# THE STRUCTURE OF A RIEMANNIAN MANIFOLD ADMITTING A PARALLEL FIELD OF TANGENT VECTOR SUBSPACES

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**1. Introduction.** Let $M$ be an $n$-dimensional connected complete Riemannian manifold of class $C^2$ admitting a parallel field of $r$-dimensional tangent vector subspaces. Then, $M$ admits the parallel field of $s$-dimensional tangent vector subspaces, where $s = n - r$, orthogonal to the given field. $M$ is also regarded as a Riemannian manifold whose homogeneous holonomy group fixes an $r$- (or $s$-) dimensional tangent vector subspace. The purpose of this note is to treat of the global structure of $M$. In a case where $r = n - 1$, i.e. $s = 1$, the author [3] already attempted to clarify geometrically the global structure. Here let us discuss the structure in the case where $1 \leqq r$, $s \leqq n - 1$, from the view-point of fibre bundle. For the main results, see Theorems 1—7. Especially Theorem 3 shows a general structure of $M$ and from the other theorems we may know structures in respective cases. Notice that these theorems all hold good even if $R$ and $S$ in these theorems are exchanged for each other (see Remark 1).

From now on the word "$k$-dimensional" is abbreviated as "$k$-", say like $k$-space (but, such a prefix does not necessarily mean dimension). Let us suppose that indices run as follows: $a, b = 1, 2,\dots, r$; $i, j = r + 1, r + 2,\dots, n$; $\alpha = 1, 2,\dots, n$. The following conventions in a Riemannian manifold $X$ are also applied to all of Riemannian manifolds: The *parallelism* in $X$ means the one of Levi-Civita. A *neighborhood* in $X$ is always an open set homeomorphic to Euclidean space. Take any $x, y \in X$. Let $[x, y]$ denote a geodesic arc joining $x$ to $y$. And further, take a unit tangent vector $v$ at $x$. Given a real number $c$, $g(x, v, c)$ is defined to be the geodesic arc issuing from $x$, whose length is $|c|$ and whose initial vector is $v$ or $-v$ according as $c > 0$ or $< 0$. Let $(x, v, c)$ denote its terminal point. Note that a geodesic arc is not necessarily simple and sometimes may be closed. Let a curve $\alpha : x(t)$ (say, $0 \leqq t \leqq 1$) be given in $X$. At $x_0 = x(0)$ we take a unit vector $v_0$ tangent to $X$. Corresponding to each $t$, let $v(t)$ denote the unit vector at $x(t)$ parallel to $v_0$ along $\alpha$. Moreover, if a geodesic arc $g(x_0, v_0, c)$ is given, each geodesic arc $g(x(t), v(t), c)$ is said to be *parallel* to $g(x_0, v_0, c)$ along $\alpha$. And to displace the latter arc parallelly along $\alpha$ is to obtain the former arcs. A *covering manifold* $C(X)$ of $X$ is defined to be a connected covering manifold of $X$

with the Riemannian metric naturally induced from $X$ by the covering map $p$. Especially, if $p^{-1}(x)$ $(x \in X)$ consists of just $k$ points, $C(X)$ is called a *k-covering manifold* of $X$. The notation "$\times$" always means the operation of metric product.

For models of $RS$-manifolds in Remarks $2-6$, cf. [3].

**2. Preliminaries.** As already defined, let $M$ be a connected complete Riemannian $n$-manifold $(n > 1)$ of class $C^2$ admitting a parallel field of tangent vector $r$-subspaces $(1 \leqq r \leqq n-1)$. More precisely, to each point of $M$ a tangent vector $r$-subspace is assigned so that all of them form a parallel field. We call it the *R-field* over $M$. Let us take the field of tangent vector $s$-subspaces, $s = n - r$, which is orthogonal to the $R$-field at each point of $M$. It is obvious that the field forms a parallel field over $M$. We call it the *S-field* over $M$. Throughout the whole discussion, $M$ is such a manifold which will be called an *RS-manifold* of dimension $n$. In $M$ the following fact is very well-known:

*At any $x_0 \in M$ there is a coordinate neighborhood $U$ with coordinate system $(x^\alpha)$ which satisfies the following properties:*

1) *The transformation from the system $(x^\alpha)$ to an admissible coordinate system of $M$ at $x_0$ is of class $C^2$;*

2) *The Riemannian metric in $U$ is expressed by the form completely decomposed as follows:*

$$ds^2 = g_{ab}(x^1,\ldots\ldots, x^r)\, dx^a dx^b + g_{ij}(x^{r+1},\ldots\ldots, x^n) dx^i dx^j$$

*where $g_{ab}$ and $g_{ij}$ are functions of class $C^1$ independent of $x^i$ and $x^a$ respectively;*

3) *A system of equations $x^i = const.$ expresses an integral manifold of the R-field and a system of equations $x^a = const.$ expresses one of the S-field.*

For the proof, see [1], say.

A coordinate neighborhood of $x^0 \in M$ with the same property as $U$ above is called a *reduced coordinate neighborhood* of $x_0$, if its coordinate system $(x^\alpha)$ consists of all of $(x^\alpha)$'s such that $a^\alpha < x^\alpha < b^\alpha (a^\alpha, b^\alpha$ are constants).

Let $U, U'$ be two reduced coordinate neighborhoods of $x_0$. Let $(x^\alpha)$ and $(x'^\alpha)$ be their coordinate systems respectively. Let $W$ be the connected components of $U \cap U'$ containing $x_0$. *In $W$ the coordinate systems $(x^\alpha)$ and $(x'^\alpha)$ are combined by the relations completely decomposed as follows:*

$$x'^a = f'^a(x^1,\ldots\ldots, x^r), \quad x'^i = f'^i(x^{r+1},\ldots\ldots, x^n)$$

*where $f'^a$ and $f'^i$ are functions of class $C^2$ independent of $x^i$ and $x^a$ respectively.* Moreover we can see that through $x_0 \in M$ there passes a pair of the maximal connected integral manifolds of the $R$-and $S$-fields. Let $R(x_0)$ and $S(x_0)$ denote the ones respectively. We give them the Riemannian metric which is naturally induced from $M$ and call them $R$- and *$S$-submanifolds* of $M$ respectively. They form Riemannian manifolds of class $C^1$. The following fact is well-known: *All of the $R$- and $S$-submanifolds are totally geodesic, and complete as Riemannian manifolds.* Let $I(x_0)$ denote the set $R(x_0) \cap S(x_0)$.

In $M$, suppose that there exists a connected open submanifold $M^0$ which satisfies the following conditions 1) and 2), or 1) and 3):

1) $M^0$ is a union set of $R$-submanifolds and the closure of $M^0$ is $M$;

2) $M^0$ is the maximal subset in which each point $x$ is a limit point of $I(x)$ relative to $S(x)$, or

3) $M^0$ is a maximal subspace which becomes a fibre bundle where each fibre is an $R$-submanifold. (By the word "maximal" it is meant that there are no subspaces, $\supset M^0, \neq M^0$, which have the same property.)

When $M^0$ satisfies 1) and 2), $M$ is said to be of *almost $R$-clustered type* with *kernel* $M^0$. In this case, if $M = M^0$, $M$ is simply said to be of *$R$-clustered type.*

When $M^0$ satisfies 1) and 3), $M$ is said to be of *almost $R$-fibred type* with *kernel* $M^0$. In this case, if $M = M^0$, $M$ is simply said to be of *$R$-fibred type.*

REMARK 1. Throughout this note, $R$-field and $S$-field, so $R$-submanifold and $S$-submanifold, can not be intrinsically distinguished. Accordingly, the statements all hold good even if we exchange the roles of them. If for example a definition is given, the new definition is obtained by exchanging $R$ and $S$ in it for each other. Of course it holds good. Let us suppose that the new definition is given there, although it is not explicitly stated. This is also applied to notations, lemmas, theorems, and so on. Besides, it is a matter of course that definitions, notations, and so on, in $M$ are used for any $RS$-manifolds under the same senses.

**3. Fundamental lemmas.** Take any $x_0 \in M$. An *$R$-neighborhood* of $x_0$ is a neighborhood in $R(x_0)$. An *$R$-normal vector* at $x_0$ is a unit tangent vector at $x_0$ orthogonal to $R(x_0)$. Take a connected open subset $O_R$ of $R(x_0)$ ($x_0 \in O_R$) and an $R$-normal vector $n_0$ at $x_0$. At each $x \in O_R$ we plant an $R$-normal vector $n(x)$ where $n(x_0) = n_0$. If for any $x_1, x_2 \in O_R$, $n(x_1)$ is parallel to $n(x_2)$ along any of curves of class $D^1$ in $O_R$ joining $x_1$ to $x_2$, the set $\{n(x) | x \in O_R\}$ is called the *$R$-normal vector field* over $O_R$ *parallel* to $n_0$ $(= n(x_0))$.

Again take $x_0 \in M$ and an $R$-normal vector $n_0$ at $x_0$. For a constant $c$, put $y_0 = (x_0, n_0, c)$. Then we have

LEMMA 3.1. *There is an $R$-neighborhood $U_R$ at $x_0$ which satisfies the following conditions*:

1) *Over $U_R$ the $R$-normal vector field $\{n(x) \mid x \in U_R\}$ parallel to $n_0$ does exist* ;

2) $(x, n(x), c) \in R(y_0)$ *for all* $x \in U_R$;

3) *The map*

$$f : U_R \to R(y_0) \text{ defined by } f(x) = (x, n(x), c)$$

*is an isometric into-homeomorphism of class $C^2$.*

PROOF. First let us consider the case where the geodesic $g(x_0, n_0, c)$ is contained in a reduced coordinate neighborhood $U$. Let $U_R$ be the connected component of $U \cap R(x_0)$ containing $x_0$. In $U$, let $(x_0^\alpha)$, $(y_0^\alpha)$, and $(n_0^\alpha)$ denote $x_0$, $y_0$, and $n_0$ respectively. Here, $n_0^a = 0$. It is verified that all of the vectors $n(x)$, $x \in U_R$, which have in $U$ the same components as $n_0$, form the $R$-normal vector field parallel to $n_0$. So, 1) holds good. In $U$ let $(x^\alpha)$ denote any $x \in U_R$. Here, $x^i = x_0^i$. Moreover we can see that a point $(x, n(x), c)$ is denoted by $(x^a, y_0^i)$. As $x_0^a = y_0^a$, $(x, n(x), c) \in R(y_0)$, i.e., 2) holds true. 3) is now obvious by § 2.

Next let us consider the case where the geodesic $g(x_0, n_0, c)$ is not contained in a reduced coordinate neighborhood. Take a finite system of reduced coordinate neighborhoods $U_\lambda (\lambda = 1, 2, \ldots\ldots, h)$ such that each $U_\lambda$ contains a geodesic arc $[x_{\lambda-1}, x_\lambda]$ where the product curve $[x_0, x_1] \cdot [x_1, x_2] \cdot \ldots \cdot [x_{h-1}, x_h]$ becomes $g(x_0, n_0, c)$. For each pair $U_\lambda$ and $[x_{\lambda-1}, x_\lambda]$, there is an $R$-neighborhood of $x_{\lambda-1}$ which satisfies the conditions 1)−3), as already proved. Hence, it is easy to find an $R$-neighborhood $U_\lambda$ which satisfies our conditions 1)−3).

Under the same notations as Lemma 3.1, let $x(t)$ $(a \leqq t \leqq b)$, $x(a) = x_0$, be a curve of class $D^1$ in $R(x_0)$. For each $t$, let $n(t)$ be the $R$-normal vector at $x(t)$ parallel to $n_0$ along the curve. We put $y(t) = (x(t), n(t), c)$. Let $n'(t)$ be the vector at $y(t)$ parallel to $n(t)$ along the geodesic $g(x(t), n(t), c)$. Since $g(x(t), n(t), c) \subset S(x(t))$, $n'(t)$ is an $R$-normal vector. Then we have

LEMMA 3.2. 1) *The curve $y(t)$ $(a \leqq t \leqq b)$ is a curve of class $D^1$ in $R(y_0)$*;

2) $\{n'(t) \mid a \leqq t \leqq b\}$ *consists of $R$-normal vectors parallel to one another along the curve $y(t)$.*

PROOF. For any $t_0$ $(a \leqq t_0 \leqq b)$ if we cover the geodesic $g(x(t_0), n(t_0), c)$ by a finite number of reduced coordinate neighborhoods, it is seen that in a suitable interval of $t$ containing $t_0$, 1) and 2) hold good (cf. Proof of Lemma 3.1). Accordingly, 1), 2) are proved.

In $M$, let $R, S$ be any $R$-, $S$-submanifolds respectively. Then we have

LEMMA 3.3. *The set $R \cap S$ is at most countable and non-empty.*

PROOF First note that the second countability axiom holds in $M$, $R$, $S$ respectively. Now take a countable (or finite) system of reduced coordinate neighborhoods $U_\lambda$ which cover $M$. Then $U_\lambda \cap R$ consists of a system of non-intersecting $R$-neighborhoods, which is at most countable. (Of course the system may be empty). For $U_\lambda \cap S$, too, it holds good. These properties are obvious by the second countability axiom.

Accordingly, $U_\lambda \cap R \cap S$ is at most countable. Hence $R \cap S$ is at most countable. For the assertion that $R \cap S$ is non-empty, see [2], p. 23.

**4. A general structure.** For any two points $x_1, x_2$ of the same $R$-submanifold in $M$, let $d_R(x_1, x_2)$ denote the length of a minimizing geodesic $[x_1, x_2]$ in the $R$-submanifold. Take any $x_0 \in M$ and a constant $a > 0$. Let $C_R(x_0; a)$ denote the part of $R(x_0)$ defined by the subset $\{x \mid x \in R(x_0), d_R(x_0, x) \leqq a\}$. If a set $\{x \mid x \in R(x_0), d_R(x_0, x) < a\}$ forms an $R$-neighborhood of $x_0$ which can be covered by a normal coordinate system in $R(x_0)$ with center $x_0$, this neighborhood is called a *normal $R$-neighborhood* of $x_0$ and denoted by $N_R(x_0; a)$. Moreover such a constant $a$ is called a *normal $R$-radius* at $x_0$. Let $T_R(x_0)$ denote the Euclidean vector $r$-space tangent to $R(x_0)$ at $x_0$. The map $\exp_R$ at $x_0 \in M$ is defined to be the map $T_R(x_0) \to R(x_0)$ such that $\exp_R v = x_0$ for the zero vector $v \in T_R(x_0)$ and $\exp_R v = (x_0, v/|v|, |v|)$ for any non-zero vector $v \in T_R(x_0)$, where $|v|$ denotes the length of $v$.

Again at $x_0 \in M$ let $e_R(x_0)$ denote the greatest lower bound of $\{d_R(x_0, x) \mid x \in I(x_0) - x_0\}$ if $I(x_0) - x_0$ is non-empty. If $I(x_0) - x_0$ is empty, we put $e_R(x_0) = \infty$. Accordingly, $0 \leqq e_R(x) \leqq \infty$ for any $x \in M$.

LEMMA 4.1. 1) *If $e_R(x_0) = 0$, $e_R(x) = 0$ for all $x \in S(x_0)$ (so, if $e_R(x_0) > 0$, $e_R(x) > 0$ for all $x \in S(x_0)$).*

2) *If $e_R(x_0) > 0$, there is a constant $a > 0$ such that the parts $C_R(x; a)$ for all $x \in S(x_0)$ do not intersect one another.*

3) *A necessary and sufficient condition for $e_R(x_0) > 0$ is that the topology of $S(x_0)$ coincides with the relative one induced from $M$.*

PROOF. 1) is evident by Lemma 3.1. To prove 2), at $x_0$ take a normal $R$-radius $c < e_R(x_0)$. Then for any $x \in S_0 = S(x_0)$, $C_R(x; c) \cap S_0$ consists of $x$ only. For, otherwise, there is $x \in S_0$ such that $C_R(x; c) \cap S_0$ contains a point $x'(\neq x)$. Let $[x, x']$ be a minimizing geodesic in $R(x)$. And let $[x_0, x'']$ be the geodesic parallel to $[x, x']$ along a curve in $S_0$. Hence, $x'' \in S_0$ by Lemma 3.2, $[x_0, x''] \subset C_R(x_0; c)$, and $x_0 \neq x''$. Here, $d_R(x_0, x'') \leqq c$

$< e_R(x_0)$. These results contradict with the definition of $e_R(x_0)$. So, $C_R(x;c) \cap S_0$ for each $x \in S_0$ consists of $x$ only. Now, put $a = c/2$. Then it is obvious that the constant $a$ is a constant $a$ in 2). 3) is easily proved by using 2).

Let $x_0$ be a point of $M$. Take a closed curve $\beta$ of class $D^1$ in $R_0 = R(x_0)$ starting from $x_0$. For any $v \in T_S(x_0)$, we obtain the vector $v'$ at the terminal point $x_0$ by displacing $v$ parallelly along $\beta$. Of course, $v' \in T_S(x_0)$. Then the map $f_\beta$ of $T_S(x_0)$ onto itself, defined by $f_\beta(v) = v'$, is a congruent transformation in $T_S(x_0)$. This is said to be the congruent transformation *induced* from $\beta$. All of such transformations form a group. We denote it by $G(R_0, x_0)$ or $G(R_0)$ (it being independent of $x_0$ as abstract group).

LEMMA 4.2. *$G(R_0, x_0)$ is isomorphic with a factor group of the fundamental group $\pi_1(R_0, x_0)$. Hence the order of $G(R_0, x_0)$ is at most countable.*

PROOF. Let $\beta_0$ be a closed curve of class $D^1$ in $R_0$, starting from $x_0$ and in $R_0$ homotopic to $x_0$. Then the congruent transformation $f_{\beta_0}$ in $T_S(x_0)$ induced from $\beta_0$ is the identity. For, otherwise, we can find a unit vector $v \in T_S(x_0)$ such that $f_{\beta_0}(v) \neq v$. Let $c$ be a normal $S$-radius at $x_0$. So, for a constant $\delta (0 < \delta < c)$, $g(x_0, v, \delta)$ is parallel to $g(x_0, f_{\beta_0}(v), \delta)$ along $\beta_0$. Here if we deform $\beta_0$ to $x_0$, we obtain a curve in $N_S(x_0; c)$ joining $y_{\beta_0} = (x_0, f_{\beta_0}(v), \delta)$ to $y = (x_0, v, \delta)$ as the locus of $y_{\beta_0}$. This curve is contained in $I(y)$. As $y_{\beta_0} \neq y$, this is contrary to Lemma 3.3. The fact above gives rise to the homomorphic map of $\pi_1(R_0, x_0)$ onto $G(R_0, x_0)$ naturally. So the former part is proved. The latter part is clear because $\pi_1(R_0, x_0)$ is at most countable.

In $M$, let $x_0$, $y_0$ be two points of an $S$-submanifold. Let $[x_0, y_0]$ be a geodesic arc in $S(x_0)$. Put $[x_0, y_0] = g(x_0, n_0, c)$. If $R(x_0)$ admits the $R$-normal vector field $\{n(x) \mid x \in R(x_0)\}$ parallel to $n_0$, we can consider the map

$$f: R(x_0) \to R(y_0) \text{ defined by } f(x) = (x, n(x), c)$$

by Lemma 3.2. $f$ is said to be the map *induced* from $[x_0, y_0]$.

LEMMA 4.3. *$f$ is locally an isometric homeomorphism of class $C^2$ and $f$ is also a covering map.*

PROOF. From Lemmas 3.1 and 3.2, the map $f$ is onto and locally an isometric homemorphism of class $C^2$. For any $y \in R(y_0)$ the subset $f^{-1}(y)$ of $R(x_0)$ is at most countable by Lemma 3.3. Let $x_\lambda (\lambda = 1, 2, \ldots\ldots)$ denote all of the points of $f^{-1}(y)$. Here, $f^{-1}(y)$ is contained in a compact subset $C_S(y; |c|)$. By covering $C_S(y; |c|)$ by a finite number of reduced coordinate neighborhoods, we can find an $R$-neighborhood $W_R(y)$ of $y$ and the $R$-neighborhoods $W_R(x_\lambda)$ of $x_\lambda$ for all $\lambda$, such that all $W_R(x_\lambda)$ are isometrically homeomorphic to $W_R(y)$

under $f$. Then, all of $W_R(x_\lambda)$ do not intersect one another. For, suppose that

$$W_R(x_\mu) \cap W_R(x_\nu) \neq 0 \qquad \text{for } x_\mu,\ x_\nu \in f^{-1}(y)\ (x_\mu \neq x_\nu).$$

If we take a curve $\alpha \subset W_R(x_\mu) \cup W_R(x_\nu)$ joining $x_\mu$ to $x_\nu$, then we have $f(\alpha) \subset W_R(y)$. This gives rise to a contradiction. Accordingly, our lemma is proved.

In $M$ suppose that $R_0$ is an $R$-submanifold such that $G(R_0)$ consists of the identity only. Then $R_0$ is said to be *$R$-maximal* in $M$. Here, note the following property: Take $x_0 \in R_0$, $y_0 \in S(x_0)$. Let $[x_0, y_0]$ be a geodesic in $S(x_0)$. Put $R_1 = R(y_0)$. Then, there exists the map $f: R_0 \to R_1$ induced from $[x_0, y_0]$. By Lemma 4. 3, $R_0$ is regarded as a covering manifold of $R_1$ under $f$. Moreover, if $R_1$ is $R$-maximal, it is easy to see

LEMMA 4. 4. *The map $f$ is an isometric homeomorphism of class $C^2$, of $R_0$ onto $R_1$.*

Let $V$ be a Euclidean vector $d$-space which is topologized by regarding as Euclidean space. Let $G$ be an effective group of congruent transformations in $V$, which is at most countable. We denote all of the elements of $G$ by $g_\lambda(\lambda = 0, 1, 2, \ldots\ldots)$, where $g_0$ is the identity. For each $g_\lambda(\lambda \neq 0)$, put $V_\lambda = \{v \mid g_\lambda v = v\}$. $V_\lambda$ forms a subspace of dimension $< d$ in $V$. Let $V^0$ denote $V - \bigcup_{\lambda=1}^{\infty} V_\lambda$. Then, $V^0$ is non-empty. For any $v \in V^0$, the vectors

$$v,\ g_1 v,\ g_2 v, \ldots\ldots$$

are all distinct from one another. This is easily verified. Such a vector $v$ is said to be *completely variant* under $G$. It follows that $V^0$ consists of all of the vectors completely variant under $G$.

First, suppose that $G$ is finite. Then we have

LEMMA 4. 5. 1) *$V^0$ is an open set of $V$ and the closure of $V^0$* is $V$.

2) *For any unit vectors $u, u' \in V^0$, there is a sequence of unit vectors:*

$$u_1(= u),\ u_2, \ldots\ldots,\ u_k(= u'),\ \subset V^0$$

*such that $u_\mu$, $u_{\mu+1}$ belong to the same connected component of $V^0$ or are $G$-connected* $(\mu = 1, 2, \ldots\ldots, k - 1)$.

By the word "*G-connected*" it is meant that $u_{\mu+1} = g(\mu) u_\mu$ for a suitable $g(\mu) \in$ G. $k$ suffices to be an integer $> 1$.

PROOF. As 1) is obvious, we prove 2). If $V^0$ is connected, the sequence: $u_1(= u)$, $u_2(= u')$, satisfies our condition. Accordingly we consider the case where $V^0$ is not connected. Then, among $V_\lambda(\lambda = 1, 2, \ldots\ldots, h - 1;\ h =$ the order of $G$), there is at least one of dimension $d-1$. Let us suppose $V_1$ to be such one. Take $v \in V^0$. The vector is represented by $u_1+v_1$ where $u_1(\neq 0)$ is perpendicular to $V_1$ and $v_1 \in V_1$. Then we have $g_1(u_1+v_1) = -u_1+v_1$. I. e., the vector $v$ is $G$-connected with $v' = -u_1+v_1$. The vector $v'$ belongs to the side distinct from $v$ with respect to $V_1$. Here, if $v \in V^0$ is suitably chosen, $v'$ belongs to $V^0$. From this fact, 2) is proved.

Next, suppose that $G$ is infinite (i. e., countable). Let $V'$ be the set of all of $v \in V$ such that the vectors $g_\lambda v$ ($\lambda = 0, 1, 2, \ldots\ldots$) indeed consist of a finite number of vectors distinct from one another. Any vector of $V - V'$ is said to be *infinitely variant* under $G$.

LEMMA 4. 6. 1) *If* $v \in V'$, $g_\lambda v \in V'$;
2) $V'$ *forms a vector subspace of* $V$;
3) *For the dimension* $d$ *of* $V'$, $0 \leqq d' < d-1$;
4) $V - V'$ *is a connected open subset of* $V$.

PROOF. 1) and 2) are obvious. To prove 3), suppose that $d' = d-1$. By 1), $g_\lambda \cdot V' = V'$ for any $g_\lambda \in G$. Hence, for a vector $e$ normal to $V'$, $g_\lambda e = e$ or $-e$. So, $e \in V'$. I. e., $V' = V$. This is contrary to the existence of vectors completely variant under $G$. Accordingly, 3) holds good. From 3), 4) follows immediately. This completes the proof.

THEOREM 1. *In $M$ suppose that the topology of every $R$-submanifold coincides with the relative one induced from $M$. Then there are $R$-maximal $R$-submanifolds. In all of them let $M^0$ be the subspace of $M$ which is their union set. Then $M^0$ is a connected open submanifold of $M$ whose closure is $M$ and a maximal subspace which becomes a fibre bundle where each fibre is a $R$-submanifold. In other words, $M$ is of almost $R$-fibred type with kernel $M^0$.*

PROOF. 1) For any $R$-submanifold $R$, $G(R)$ is finite. In fact, suppose that it is infinite (i. e., countable by Lemma 4. 2). Denote all of the elements of $G(R, x)$, $x \in R$, by $g_\lambda(\lambda = 0, 1, 2, \ldots\ldots)$ where $g_0$ is the identity. Then we can find a unit vector $v \in T_S(x)$ completely variant under $G(R, x)$. Let $a$ be a normal $S$-radius at $x$. For a constant $b$ $(0 < b < a)$ put $x_\lambda = (x, g_\lambda v, b)$. Then $x_\lambda(\lambda = 0, 1, 2, \ldots\ldots)$ are distinct from one another. And, $x_\lambda \in R(x_0) \cap C_S(x; b)$. $C_S(x; b)$ being compact, we have $e_S(x_0) = 0$. This contradicts with the assumption of our theorem by Lemma 4. 1. So, $G(R)$ is finite.

$2)_1$ Take an $R$-submanifold $R_0 = R(x_0)$. Let $k_0$ be the order of $G(R_0, x_0)$. By 1), $k_0$ is finite. Denote all of the elements of $G(R_0, x_0)$ by $g$ $(\lambda = 0, 1, \ldots\ldots, k_0 - 1)$ where $g_0$ is the identity. We have $e_S(x_0) > 0$. Let $a$ be a normal

$S$-radius at $x_0$ such that $0 < a < e_S(x_0)/2$. Let $v_0$ be a unit vector of $T_S(x_0)$ completely variant under $G(R_0, x_0)$ For a constant $\delta\,(0 < \delta < a)$ put $y_\lambda = (x_0, g_\lambda v_0, \delta)$. Let $g(y_0, u_0, \delta)$ denote the geodesic $[y_0, x_0]$ in $N_S(x_0\,; a)$. Put $R_1 = R(y_0)$. Then, $R_1 \cap N_S(x_0; a) = \{y_\lambda \mid \lambda = 0, 1, \ldots\ldots, k_0 - 1\}$. For, take $y \in R_1 \cap N_S(x_0; a)$ and displace $[y_0, x_0]$ parallelly along a curve of class $D^1$ in $R_1$ joining $y_0$ to $y$. At $y$, we obtain the geodesic $[y, x_0']$ in $S(x_0)$. Here

$$d_S(x_0, x_0') \leqq d_S(x_0, y) + d_S(y, x_0') < a + \delta < e_S(x_0).$$

So, $x_0 = x_0'$. From this manner, we can see that $y \in \{y_\lambda \mid \lambda = 0, 1, \ldots\ldots, k_0 - 1\}$ by Lemma 3. 2. Accordingly, $R_1 \cap N_S(x_0; a)$ consists of $y_\lambda (\lambda = 0, 1, \ldots\ldots, k_0 - 1)$ only.

Now over $R_1$ there is the $R$-normal vector field parallel to $u_0$. For, otherwise, by displacing $u_0$ parallelly along a suitable closed curve in $R_1$ we can obtain a vector $u_0'$ at $y_0$ distinct from $u_0$. Of course, $u_0' \in T_S(y_0)$. By Lemma 3. 2, $x_0' = (y_0, u_0', \delta) \in R(x_0)$. So $x_0' \in I(x_0)$. Here, $x_0 \neq x_0'$ and we have

$$d_S(x_0, x_0') \leqq d_S(x_0, y_0) + d_S(y_0, x_0') \leqq 2\,\delta < e_S(x_0).$$

This is contrary to the definition of $e_S(x_0)$. So our assertion is true. Hence there is the map $f\colon R_1 \to R_0$ induced from the geodesic $[y_0, x_0]$. By Lemma 4. 3, $R_1$ is a $k_0$-covering manifold of $R_0$ under $f$.

$2)_2$ We prove that $R_1$ is $R$-maximal. Denote all of the elements of $G(R_1, y_0)$ by $h_\mu (\mu = 0, 1, \ldots\ldots, k_1 - 1)$ where $h_0$ is the identity and $k_1$ is the order. By 1), $1 \leqq k_1 < \infty$. Now suppose $k_1 > 1$. We take a constant $\varepsilon < e_S(y_0)/2$, which becomes at each $y_\lambda$ a normal $S$-radius, such that all $N_S(y_\lambda; \varepsilon)$ are contained in $N_S(x_0; a)$ and do not intersect one another. Here we can find a unit vector $w_0 \in T_S(y_0)$ completely variant under $G_S(R_1, y_0)$ and perpendicular to $u_0$. All of the vector $h_\mu w_0$ are perpendicular to $u_0$. Put $z_0 = (y_0, w_0, \varepsilon')$ for a constant $\varepsilon'$, $0 < \varepsilon' < \varepsilon$. Then there is a map $f'\colon R(z_0) \to R_1$ induced from the geodesic $[z_0, y_0]$ in $N_S(y_0; \varepsilon)$. Under $f'$, $R(z_0)$ is the $k_1$-covering manifold of $R_1$. This is verified by the same way as $2)_1$. Let $[z_0, x_0]$ be the geodesic in $N_S(x_0; a)$. Then, there is a map $f''\colon R(z_0) \to R_0$ induced from $[z_0, x_0]$. Under the map $f''$, $R(z_0)$ is a $k_0 k_1$-covering manifold of $R_0$. This is easily verified, too. These results implies that $G(R_0, x_0)$ has order $\geqq k_0 k_1$, so $> k_0$. This being a contradiction, $k_1$ must be one. I.e., $R_1$ is $R$-maximal.

$2)_3$ In the case where $R_0$ is $R$-maximal, we can see by $2)_2$ that the $R$-submanifolds $R(y)$ for all $y \in N_S(x_0\,;\, a)$ are $R$-maximal.

Let us consider the case where $R_0$ is not $R$-maximal and where there is a unit vector $v \in T_S(x_0)$ which is not completely variant under $G(R_0, x_0)$. Put $y = (x_0, v, \delta)$ where $0 < \delta < a$. Then, $R(y)$ is not $R$-maximal. To prove this, at $y$ take a normal $S$-radius $b < e_S(y)/2$. On the other hand, there is $g\,(\neq g_0) \in$

$G(R_0, x_0)$ such that $gv = v$. Hence, we can find a unit vector $v^* \in T_S(x_0)$ such that $y^* \neq y_g^*$ and $y^*, y_g^* \in N_S(y; b)$ for $y^* = (x_0, v^*, \delta), y_g^* = (x_0, gv^*, \delta)$. Let $[y^*, y]$ be the geodesic in $N_S(y ; b)$. Displace $[y^*, y]$ parallelly along a curve of class $D^1$ in $R(y^*)$ joining $y^*$ to $y_g^*$. At $y_g^*$ we obtain the geodesic $[y_g^*, y']$. Of course $[y_g^*, y'] \subset S(x_0)$. As $d_S(y, y') < e_S(y)$, we have $y = y'$. This means that the order of $G(R(y), y)$ is not one. So, $R(y)$ is not $R$-maximal.

3) Take an $S$-submanifold $S$ of $M$. In $S$ let $S^0$ be the subspace consisting of all $x \in S$ such that $R(x)$ is $R$-maximal. By $2)_3$, $S^0$ is open in $S$. We prove that in $S$ the closure of $S^0$ is $S$. It suffices to consider the case only where $S - S^0$ is non-empty. Take $x_0 \in S - S^0$ and at $x_0$ a normal $S$-radius $c < e_S(x_0)/2$. We put $R_0 = R(x_0)$. The order of $G(R_0, x_0)$ is greater than one. Let $V^0$ be the set of all vectors of $T_S(x_0)$, each of which has length $< c$ and is completely variant under $G(R_0, x_0)$. Then, $\exp_S V^0 = S^0 \cap N_S(x_0 ; c)$ by $2)_2$, $2)_3$. From Lemma 4. 5, we can see that in $N_S(x_0 ; c)$ the closure of $\exp_S V^0$ is $N_S(x_0 ; c)$. So, $x_0$ is contained in the closure in $S$ of $S^0$. Accordingly our assertion is proved.

Now, by Lemma 3. 3 $M^0$ is regarded as the union set of $\{R(x) | x \in S^0\}$. From the above facts and Lemma 3. 1, $M^0$ is an open submanifold of $M$ whose closure is $M$.

Next, we prove that $M^0$ is connected. For this, it suffices to show that any two points $x_1, x_2 \in S^0$ are joined by a curve in $M^0$. Let $\alpha$ be a curve in $S$ joining $x_1$ to $x_2$. Cover $\alpha$ by a finite number of normal $S$-neighborhoods $N_S(y_\lambda : a_\lambda)$ $(\lambda = 1, 2, \ldots\ldots, h)$ where $y_\lambda \in \alpha$, $a_\lambda < e_S(y_\lambda)/2$. For some $\lambda$ ; if $y_\lambda \in S^0$, $N_S(y_\lambda ; a_\lambda) \subset S^0$ by $2)_3$. If $y_\lambda \notin S^0$, we denote by $W_\lambda$ the subspace of $M$ which is the union set of $\{R(x) | x \in S^0 \cap N_S(y_\lambda ; a_\lambda)\}$. Moreover let $V_\lambda^0$ be the set of all vectors of $T_S(y_\lambda)$, each of which has length $< a_\lambda$ and is completely variant under $G(R(y_\lambda), y_\lambda)$. Here, $\exp_S V_\lambda^0 = S^0 \cap N_S(y_\lambda ; a_\lambda)$ If we give $T_S(y_\lambda)$ the topology by regarding as Euclidean space, $V_\lambda^0$ has the same property as the part of $V^0$ in Lemma 4. 5. Hence, we can see that $W_\lambda$ is open in $M$ and connected. These facts, together with the property that in $S$ the closure of $S^0$ is $S$, show that $x_1$, $x_2$ are joined by a curve in $M^0$. So, $M^0$ is connected.

4) Take $x_0 \in S^0$ and at $x_0$ a normal $S$-radius $a < e_S(x_0)/2$. Then, $N_S(x_0 ; a) \subset S^0$ by $2)_3$. For $z \in N_S(x_0 ; a)$ let $[x_0, z]$ be the geodesic contained in $N_S(x_0 ; a)$. Then there is the map $f_z : R(x_0) \to R(z)$ induced from $[x_0, z]$. This map $f_z$ is an isometric homeomorphism by Lemma 4. 4. Denote $R(x_0) \times N_S(x_0 ; a)$ by $V(x_0)$. Hence, any $x \in V(x_0)$ is represented by a pair $(y, z)$ where $y \in R(x_0)$, $z \in N_S(x_0 ; a)$. Define a map

$$f : V(x_0) \to M^0 \text{ by } f(x) = f_z(y).$$

The map $f$ is one-to-one. For, otherwise, there are $x_1, x_2 \in V(x_0)$, $x_1 \neq x_2$, such that $f(x_1) = f(x_2)$. Represent $x_1$ by $(y_1, z_1)$ and $x_2$ by $(y_2, z_2)$. Hence $f_{z_1}(y_1) = f_{z_2}(y_2)$, so $R(z_1) = R(z_2)$. As $z_1, z_2 \in N_S(x_0 ; a)$ and $R(z_1) \cap N_S(x_0 ; a)$ consists of $z_1$ only, we have $z_1 = z_2$. It follows that $y_1 = y_2$. I. e., $x_1 = x_2$. This is a contradiction. So, $f$ is one-to-one. It is verified that $f$ is an isometric into-homeomorphism such that $f(R(x_0), z) = R(z)$ for all $z \in N_S(x_0 ; a)$, $f(y, N_S(x_0 ; a)) \subset S(y)$ for all $y \in R(x_0)$.

In $S^0$, if $x, y \in S^0$ belong to the same $R$-submanifold, we say that they are equivalent to each other. By this equivalence relation, we construct the quotient space of $S^0$ and denote it by $B$. Then, $B$ becomes a manifold and over $B$ a Riemannian metric is naturally indued from $S^0$. Thus $B$ is regarded as a connected Riemannian $s$-manifold of class $C^1$. Next, for any $x \in M^0$, let $[x]$ denote the point of $B$ representing $R(x) \cap S^0$. Then the map

$$\pi : M^0 \to B \text{ defined by } \pi(x) = [x]$$

is an onto-map. Thus we can prove that $M^0$ becomes a fibre bundle where each fibre is an $R$-submanifold, the base space is $B$, and the projection is $\pi$. The proof is omitted, as it is too long to give here (cf. [5]).

5) If $M = M^0$, our theorem holds good, $M$ being of $R$-fibred type. So it remains to consider the case where $M \neq M^0$. For $x \in M - M^0$, the order of $G(R(x), x)$ is not one. Hence by $2)_2$ it follows that any $S$-neighborhood of $x$ contains at least two points of an $R$-submanifold which is contained in $M^0$. This shows that there is no subspace, $\supset M^0$, $\neq M^0$, which is a union set of $R$-submanifolds and a fibre bundle where each fibre becomes an $R$-submanifold. Accordingly, $M$ is of almost $R$-fibred type with kernel $M^0$. This completes the proof of our theorem.

THEOREM 2. *In $M$ suppose that the topology of at least one $R$-submanifold does not coincide with the relative one induced from $M$. In all of such $R$-submanifolds let $M^0$ be the subspace of $M$ which is their union set. Then $M^0$ is a connected open submanifold of $M$ whose closure is $M$, and the maximal subset of $M$ in which each point $x$ is a limit point of $I(x)$ relative to $S(x)$. In other words, $M$ is of almost $R$-clustered type with kernel $M^0$.*

PROOF. In the case where the topology of every $R$-submanifold does not coincide with the relative one induced from $M$, $e_S(x) = 0$ for all $x \in M$ by Lemma 4. 1. So, $M$ is of $R$-clustered type. Our theorem holds good. Accordingly consider the other case. Then, there is at least one $R$-submanifold $R_0$ whose topology coincides with the relative one. Let $R_1$ denote an $R$-submanifold whose topology does not coincide with the relative one.

1) For $x_0 \in R_0$, we have $e_S(x_0) > 0$ by Lemma 4. 1. Let us prove that

$G(R_0)$ is infinite, i. e., countable. Take $y_0 \in R_1 \cap S(x_0)$, so $e_S(y_0) = 0$. Let $[y_0, x_0]$ be a minimizing geodesic in $S(x_0)$. Put $L = d_S(y_0, x_0)$. Let $a$ be a normal $S$-radius at $y_0$. We denote $N_S(y_0 ; a) \cap R_1$ by $\{y_\lambda | \lambda = 0, 1, 2, \ldots\ldots\}$, it being countable. For each $\lambda$ let $\beta_\lambda$ be a curve of class $D^1$ in $R_1$ joining $y_0$ to $y_\lambda$. Displace $[y_0, x_0]$ parallelly along $\beta_\lambda$. As the locus of the terminal point $x_0$ we obtain a curve $\alpha_\lambda$ and at $y_\lambda$ the geodesic $[y_\lambda, x_\lambda]$. $x_\lambda$ is the terminal point of $\alpha_\lambda$ and by Lemma 3. 2, $\alpha_\lambda \subset R_0$. Moreover, $\{x_\lambda | \lambda = 0, 1, 2, \ldots\ldots\} \subset I(x_0)$, and $\subset C_S(y_0, a + L)$. From the compactness of $C_S(y_0, a + L)$ and Lemma 4. 1, the set $\{x_\lambda | \lambda = 0, 1, 2, \ldots\ldots\}$ must be finite. Hence, there is an infinite subset $\{\lambda_1, \lambda_2, \ldots\ldots, \lambda_k, \ldots\ldots\}$ of $\{0, 1, 2, \ldots\ldots\}$ such that $x_{\lambda_1} = x_{\lambda_2} = \ldots\ldots = x_{\lambda_k} = \ldots\ldots$ Displace the geodesic $[y_{\lambda_1}, x_{\lambda_1}]^{-1} = g(x_{\lambda_1}, v_0', c)$ $(c > 0)$ parallelly along each product curve $\alpha_{\lambda_1}^{-1} \cdot \alpha_{\lambda_k}$. At $x_{\lambda_k} = x_{\lambda_1}$, we obtain the geodesic $[y_{\lambda_k}, x_{\lambda_k}]^{-1}$. $\{y_{\lambda_k}\}$ being however infinite, it follows that the vector $v_0'$ is infinitely variant under $G(R_0, x_{\lambda_1})$. So, $G(R_0, x_{\lambda_1})$, i. e., $G(R_0)$ is infinite.

From this proof, it is seen that if we put $[y_0, x_0]^{-1} = g(x_0, v_0, c)$, the vector $v_0$ is infinitely variant under $G(R_0, x_0)$.

2) Take any $S$-submanifold $S$. Let $S^0$ be the maximal subset of $S$ such that each point $x$ satisfies $e_S(x) = 0$. In our case, $S^0 \neq S$. For any $x_0 \in S - S^0$, $e_S(x_0) > 0$ and $G(R_0', x_0)$, $R_0' = R(x_0)$, is infinite by 1). Take a normal $S$-radius $a$ at $x_0$. Let $V^0$ be the set of all vectors in $T_S(x_0)$ with lengths $< a$, infinitely variant under $G(R_0', x_0)$. Then, $\exp_S V^0 = N_S(x_0 ; a) \cap S^0$. For, it is obvious that $\exp_S V^0 \subset N_S(x_0 ; a) \cap S^0$. Take any $y_0 \in N_S(x_0 ; a) \cap S^0$. If $g(x_0, v_0, \delta)$ is the geodesic $[x_0, y_0]$ in $N_S(y_0 ; a)$, the vector $v_0$ is infinitely variant under $G(R_0', x_0)$ by 1). I. e., $y_0 \in \exp_S V^0$. So, our assertion is true. Here, by using Lemma 4. 6, it is shown that $\exp_S V^0$ is a connected open subset of $N_S(x_0 ; a)$ and its closure in $N_S(x_0 ; a)$ contains $x_0$. Accordingly, in $S$ the closure of $S^0$ is $S$.

Moreover, $S^0$ is open in $S$. For, if $y_0 \in S^0$ is not an inner point of $S^0$ relative to $S$, we can find $x_0 \in S - S^0$ and a normal $S$-radius $a$ at $x_0$ such that $y_0 \in N_S(x_0 ; a)$. However, $N_S(x_0 ; a) \cap S^0$ is a connected open subset of $N_S(x_0 ; a)$ containing $y_0$. This is a contradiction. So, $S^0$ is open in $S$.

Next, we prove that $S^0$ is a connected subset of $S$. In fact take two points $x_1, x_2 \in S^0$. Let $\alpha$ be a curve in $S$ joining $x_1$ to $x_2$. Cover $\alpha$ by a finite number of normal $S$-neighborhoods $N_S(y_\lambda ; a_\lambda)$, $y_\lambda \in \alpha$ $(\lambda = 1, 2, \ldots\ldots, h)$, such that if $y_\lambda \in S^0$ for some $\lambda$, $N_S(y_\lambda ; a_\lambda) \subset S^0$. Then, by the properties above, we can verify that $x_1, x_2$ are joined by a curve in $S^0$. So, $S^0$ is connected.

3) By Lemmas 3.3 and 4.1, $M^0$ is regarded as the union set of $\{R(x) | x \in S^0\}$ In other words, $M^0$ is the maximal subset of $M$ in which each point $x$ is a limit point of $I(x)$ relative to $S(x)$. From 2), it follows that $M^0$ is a connected open submanifold of $M$ whose closure is $M$. Therefore $M$ is of almost $R$-

clustered type with kernel $M^0$.

Summing up Theorems 1, 2, we have

THEOREM 3. *M is of almost R-fibred type or almost R-clustered type.*

REMARK 2. There exist $RS$-manifolds of the following respective type: $R$-fibred type; almost $R$-fibred type (not $R$-fibred type); $R$-clustered types; almost $R$-clustered type (not $R$-clustered type).

5. **Fundamental groups and structures.** Take any $x_0 \in M$ and put $R_0 = R(x_0)$, $S_0 = S(x_0)$. Let $i_R\colon R_0 \to M$ be the inclusion map. Let $i_R^*\colon \pi_1(R_0, x_0) \to \pi_1(M, x_0)$ be the homomorphism induced by the map $i_R$ ([4], p. 75). It is already known that the *map* $i_R^*$ *is into-isomorphic* ([2], p. 22). We denote the image $i_R^* \cdot \pi_1(R_0, x_0)$ by $i\pi_1(R_0, x_0)$. This is the subgroup of $\pi_1(M, x_0)$. Let $U(M)$ denote the universal covering manifold of $M$. Let $p$ denote the covering map. So $U(M)$ becomes naturally an $RS$-manifold of dimension $n$. Take a point $\tilde{x}_0 \in p^{-1}(x_0)$. For the $R$-, $S$-submanifolds $R(\tilde{x}_0)$, $S(\tilde{x}_0)$ of $U(M)$, $R(\tilde{x}_0) \times S(\tilde{x}_0)$ is a Riemannian manifold of class $C^1$. Then the following theorem is well-known: *There is the isometric homeomorphism*

$$j\colon R(\tilde{x}_0) \times S(\tilde{x}_0) \to U(M)$$

*of class* $C^2$ *such that* $j(\tilde{x}, \tilde{x}_0) = \tilde{x}$ *for all* $\tilde{x} \in R(\tilde{x}_0)$ *and* $j(\tilde{x}_0, \tilde{x}) = \tilde{x}$ *for all* $\tilde{x} \in S(\tilde{x}_0)$ [1]. Hence, $j(R(\tilde{x}_0), \tilde{x}) = R(\tilde{x})$ for $\tilde{x} \in S(\tilde{x}_0)$ and $j(\tilde{x}, S(\tilde{x}_0)) = S(\tilde{x})$ for each $\tilde{x} \in R(\tilde{x}_0)$. Such a map is always denoted by $j$. The fact above shows that $U(M)$ is completely decomposed with respect to the $R$-, $S$-submanifolds. Now, using these notations, let us prove the following lemma.

LEMMA 5.1. 1) *The R-submanifold* $R(\tilde{x}_0)$ *of* $U(M)$ *is a universal covering manifold of* $R_0$, *where* $p_R = p\,|\,R(\tilde{x}_0)$ *is the covering map.*

2) *The subgroups* $i\pi_1(R_0, x_0)$, $i\pi_1(S_0, x_0)$ *have no common element except the identity of* $\pi_1(M, x_0)$.

3) *If* $I(x_0)$ *is infinite,* $\pi_1(M)$ *is infinite.*

PROOF. To prove 1) it suffices to show $p(R(\tilde{x}_0)) = R_0$, $R(\tilde{x}_0)$ being simply-connected. For $\tilde{x} \in R(\tilde{x}_0)$, take a curve $\tilde{\alpha}$ in $R(\tilde{x}_0)$ joining $\tilde{x}_0$ to $\tilde{x}$. Then we can see $p(\tilde{\alpha}) \subset R_0$. Hence $p(R(\tilde{x}_0)) \subset R_0$. Conversely for $x \in R_0$ if we take a curve $\beta$ in $R_0$ joining $x_0$ to $x$, we can find the curve $\tilde{\beta}$ in $R(\tilde{x}_0)$ with the initial point $\tilde{x}_0$ such that $p(\tilde{\beta}) = \beta$. Hence, $p(R(\tilde{x}_0)) \supset R_0$. So, $p(R(\tilde{x}_0)) = R_0$. I. e., 1) is proved.

To prove 2) suppose that $i\pi_1(R_0, x_0)$, $i\pi_1(S_0, x_0)$ have a common element

$A \in \pi_1(M, x_0)$ which is not the identity. Let $\alpha_R$, $\alpha_S$ be two closed curves in $R_0$, $S_0$ respectively, starting from $x_0$ and representing $A$. Then the curves $\tilde{\alpha}_R$, $\tilde{\alpha}_S$ starting from $\tilde{x}_0$ such that $p(\tilde{\alpha}_R) = \alpha_R$, $p(\tilde{\alpha}_S) = \alpha_S$, must have the same terminal point. Moreover, this point is not $\tilde{x}_0$, and $\tilde{\alpha}_R \subset R(\tilde{x}_0)$, $\tilde{\alpha}_S \subset S(\tilde{x}_0)$. This contradicts with the fact that $U(M)$ is completely decomposed. So, 2) is true.

To prove 3) let us denote $I(x_0)$ by $\{x_\lambda \mid \lambda = 0, 1, 2, \ldots\ldots\}$, $I(x_0)$ being countable by Lemma 3. 3. For each $\lambda$, take $\tilde{x}_\lambda \in p^{-1}(x_\lambda) \cap R(\tilde{x}_0)$. As $x_0 \in S(x_\lambda)$, $p^{-1}(x_0) \cap S(\tilde{x}_\lambda)$ is non-empty. However, all of $\tilde{x}_\lambda$ are distinct from one another. Hence all of $S(\tilde{x}_\lambda)$ are distinct from one another by the fact that $U(M)$ is completely decomposed. Accordingly $p^{-1}(x_0)$ is infinite and so $\pi_1(M)$ is infinite.

THEOREM 4. *In $M$ suppose that $\pi_1(M)$ is finite. Then $M$ is of almost $R$-fibred type and further almost $S$-fibred type.*

PROOF. For any $x_0 \in M$, $I(x_0)$ is finite by Lemma 5. 1. Hence, $e_R(x_0) > 0$ and $e_S(x_0) > 0$. By Lemma 4. 1 and Theorem 1, our theorem is evident.

REMARK 3. There exist $RS$-manifolds, whose fundamental groups are finite, of the following respective types: $R$-fibred type and further $S$-fibred type; almost $R$-fibred type (not $R$-fibred type) and further $S$-fibred type; $R$-fibred type and further almost $S$-fibred type (not $S$-fibred type).

In $M$ suppose that all the $R$-submanifolds are simply-connected. Moreover if $M$ is of almost $R$-fibred type, we have

LEMMA 5. 2. *$M$ is of $R$-fibred type.*

PROOF For any $R$-submanifold $R$, $G(R)$ consists of the identity only by Lemma 4. 2. Hence, all the $R$-submanifolds are $R$-maximal. As $M$ satisfies the assumption of Theorem 1, $M$ is of $R$-fibred type.

THEOREM 5. *In $M$ suppose that the order of $\pi_1(M)$ is finite and prime. Then $M$ is of one of the following three structures:*

1) *$R$-fibred type, where all the $R$-submanifolds are simply-connected and $\pi_1(S_0)$ for at least one $S$-submanifold $S_0$ is isomorphic to $\pi_1(M)$.*

2) *$S$-fibred type, where all the $S$-submanifolds are simply-connected and $\pi_1(R_0)$ for at least one $R$-submanifold $R_0$ is isomorphic to $\pi_1(M)$.*

3) *$R$-fibred type and further $S$-fibred type, where all the $R$-, $S$-submanifolds are simply-connected.*

PROOF. For an $S$-submanifold $S_0$, suppose that $S_0$ is not simply-connected.

Then it follows that $i\pi_1(S_0, x_0) = \pi_1(M, x_0)$ for any $x_0 \in S_0$. So, $R(x_0)$ is simply-connected by Lemma 5. 1. As $x_0$ is any point of $S_0$, all the $R$-submanifolds are simply-connected by Lemma 3. 3. By Theorem 4 and Lemma 5. 2, $M$ is of $R$-fibred type. So, $M$ is of the structure 1). Similarly, if we suppose that an $R$-submanifold $R_0$ is not simply-connected, we have the structure 2).

Finally, suppose that all the $R$-, $S$-submanifolds are simply-connected. Then, by Theorem 4 and Lemma 5. 2, $M$ is of the structure 3). This completes the proof of our theorem.

REMARK 4. There exist $RS$-manifolds, in which the orders of the fundamental groups are finite and prime, such that the conditions 1), 2), 3) of Theorem 5 hold good respectively. (Especially, for a model in the case 3) see § 4, [2].)

THEOREM 6. *In $M$ suppose that $\pi_1(M)$ is infinite cyclic. Then $M$ is of one of the following structures:*

1) *$R$-fibred type, where all the $R$-submanifolds are simply-connected and $\pi_1(S_0)$ for at least one $S$-submanifold $S_0$ is infinite cyclic.*

2) *$S$-fibred type, where all the $S$-submanifolds are simply-connected and $\pi_1(R_0)$ for at least one $R$-submanifold $R_0$ is infinite cyclic.*

3) *All the $R$-, $S$-submanifolds are simply-connected.*

PROOF. For any $R$-submanifold $R$, $\pi_1(R)$ is the group of identity only or an infinite cyclic group, being isomorphic into $\pi_1(M)$. This holds good for any $S$-submanifold, too. Moreover, there is not a pair of $R$-, $S$-submanifolds whose fundamental groups both are infinite cyclic. For, if such a pair $(R_0, S_0)$ does exist, we can find $A \in \pi_1(M, x_0)$, $x_0 \in R_0 \cap S_0$, which is not the identity and belongs to both of $i\pi_1(R_0, x_0)$ and $i\pi_1(S_0, x_0)$. This is contrary to Lemma 5. 1. So, by Lemma 3. 3 the following three cases are considered:

a) All the $R$-submanifolds are simply-connected and $\pi_1(S_0)$ for at least one $S$-submanifold $S_0$ is infinite cyclic.

b) All the $S$-submanifolds are simply-connected and $\pi_1(R_0)$ for at least one $R$-submanifold $R_0$ is infinite cyclic.

c) All the $R$-, $S$-submanifolds are simply-connected.

The case c) being the same as 3), it suffices to prove that $M$ in the case a) is of $R$-fibred type. To prove this, take any $x_0 \in S_0$. Let $\alpha$ be a closed curve with endpoint $x_0$ which is a geodesic arc representing a generator of $\pi_1(M, x_0)$. As $i\pi_1(S_0, x_0)$ is infinite cyclic, we can find an integer $m > 0$ such that the product curve $\alpha^m$ represents a generator of $i\pi_1(S_0\,;\,x_0)$. Let $\tilde{x}_0 \in U(M)$ be a point of $p^{-1}(x_0)$. Let $\widetilde{\beta}_1$ be the curve starting from $\tilde{x}_0$ such that $p(\widetilde{\beta}_1) = \alpha^m$. Here, the terminal point $\tilde{x}_m$ of $\widetilde{\beta}_1$ is contained in the $S$-submanifold $S(\tilde{x}_0)$ of $U(M)$ and $p^{-1}(x_0) \cap \tilde{\beta}_1$ consists of $m+1$ points. Accordingly, we can find a part $C_R(\tilde{x}_0\,;\,c) \subset R(\tilde{x}_0)$ such that $j(C_R(\tilde{x}_0\,;\,c) \times S(\tilde{x}_0)) \supset p^{-1}(x_0) \cap \tilde{\beta}_1$. Next $\tilde{\beta}_2$ be the curve startinng from $\tilde{x}_m$ such that $p(\tilde{\beta}_2) = \alpha^m$. The terminal point of $\tilde{\beta}_2$ is also contained in $S(\tilde{x}_0)$. Hence, $j(C_R(\tilde{x}_0\,;\,c) \times S(\tilde{x}_0)) \supset p^{-1}(x_0) \cap \tilde{\beta}_2$. Thus we can verify that $j(C_R(\tilde{x}_0\,;\,c) \times S(\tilde{x}_0)) \supset p^{-1}(x_0)$.

Now suppose that $e_S(x_0) = 0$. For any constant $d > 0$, we can find a countable subset $\{x_\lambda \,|\, \lambda = 0, 1, 2, \ldots\ldots\}$ of $I(x_0)$ such that $d_S(x_0, x_\lambda) < d$. For each $x_\lambda$, there is $\tilde{x}_\lambda \in S(\tilde{x}_0) \cap p^{-1}(x_\lambda)$ where $d_S(\tilde{x}_0, \tilde{x}_\lambda) < d$. By Lemma 5. 1, $R(\tilde{x}_\lambda)$ contains a point of $p^{-1}(x_0)$. Here, all of $\tilde{x}_\lambda$ are distinct from one another. Hence, all of $R(\tilde{x}_\lambda)$ are distinct from one another by the fact that $U(M)$ are completely decomposed. Accordingly, a part $j(C_R(\tilde{x}_0\,;\,c) \times C_S(\tilde{x}_0\,;\,d))$ of $U(M)$ contains an infinite subset of $p^{-1}(x_0)$. It being however compact, this contradicts with the property of covering. So, $e_R(x_0) > 0$. Since $x_0$ is any point of $S_0$ and the $R$-submanifolds of $M$ are all simply-connected, $M$ is of $R$-fibred type by Lemmas 4. 1, 5. 2, and Theorem 1. This completes the proof of our theorem.

REMARK 5. There exist $RS$-manifolds, whose fundamental groups are infinite cyclic, such that the conditions 1), 2), 3) of Theorem 6 hold good respectively

In Euclidean $d$-space $E^d$ suppose that there are given a point set $Z = \{P_\lambda \,|\, \lambda = \text{integer}\}$ and a congruent transformation $T$ leaving $P_0$ fixed, such that the vector $\overrightarrow{P_\lambda P_{\lambda+1}}$ is equal to the vector $T^\lambda \cdot \overrightarrow{P_0 P_1}$ for each $\lambda$. ($P_\lambda$'s are not necessarily distinct from one another.) Then we have

LEMMA 5. 3. *There are two cases where $Z$ is bounded or unbounded. In the latter case, $P_0$ is not limit point of $Z$.*

PROOF. We take an orthogonal coordinate system in $E^d$ with origin $P_0$. where $T$ is represented by the following matrix:

$$\begin{pmatrix} E_1 & & & & & \\ & -E_2 & & & 0 & \\ & & (\theta_1) & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & 0 & & & & \cdot \\ & & & & & (\theta_k) \end{pmatrix}$$

where $E_1$, $E_2$ denote the unit matrices of degrees $r_1$, $r_2$ respectively and

$$(\theta_\nu) = \begin{pmatrix} \cos\theta_\nu & -\sin\theta_\nu \\ \sin\theta_\nu & \cos\theta \end{pmatrix}$$

for $0 < \theta_\nu < \pi$ ($\nu = 1, 2, \ldots\ldots, k$ ; $r_1 + r_2 + 2k = d$). So, the matrix representation of $T^\lambda$ is immediately obtained. Put $v_\lambda = \overrightarrow{P_\lambda P_{\lambda+1}}$. Let $(v_\lambda^m)$ denote the vector $v_\lambda$ and let $(P_\lambda^m)$ denote the point $P_\lambda$, where $m = 1, 2, \ldots\ldots, d$. Here, if $\lambda > 0$, $P_\lambda^m = v_0^m + \ldots\ldots + v_{\lambda-1}^m$ ; if $\lambda < 0$, $P_\lambda^m = -(v_{-1}^m + \ldots\ldots + v_\lambda^m)$. Then for all $\lambda$, we can verify the following facts: a) $P_\lambda^\beta = \lambda\ v_0^\beta$ ($\beta = 1, \ldots\ldots, r_1$), b) $|P_\lambda^\gamma| < N$ ($\gamma = r_1 + 1, \ldots\ldots, d$) where $N$ is a constant independent of $\gamma$, $\lambda$. Hence, in the case $r_1 \neq 0$ ; if $v_0^\beta = 0$ for all $\beta$, $Z$ is bounded, and if $v_0^\beta \neq 0$ for some $\beta$, $Z$ is unbounded and $P_0$ is not limit point of $Z$. Next, in the case $r_1 = 0$, $Z$ is bounded. So our lemma holds good.

THEOREM 7. *In $M$ suppose that $\pi_1(M)$ is infinite cyclic and that all the $R$-submanifolds are Euclidean space forms. Then $M$ is of $R$-fibred type or $S$-fibred type, both having simply-connected fibres.*

PROOF. It suffices to prove our theorem in the case 3) of Theorem 6. Accordingly, suppose that all the $R$-, $S$-submanifolds are simply-connected. For $x_0 \in M$ we take a closed curve $\alpha$ issuing from $x_0$ which is a geodesic arc representing a generator of $\pi_1(M, x_0)$. Let $\widetilde{\beta}$ be the curve in $U(M)$ such that $p(\widetilde{\beta})$ is the product curve

$$\cdots\cdots\alpha\cdot\cdots\cdots\alpha\cdot\alpha\cdot\alpha\cdot\cdots\cdots\alpha\cdot\cdots\cdots.$$

Then $p^{-1}(x_0) \subset \widetilde{\beta}$. We denote all of the points of $p^{-1}(x_0)$ by $\tilde{x}_\lambda$($\lambda$ = integer), where the subarc of $\widetilde{\beta}$ from $\tilde{x}_\lambda$ to $\tilde{x}_{\lambda+1}$ is mapped to the arc $\alpha$ by $p$. Any $\tilde{x} \in U(M)$ is represented by $j(P, Q)$ where $P \in R(\tilde{x}_0)$, $Q \in S(\tilde{x}_0)$. Define a map

$$f: U(M) \to R(\tilde{x}_0) \text{ by } f(\tilde{x}) = P.$$

We put $P_\lambda = f(\tilde{x}_\lambda)$. The curve $f(\widetilde{\beta})$ contains $P_\lambda$ and is a broken line in the Euclidean $r$-space $R(\tilde{x}_0)$. (Note that in our case all the $R$-submanifolds are Euclidean $r$-spaces.) Moreover we can see that the point set $Z = \{P_\lambda | \lambda = \text{integer}\}$ satisfies the condition of Lemma 5.3. Here $T$ is the same as the congruent transformation in $T_R(x_0)$ which is induced from the element associated with $\alpha$ (or $\alpha^{-1}$) of the homogeneous holonomy group of $M$ at $x_0$.

1) The case where $Z$ is bounded. Take a part $C_R(\tilde{x}_0 ; c)$ of $R(\tilde{x}_0)$ which contains $Z$. Hence, a part $j(C_R(\tilde{x}_0 : c) \times S(\tilde{x}_0))$ of $U(M)$ contains $p^{-1}(x_0)$. Then $e_S(y_0) > 0$, for any $y_0 \in S(x_0)$. For, otherwise, we can find a countable subset $\{y_\lambda | \lambda = 0, 1, 2, \ldots\ldots\}$ of $I(y_0)$ such that $d_S(y_0, y_\lambda) < d$ for a constant $d > 0$. For each $y_\lambda$ there is $\tilde{y}_\lambda \in S(\tilde{x}_0) \cap p^{-1}(y_\lambda)$ such that $d_S(\tilde{y}_0, \tilde{y}_\lambda) < d$. By Lemma 5.1, $R(\tilde{y}_\lambda)$ contains a point of $p^{-1}(y_0)$. Here all of $\tilde{y}_\lambda$ are distinct from one another. Hence all of $R(\tilde{y}_\lambda)$ are distinct from one another by the fact that

$U(M)$ is completely decomposed. Accordingly a part $j(C_R(\tilde{x}_0\ ;\ c) \times C_S(\tilde{y}_0\ ;\ d))$ of $U(M)$ contains an infinite subset of $p^{-1}(y_0)$. It being however compact, this contradicts with the property of covering. So, $e_S(y_0) > 0$ for any $y_0 \in S(x_0)$. By Lemmas 4. 1, 5. 2, and Theorem 1, $M$ is of $R$-fibred type.

2) The case where $Z$ is unbounded. Then by Lemma 5. 3, $P_0$ is not limit point of $Z$. So there is a part $C_R(\tilde{x}_0; c)$ of $R(\tilde{x}_0)$ such that $R(\tilde{x}_0) - C_R(\tilde{x}_0\ ;\ c) \supset Z$. Take a positive constant $d < c/3$. By using the property of covering, we can see that $e_R(x) > 0$ for any $x \in C_R(x_0\ ;\ d)$. From Lemma 4. 1, Theorems 1 and 2, and Lemma 5. 2, $M$ is of $S$-fibred type. This completes the proof.

REMARK 6. There exist $RS$-manifolds, whose fundamental groups are infinite cyclic and whose $R$-submanifolds are Euclidean space forms, of the following respective types: $R$-fibred type (not $S$-fibred type) ; $S$-fibred type (not $R$-fibred type) ; $R$-fibred type and further $S$-fibred type.

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