}{\nu}$ and using Abel transform,

$$J_{22} = \frac{2}{\omega\eta^3} \sum_{|n-\omega|>1} \frac{nb_n}{n^2-\omega^2}$$
$$\leqq \frac{2}{\omega\eta^3} \sum_{n=1}^{\rho} \frac{nb_n}{n^2-\omega^2} + \frac{1}{\omega\eta^3}\left(\sum_{n=\rho+3}^{\infty} \frac{b_n}{n-\omega} + \sum_{n=\rho+3}^{\infty} \frac{b_n}{n+\omega} \right)$$
$$= O\left(\frac{1}{\omega} \sum_{n=1}^{\rho} \frac{1}{n} \right) + \frac{1}{\omega\eta^3} \left\{ \frac{(\rho+3)r_{\rho+3}}{\rho+3-\omega} - \sum_{n=\rho+3}^{\infty} r_{n+1}\left(\frac{n}{n-\omega} - \frac{n+1}{n+1-\omega} \right) \right.$$
$$\left. + \frac{(\rho+3)r_{\rho+3}}{\rho+3+\omega} - \sum_{n=\rho+3}^{\infty} r_{n+1}\left(\frac{n}{n+\omega} - \frac{n+1}{n+1+\omega} \right) \right\}$$
$$= o(1) + o\left(\sum_{n=\rho+3}^{\infty} \frac{1}{(n-\omega)^2} \right) + o\left(\sum_{n=\rho+3}^{\infty} \frac{1}{(n+\omega)^2} \right)$$
$$= o(1).$$

Therefore, summing up the above results, we have

$$\sum_{n<\omega} n b_n \left(1 - \frac{n}{\omega}\right) = o(1) \qquad \text{as } \omega \to \infty,$$

which is equivalent to that the series $\sum_{n=0}^{\infty} nb_n$ is evaluable $(C, 1)$ to zero.

LEMMA 5 *Let* $a_n = o(1)$, $b_n = o(1)$, $\alpha_n = O(n^{-3})$ *and* $\beta_n = O(n^{-3})$. *Let the formal product of the two series*

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

and

$$\frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos nt + \beta_n \sin nt)$$

where the last series converge to, say, $S(t)$, *be*

$$\frac{1}{2} A_0 + \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt),$$

where, putting $a_{-n} = a_n$, $b_{-n} = b_n$, $\alpha_{-n} = \alpha_n$ *and* $\beta_{-n} = \beta_n$,

$$A_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} (a_m \alpha_{n-m} - b_m \beta_{n-m}) \text{ and } B_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} (a_m \beta_{n-m} + b_m \alpha_{n-m}).$$

Then

$$\frac{1}{2} A_0 + \sum_{n=1}^{N} (A_n \cos nt + B_n \sin nt) - S(t)\Big(\frac{1}{2} a_0 + \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt)\Big)$$

tends to zero, as $N \to \infty$, *uniformly in* t *of the interval* $[0, 2\pi]$.

This Lemma is due to A. Zygmund [9; p. 60, Theorem III]. See also G. H. Hardy [1; p. 366].

LEMMA 6. *If the series* $\sum_{n=0}^{\infty} a_n$ *is evaluable* (C, β) *to* s *and if*

$$\sigma_n^\gamma = s_n^\gamma / A_n^\gamma = o(\log^\gamma n), \qquad 0 < \gamma < \beta,$$

then the series $\sum_{n=0}^{\infty} a_n$ *is evaluable* $(\log n, \gamma)$ *to* s.

This is due to A. Zygmund [10].

3. Proof of Theorem 1.

3. 1. We are given that the series $\sum_{n=0}^{\infty} a_n$ is evaluable $(R, 2, \alpha)$ to s, and we may suppose, without loss of generality, that $s = 0$. Then

$$F(t) = \sum_{n=1}^{\infty} \frac{s_n^\alpha}{n^2} \sin^2 \frac{1}{2} nt$$

converges for t in some interval $(0, \xi)$, and $F(t) = o(t^{1-\alpha})$. Since the series $\sum_{n=1}^{\infty} \frac{s_n^\alpha}{n^2}$ is convergent, by Lemma 1, we may write our series as

$$\frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nt, \tag{3.1}$$

where

$$(3.2) \qquad \alpha_n = -\frac{1}{2} n^{-2} s_n^{\alpha} \ (n > 0), \ \alpha_0 = -2\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} n^{-2} s_n^{\alpha}.$$

This series (3.1) converges to $F(t)$ for t in the interval $(0, \xi)$. We shall first prove that Theorem is true generally provided that Theorem is true in the two particular cases;

(3.3) the series (3.1) is a Fourier series

and

(3.4) the series (3.1) converges uniformly to zero in an interval of t including origin.

We suppose that the series (3.1) is convergent and bounded for $|t| \leqq \xi$, choose a positive η less than $\frac{1}{2}\xi$, and take an even periodic function $\lambda(t)$ which equals to 1 for $|t| \leqq \eta$, and equals to 0 for $2\eta \leqq |t| \leqq \pi$, and whose first three derivatives exist and continuous. If

$$(3.5) \qquad \lambda(t) \sim \frac{1}{2}\beta_0 + \sum_{n=1}^{\infty} \beta_n \cos nt,$$

then $\beta_n = O(n^{-3})$. Since $\alpha_n = o(1)$, it follows from Lemma 5 that if

$$(3.6) \qquad \frac{1}{2}\gamma_0 + \sum_{n=1}^{\infty} \gamma_n \cos nt$$

is the formal product of the series (3.1) and (3.5), then

$$(3.7) \qquad \frac{1}{2}\gamma_0 + \sum_{n=1}^{N} \gamma_n \cos nt - \lambda(t)\left\{\frac{1}{2}\alpha_0 + \sum_{n=1}^{N} \alpha_n \cos nt\right\} \to 0$$

as $N \to \infty$, uniformly in t. Since the series (3.1) converges to $F(t)$ for $|t| \leqq 2\eta < \xi$, and $\lambda(t) = 0$ for $2\eta \leqq |t| \leqq \pi$, it follows from (3.7) that the series (3.6) converges for all t, and to a sum $F^*(t)$ defined by

$$F^*(t) = \begin{cases} F(t) & |t| \leqq \eta \\ \lambda(t)\ F(t) & \eta \leqq |t| \leqq 2\eta \\ 0 & 2\eta \leqq |t| \leqq \pi. \end{cases}$$

Since $F^*(t)$ is bounded, the series (3.6) is the Fourier series of $F^*(t)$, by a known theorem. We shall now define a series $\sum_{n=0}^{\infty} c_n$ by

$$\gamma_n = -\frac{1}{2} n^{-2} \tau_n^{\alpha} \quad (n > 0), \quad \gamma_0 = -2\sum_{n=1}^{\infty} \gamma_n = \sum_{n=1}^{\infty} n^{-2}\tau_n^{\alpha},$$

where τ_n^{α} is the (C, α) sum of the series $\sum_{n=0}^{\infty} c_n$ with $c_0 = 0$. Then the series (3.6) is related to the series $\sum_{n=0}^{\infty} c_n$ as the series (3. 1) to the series $\sum_{n=0}^{\infty} a_n$. Thus, by $F^*(t) = F(t)$ for small t, the series $\sum_{n=0}^{\infty} c_n$ is evaluable $(R, 2, \alpha)$ to zero. From this, and our assumption of Theorem in case (3. 3), it follows that the series $\sum_{n=0}^{\infty} c_n$ is evaluable (C, β), $\beta > 2$, to zero.

On the other hand, since $\lambda(t) = 1$ for $|t| \leqq \eta$, it follows from (3. 7) that

$$\frac{1}{2}(\gamma_0 - \alpha_0) + \sum_{n=1}^{\infty}(\gamma_n - \alpha_n)\cos nt = \sum_{n=1}^{\infty}(\tau_n^{\alpha} - s_n^{\alpha})\frac{\sin^2 nt/2}{n^2}$$

converges uniformly to zero for $|t| \leqq \eta$ and that the series $\sum_{n=0}^{\infty}(c_n - a_n)$ is evaluable $(R, 2, \alpha)$ to zero. From this, and our assumption of Theorem in case (3. 4), it follows that the series $\sum_{n=0}^{\infty}(c_n - a_n)$ is evaluable (C, β), $\beta > 2$, to zero. Finally, since $a_n = c_n - (c_n - a_n)$, the series $\sum_{n=0}^{\infty} a_n$ is evaluable (C, β), $\beta > 2$, to zero. Thus Theorem is proved provided that Theorem is true in the two particular cases (3. 3) and (3. 4).

3. 2. Proof for the Case (3. 3). For the proof, we may suppose that $2 < \beta < 3$ and choose γ such that $\beta = \alpha + \gamma + 1$. Then $0 < \gamma < 3$. Let

$$\varphi(t) \sim \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nt. \tag{3. 8}$$

Since this series converges to $F(t)$ for small t, we have $\varphi(t) = F(t)$ for almost all such t, and we may suppose that this is true for all such t. Hence

$$\varphi(t) = o(t^{1-\alpha}). \tag{3. 9}$$

Now, S_n denoting the (C, γ) sum of the twice formally differentiated series of the series (3. 8) at $t = 0$, we have

$$S_n^{\gamma} = \frac{2}{\pi} A_n^{\gamma} \int_0^{\pi} \varphi(t) \frac{d^2}{dt^2} K_n^{\gamma}(t)\, dt, \tag{3. 10}$$

where $K_n^{\gamma}(t)$ is the (C, γ) mean of the series

$$\frac{1}{2} + \cos t + \cos 2t + \cos 3t + \cdots\cdots.$$

Since, denoting by C constants independent of n and t,

$$\left|\frac{d^m}{dt^m} K_n^\gamma(t)\right| \leqq Cn^{m+1} \qquad (0 \leqq t \leqq \pi) \tag{3.11}$$

and

$$\left|\frac{d^m}{dt^m} K_n^\gamma(t)\right| \leqq Cn^{m-\gamma}t^{-\gamma-1} \quad \left(\frac{1}{n} \leqq t \leqq \pi\right), \tag{3.12}$$

for $0 \leqq \gamma \leqq m+1$, $m, n = 1, 2, 3, \ldots\ldots$ (See A. Zygmund [11; p. 60]) Hence, by (3.9),

$$\begin{aligned} S_n^\gamma &= \frac{2}{\pi} A_n^\gamma \left(\int_0^{1/n} + \int_{1/n}^\pi\right) \varphi(t) \frac{d^2}{dt^2} K_n^\gamma(t)\, dt \\ &= o\left(n^\gamma \int_0^{1/n} n^3\, t^{1-\alpha}\, dt\right) + o\left(n^\gamma \int_{1/n}^\pi n^{2-\gamma}\, t^{-\gamma-\alpha}\, dt\right) \\ &= o(n^{\gamma+3} \cdot n^{\alpha-2}) + o(n^2 \cdot n^{\gamma+\alpha-1}) \\ &= o(n^{\alpha+\gamma+1}). \end{aligned}$$

Since, by (3.2), remembering that $a_0 = 0$,

$$S_n^\gamma = -\sum_{\nu=0}^n A_{n-\nu}^\gamma\, \nu^2\, \alpha_\nu = \frac{1}{2}\sum_{\nu=0}^n A_{n-\nu}^\gamma\, s_\nu^\alpha = \frac{1}{2}\, s^{\alpha+\gamma+1}, \tag{3.13}$$

we get $s_n^{\alpha+\gamma+1} = o(n^{\alpha+\gamma+1})$ and then we see, by $\beta = \alpha+\gamma+1$, that the series $\sum_{n=0}^\infty a_n$ is evaluable (C, β) to zero.

3.3. Proof for the Case (3.4). This is shown as a corollary of Lemma 2. If

$$\frac{1}{2}\alpha_0 + \sum_{n=1}^\infty \alpha_n \cos nt = 0$$

uniformly in an interval including origin, then, by Lemma 2, we have

$$\sum_{n=1}^\infty n^2\, \alpha_n = 0 \ (C, 2),$$

that is, by (3.2),

$$\sum_{\nu=1}^n A_{n-\nu}^2 \nu^2 \alpha_\nu = -\frac{1}{2}\sum_{\nu=0}^n A_{n-\nu}^2 s_\nu^\alpha = -\frac{1}{2}\, s_n^{\alpha+3} = o(n^2).$$

Thus we get $s_n^{\alpha+3} = o(n^2)$. Since $\alpha + 3 \geqq 2$, we can easily see that $s_n^2 = o(n^2)$, and that the series $\sum_{a=0}^\infty a_n$ is evaluable $(C, 2)$ to zero. Therefore, evidently, the series $\sum_{n=0}^{\infty} a_n$ is evaluable (C, β), $\beta > 2$, to zero.

Thus, Theorem 1 is completely proved.

4. Proof of Theorem 2. From the argument in the paragraph 3. 1, it is sufficient to prove Theorem in the two particular cases (3. 3) and (3. 4). Since the summability $(C, 2)$ implies the summability $(\log n, 2)$, it remains to prove Theorem for the case (3. 3). Since, by Theorem 1, the series $\sum_{n=0}^{\infty} a_n$ is evaluable (C, β), $\beta > 2$, it is sufficient, by Lemma 6, to prove that

$$\sigma_n^2 = s_n^2 / A_n^2 = o(\log^2 n).$$

For this, by $1 - \alpha > 0$, using (3. 9), (3. 10), (3. 11) and (3. 12), we have

$$\begin{aligned} S_n^{1-\alpha} &= \frac{2}{\pi} A_n^{1-\alpha} \left(\int_0^{1/n} + \int_{1/n}^{\pi} \right) \varphi(t) \frac{d^2}{dt^2} K_n^{1-\alpha}(t)\, dt \\ &= o\left(n^{1-\alpha} \int_0^{1/n} n^3 t^{1-\alpha}\, dt \right) + o\left(n^{1-\alpha} \int_{1/n}^{\pi} n^{1+\alpha} t^{-1}\, dt \right) \\ &= o(n^2 \log n). \end{aligned}$$

Hence, by (3. 13) for $\gamma = 1 - \alpha$,

$$s_n^2 = o(n^2 \log n),$$

which is the required result. Thus, Theorem is proved.

5. Proof of Theorem 3. The first stage of the proof is like that of the proof of Theorem 1. From our assumption,

$$F(t) = \sum_{n=1}^{\infty} \frac{s_n^{\alpha}}{n} \sin nt \tag{5. 1}$$

converges for t in some interval $(0, \xi)$, and $F(t) = o(t^{-\alpha})$. Now, as in the paragraph 3. 1, we can easily see that Theorem is true generally provided that Theorem is true in the two particular cases;

(5. 2) the series in (5. 1) is a Fourier series

and

(5. 3) the series in (5. 1) converges uniformly to zero in an interval of t including origin.

For the case (5. 3), by Lemma 4, the series $\sum_{n=1}^{\infty} s_n^{\alpha}$ is evaluable $(C, 1)$ to zero, and then

$$\sum_{\nu=0}^{n} A_{n-\nu}^{1} s_{\nu}^{\alpha} = s_{n}^{\alpha+2} = o(n).$$

Since $\alpha + 2 \geqq 1$, we see that $s_n^1 = o(n)$. This shows that the series $\sum_{n=0}^{\infty} a_n$ is evaluable $(C, 1)$ to zero. Hence, evidently, the series $\sum_{n=0}^{\infty} a_n$ is evaluable $(C\ \beta)$, $\beta > 1$, to zero. We shall now prove Theorem for the case (5. 2). We may suppose that $1 < \beta < 2$ and choose γ such that $\beta = \alpha + \gamma + 1$. Let

$$\varphi(t) \sim \sum_{n=1}^{\infty} \frac{s_n^{\alpha}}{n} \sin nt. \tag{5. 4}$$

Then, by our assumption, we may suppose that $\varphi(t) = o(t^{-\alpha})$. Now, S_n^{γ} denoting the (C, γ) sum of the formally differentiated series of the series (5. 4) at $t = 0$, we have

$$S_n^{\gamma} = -\frac{2}{\pi} A_n^{\gamma} \int_0^{\pi} \varphi(t) \frac{d}{dt} K_n^{\gamma}(t)\ dt. \tag{5. 5}$$

Therefore, using (3. 11) and (3. 12),

$$\begin{aligned} S_n^{\gamma} &= -\frac{2}{\pi} A_n^{\gamma} \left(\int_0^{1/n} + \int_{1/n}^{\pi} \right) \varphi(t) \frac{d}{dt} K_n^{\gamma}(t)\ dt \\ &= o\left(n^{\gamma} \int_0^{1/n} n^2 t^{-\alpha}\ dt \right) + o\left(n^{\gamma} \int_{1/n}^{\pi} t^{-\alpha} \cdot n^{1-\gamma} t^{-1-\gamma}\ dt \right) \\ &= o(n^{\alpha+\gamma+1}) + o(n \cdot n^{\alpha+\gamma}) \\ &= o(n^{\alpha+\gamma+1}) = o(n^{\beta}), \end{aligned}$$

by $\beta = \alpha + \gamma + 1$. Since, remembering that $a_0 = 0$,

$$S_n^{\gamma} = \sum_{\nu=0}^{n} A_{n-\nu}^{\gamma}\ s_{\nu}^{\alpha} = s_n^{\alpha+\gamma+1}, \tag{5. 6}$$

we have $s_n^{\beta} = o(n^{\beta})$, which is the requiredresult.

6. Proof of Theorem 4. From the argument in the former section, it is sufficient to prove Theorem in the two particular cases (5. 2) and (5. 3). For the case (5. 3), since the summability $(C, 1)$ implies the summability $(\log n, 1)$, it remains to prove Theorem for the case (5. 2). For this, by Lemma 6, it is sufficient to prove that

$$\sigma_n^1 = s_n^1 / A_n^1 = o(\log\ n).$$

Since, by (5. 5),

$$\begin{aligned} S_n^{-\alpha} &= -\frac{2}{\pi} A_n^{-\alpha}\int_0^{\pi} \varphi(t)\frac{d}{dt} K_n^{-\alpha}(t)\, dt \\ &= -\frac{2}{\pi} A_n^{-\alpha}\left(\int_0^{1/n} + \int_{1/n}^{\pi}\right)\varphi(t)\frac{d}{dt} K_n^{-\alpha}(t)\, dt \\ &= o\left(n^{-\alpha}\int_0^{1/n} t^{-\alpha}\cdot n^2\, dt\right) + o\left(n^{-\alpha}\int_{1/n}^{\pi} t^{-\alpha}\cdot n^{\alpha+1} t^{\alpha-1}\, dt\right) \\ &= o(n^{2-\alpha} n^{\alpha-1}) + o(n\log n) \\ &= o(n\log n), \end{aligned}$$

we have, by (5.6) for $\gamma = -\alpha$, $s_n^1 = o(n\log n)$, which is equivalent to that $s_n^1/A_n^1 = o(\log n)$. Thus Theorem is proved.

7. Proof of Theorem 5. The method of the proof is also similar to that of Theorem 3. Theorem for the two particular cases (5.2) and (5.3) is obvious. For, if

$$\sum_{n=1}^{\infty}\frac{s_n^{\alpha}}{n}\sin nt = F(t) = o(t^{-\alpha}), \tag{7.1}$$

and if the series is a Fourier series, then

$$\sum_{n=1}^{\infty}\frac{s_n^{\alpha}}{n^2}(1-\cos 2nt) = 2\sum_{n=1}^{\infty}\frac{s_n^{\alpha}}{n^2}\sin^2 nt = \int_0^{2t} F(u)\, du = o(t^{1-\alpha}), \tag{7.2}$$

because a Fourier series can be integrated term by term; and if the series in (7.1) converges uniformly to zero for small t, then the series in (7.2) converges to zero for small t. Thus the series $\sum_{n=0}^{\infty} a_n$ is evaluable $(R, 2, \alpha)$ to zero.

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