RIEMANN METHODS FOR EVALUATION OF SERIES*

RALPH PALMER AGNEW

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1. Introduction. Two similar but different families of methods for evaluation of series stem from work of Riemann on trigonometric series. Let p be a positive integer, and let $u_0 + u_1 + \ldots\ldots$ be a series of real or complex terms with partial sums $s_0, s_1, \ldots\ldots$ The series is said to be evaluable $R(p)$ to L if the series in

$$(1.1) \qquad \sigma_1(p, t) = \sum_{k=0}^{\infty} \left(\frac{\sin kt}{kt} \right)^p u_k$$

converges over some interval $0 < t < t_0$ and $\sigma_1(p, t) \to L$ as $t \to 0$. Here and elsewhere, $(\sin kt)/kt$ is interpreted to have the value 1 when $k = 0$. The series is said to be evaluable R_p to L if the series in

$$(1.2) \qquad \sigma_2(p, t) = c_p t \sum_{k=0}^{\infty} \left(\frac{\sin kt}{kt} \right)^p s_k$$

converges over some interval $0 < t < t_0$ and $\sigma_2(p, t) \to L$ as $t \to 0$. The constant c_p in (1.2) is defined so that

$$(1.21) \qquad \lim_{t \to 0} c_p t \sum_{k=0}^{\infty} \left(\frac{\sin kt}{kt} \right)^p = 1$$

and hence

$$(1.22) \qquad c_p \int_0^{\infty} \left(\frac{\sin x}{x} \right)^p dx = 1.$$

In particular, $c_1 = 2/\pi$. It is well known that $R(p)$ and R_p are not regular when $p = 1$ but are regular when $p \geqq 2$. The method $R(1)$ is sometimes called thc Lebesgue method.

The methods $R(p)$ and R_p have been the subject of many investigations which show that there are respects in which they have identical properties. It has often happened that one author has proved that one of $R(p)$ and R_p has a particular property, and then the same or another author has proved that the other method has the same property. It seems that similarities

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between $R(p)$ and R_p have been emphasized much more than differences between $R(p)$ and R_p. It has, however, been shown respectively by Kuttner [24] Marcinkiewicz [28], and Kuttner [24] that if p is 1 or 2 or 3, then neither of $R(p)$ and R_p includes the other. See also Hardy and Rogosinski [14], [15].

In order to exhibit differences as well as similarities between $R(p)$ and R_p, this paper investigates the behaviors of the $R(p)$ and R_p transforms of series Σu_n satisfying the Tauberian condition

$$\limsup_{n\to\infty} |nu_n| \leqq M < \infty. \tag{1.3}$$

Section 2 gives optimal relations between the $R(p)$ transforms and partial sums of series satisfying (1. 3). Section 3 gives lemmas on R_p transforms, and section 4 gives optimal relations between the R_p transforms and partial sums of series satisfying (1. 3). Section 5 gives optimal relations between the $R(p)$ and R_p transforms of series satisfying (1. 3). In section 6 we suppose that T_p is $R(p)$ or R_p and study various questions concerning T_pC_r, $T_pC_r^{-1}$ and relations between T_p and C_r where C_r is the Cesàro transformation of order r.

In connection with section 2, we observe that Zygmund [57] has obtained an optimal relation between the partial sums and $R(2)$ transforms of series having bounded partial sums. While he formulated his result in a different way in terms of real series, his work shows that if z is a real (or complex) number then the inequality

$$\limsup_{t\to\infty} |\sigma_1(2, t) - z| \leqq \frac{e^2-5}{2} \limsup_{n\to\infty} |s_n - z| \tag{1.4}$$

holds whenever Σu_n is a real (or complex) series having bounded partial sums. Moreover the numerical constant in (1. 4) is the least for which (1. 4) is always valid.

We now make some remarks showing one way in which $R(1)$ and R_1 can be originated in mathematical analysis and brought into contact with problems involving trigonometric series. Suppose that the series Σu_n is such that, as a standard notation

$$\sum_{k=0}^{\infty} (\cos kt) u_k \sim f(t) \tag{1.5}$$

indicates, the series in (1. 5) is the Fourier series of a function $f(t)$ which is integrable (Lebesgue) over $0 \leqq t \leqq 2\pi$. Then, since Fourier series can be integrated termwise, integration gives

$$\sigma_1(1, t) = \sum_{k=0}^{\infty} \frac{\sin kt}{kt} u_k = \frac{1}{t}\int_0^t f(x)\,dx. \tag{1.51}$$

It is now easy to formulate various conditions under which Σu_n is evaluable $R(1)$ to s. Suppose next that the series Σu_n converges to s and is such that the series in

$$\sum_{k=0}^{\infty} (\cos kt)(s_k - s) \sim g(t) \tag{1.6}$$

is the Fourier series of a function $g(t)$ integrable over $(0, 2\pi)$. Then integration gives

$$t \sum_{k=0}^{\infty} \frac{\sin kt}{kt} (s_k - s) = \int_0^t g(x)dx = o(1) \tag{1.61}$$

as $t \to 0$. Since

$$\frac{2t}{\pi} \sum_{k=0}^{\infty} \frac{\sin kt}{kt} = 1 + o(1), \tag{1.62}$$

we can multiply (1.61) by $2/\pi$ and conclude that

$$\sigma_2(1, t) = \frac{2t}{\pi} \sum_{k=0}^{\infty} \frac{\sin kt}{kt} s_k = s + o(1) \tag{1.63}$$

and hence that Σu_n is evaluable R_1 to s. It must be recognized, however that the full scopes of the classes of series evaluable $R(1)$ and R_1 cannot be discovered by studying series Σu_n for which the series in (1.5) and (1.6) are Fourier series of integrable functions. In fact the condition $\lim u_n = 0$ is not necessary for evaluability $R(1)$ and R_1. The simplest example illustrating this fact is the following. The elementary formulas

$$\frac{t}{2} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kt}{k}, \qquad \frac{\pi - 2t}{4} = \sum_{k=1}^{\infty} \frac{\sin 2kt}{2k}, \tag{1.7}$$

which are valid when $0 < t < \pi$, show that for series the $1 - 1 + 1 - 1 + \ldots\ldots$ for which $u_n = (-1)^n$, $s_{2n} = 1$, and $s_{2n+1} = 0$ we have $\sigma_1(1, t) = 1/2$ and $\sigma_2(1, t) = 1/2 + t/\pi$; hence this classic divergent series is evaluable $R(1)$ and R_1 to the classic value $1/2$.

2. $R(p)$ transforms and partial sums. Let $n = n(\alpha)$ and $t = t(\alpha)$ be positive functions, defined for $\alpha > 0$, such that $n(\alpha)$ is an integer for each α and $n(\alpha) \to \infty$ and $t(\alpha) \to 0$ as $\alpha \to \infty$. It is the purpose of this section to determine the least constant A, which depends upon p and the functions $n(\alpha)$ and $t(\alpha)$ and which is either finite or $+ \infty$, such that

$$\limsup_{\alpha \to \infty} |\sigma_1(p, t) - s_n| \leqq A \limsup_{n \to \infty} |nu_n| \tag{2.1}$$

whenever Σu_n is a series for which $\limsup |nu_n| < \infty$. In particular, we characterize the pairs of functions $n(\alpha)$ and $t(\alpha)$ for which A is finite and those for which A attains its minimum value A_0.

When $|nu_n| < M$, the series in (1. 1) which defines the $R(p)$ transform is necessarily convergent because (1. 1) can be put in the form

$$\sigma_1(p, t) = u_0 + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{\sin kt}{kt}\right)^p ku_k \tag{2.11}$$

and the last series is dominated for each $t > 0$ by the convergent series $\Sigma M t^{-p} k^{-2}$.

Setting $x_k = ku_k$,

$$a_k(t) = -\frac{1}{k}\left[1 - \left(\frac{\sin kt}{kt}\right)^p\right], \qquad 1 \leqq k \leqq n \tag{2.2}$$

and

$$a_k(t) = \frac{1}{k}\left(\frac{\sin kt}{kt}\right)^p, \qquad k > n \tag{2.21}$$

we find from (1. 1) that

$$\sigma_1(p, t) - s_n = \sum_{k=1}^{\infty} a_k(t) x_k, \tag{2.22}$$

the series being convergent when x_n is bounded. Use of lemmas set forth by Agnew [2, section 7] shows that the least constant A for which (2. 1) holds is the constant A defined by

$$A = \limsup_{\alpha\to\infty} F_1(\alpha) \tag{2.3}$$

where

$$F_1(\alpha) = \sum_{k=1}^{\infty} |a_k(t)| \tag{2.31}$$

and hence

$$F_1(\alpha) = \sum_{k=1}^{n} \frac{1}{k}\left[1 - \left(\frac{\sin kt}{kt}\right)^p\right] + \sum_{k=n+1}^{\infty} \frac{1}{k}\left|\frac{\sin kt}{kt}\right|^p. \tag{2.32}$$

It turns out that the behavior as $\alpha \to \infty$ of $F_1(\alpha)$ depends only upon the product $q(\alpha)$ defined by

$$q(\alpha) = n(\alpha)t(\alpha). \tag{2.33}$$

Considering first the case in which

$$0 < q_1 = \liminf_{\alpha\to\infty} q(\alpha) \leqq \limsup_{\alpha\to\infty} q(\alpha) = q_2 < \infty \tag{2.34}$$

we put (2. 32) in the form

$$(2.35)\qquad F_1(\alpha) = \sum_{t\leqq kt\leqq q} \frac{1}{kt}\left[1 - \left(\frac{\sin kt}{kt}\right)^p\right] t + \\ + \sum_{kt>q} \frac{1}{kt}\left|\frac{\sin kt}{kt}\right|^p t.$$

For each α we may set $kt = x_k$ and $t = x_{k+1} - x_k$ to see that the right member of (2.35) is much like a Riemann sum. It is complicated by the fact that the function for which the Riemann sum is formed depends upon the variable q and by the fact that the points $x_k = kt$ are not confined to a finite interval, but it is nevertheless not difficult to show that if (2.34) holds, then, as $\alpha \to \infty$,

$$(2.4)\qquad F_1(\alpha) = o(1) + G_1(q)$$

where

$$(2.41)\qquad G_1(q) = \int_0^q \frac{1}{x}\left[1 - \left(\frac{\sin x}{x}\right)^p\right] dx + \int_q^\infty \frac{1}{x}\left|\frac{\sin x}{x}\right|^p dx.$$

The last terms in (2.35) and (2.41) and the infinite interval of integration cause no trouble because

$$(2.42)\qquad \sum_{kt>q_3} \frac{1}{kt}\left|\frac{\sin kt}{kt}\right|^p t \leqq \sum_{kt>q_3} \frac{1}{(kt)^2} t \leqq \int_{q_3-1} \frac{1}{x^2} dx,$$

$$(2.43)\qquad \int_{q_3}^\infty \frac{1}{x}\left|\frac{\sin x}{x}\right|^p dx \leqq \int_{q_3}^\infty \frac{1}{x^2} dx,$$

and, for each $\varepsilon > 0$, we can choose a constant q_3 such that $q_3 > q_2$ and the last members of (2.42) and (2.43) are less than ε when $0 < t < 1$ and hence when α is sufficiently great.

The function $G_1(q)$ in (2 41) is positive when $q > 0$ and attains its absolute minimum when and only when q is the unique positive number q_0 for which

$$(2.44)\qquad \frac{\sin q_0}{q_0} = \left(\frac{1}{2}\right)^{1/p}.$$

Moreover $G_1(q) \to \infty$ as $q \to 0$, $G_1(q) \to \infty$ as $q \to \infty$, $G_1(q)$ is decreasing over $0 < q < q_0$, and $G_1(q)$ is increasing over $q_0 < q < \infty$. These facts and (2.1) imply that if $n(\alpha)$ and $t(\alpha)$ are such that (2.34) holds, then the least constant A for which (2.1) is valid is finite and is the maximum of $G_1(q_1)$ and $G_1(q_2)$. This maximum is a minimum when and only when $q_1 = q_2 = q_0$. In case lim inf $q(\alpha) = 0$, consideration of the last term in (2.35) shows that lim sup $F_1(\alpha) = \infty$ and hence that $+\infty$ is the least constant A for which (2.1) is valid. In case lim sup $q(\alpha) = \infty$, consideration of the first term in the right member of (2.35) produces the same result. Therefore the least constant A for which (2.1) is valid is finite if and only if the functions $n(\alpha)$ and $t(\alpha)$ are such that

(2.45) $$0 < \liminf_{\alpha\to\infty} n(\alpha)t(\alpha) \leqq \limsup_{\alpha\to\infty} n(\alpha)t(\alpha) < \infty.$$

A consequence of the results which we have obtained is set forth in the following theorem.

THEOREM 2 5. *The constant* $G_1(q_0)$ *determined by* (2.41) *and* (2.44) *is the least constant with the following property. There exist functions* $n(\alpha)$ *and* $t(\alpha)$ *such that* $n(\alpha)\to\infty$, $t(\alpha)\to 0$, *and*

(2.51) $$\limsup_{\alpha\to\infty} |\sigma_1(p,t) - s_n| \leqq G_1(q_0) \limsup_{n\to\infty} |nu_n|$$

whenever Σu_n *is a series for which* $\limsup |nu_n| < \infty$. *Moreover, the functions* $n(\alpha)$ *and* $t(\alpha)$ *are such that* (2.51) *is valid if and only if*

(2.52) $$\lim_{\alpha\to\infty} n(\alpha)t(\alpha) = q_0.$$

Our results also show that for each $q > 0$ the inequalities

(2.61) $$\limsup_{n\to\infty} |\sigma_1(p, q/n) - s_n| \leqq G_1(q) \limsup_{n\to\infty} |nu_n|$$

and

(2.62) $$\limsup_{t\to 0} |\sigma_1(p,t) - s_{[q/t]}| \leqq G_1(q) \limsup_{n\to\infty} |nu_n|$$

are optimal inequalities, and that the right members are minimal when $q = q_0$. It is not difficult to modify the above work to obtain the following theorem.

THEOREM 2.7. *In order that functions* $n(\alpha)$ *and* $t(\alpha)$ *be such that*

(2.71) $$\lim_{\alpha\to\infty} |\sigma_1(p,t) - s_n| = 0$$

whenever Σu_n *is a series for which* $\lim nu_n = 0$, *it is necessary and sufficient that*

(2.72) $$0 < \liminf_{\alpha\to\infty} n(\alpha)t(\alpha) \leqq \limsup_{\alpha\to\infty} n(\alpha)t(\alpha) < \infty.$$

It is a corollary of Theorem 2.7 that if $\lim nu_n = 0$, then Σu_n is evaluable $R(p)$ to s if and only if it converges to s. This corollary is a Tauberian theorem of classical type, and numerous theorems of this type are given in papers cited in the bibliography.

The appropriateness of the Tauberian condition $\limsup |nu_n| < \infty$ which we have employed is shown by the following facts. If $\phi(n)$ is a nonnegative function of n for which $\lim n/\phi(n) = \infty$, then $+\infty$ is the least constant A^* such that

(2.8) $$\limsup_{\alpha\to\infty} |\sigma_1(p,t) - s_n| \leqq A^* \limsup_{n\to\infty} |\phi(n)u_n|$$

whenever Σu_n is a series for which $\limsup |\phi(n)u_n| < \infty$. To prove this, we follow the procedure used above to find that $A^* = \limsup F_1^*(\alpha)$ where

$$F_1^*(\alpha) = \sum_{t \leqq kt \leqq q} \frac{k}{\phi(k)} \frac{1}{kt} \left[1 - \left(\frac{\sin kt}{kt} \right)^p \right] t + \sum_{kt > q} \frac{k}{\phi(k)} \frac{1}{kt} \left| \frac{\sin kt}{kt} \right|^p t. \tag{2.81}$$

For each positive constant M there is an index $k(M)$ such that $k/\phi(k) > M$ when $k > k(M)$. It therefore follows from (2. 81) that, as $\alpha \to \infty$, $F^*(\alpha) \geqq o(1) + MF_1(\alpha)$ where $F_1(\alpha)$ is defined by (2. 35). Since limsup $F_1(\alpha) > 0$, it follows that $A^* = \infty$. If on the other hand $\lim n/\phi(n) = 0$, then there exist functions $n(\alpha)$ and $t(\alpha)$ such that (2. 8) holds with $A^* = 0$.

3. R_p transforms. For use later, we need a formula which gives the R_p transform $\sigma_2(p, t)$ defined by (1. 2) in terms of the terms u_n instead of the partial sums s_n of a given series Σu_n for which lim sup$|nu_n| < \infty$. While we could use the formula of the first lemma of this section, it is much more convenient to use the formula of the second lemma which is obtained with the aid of the first lemma. If Σu_n is a series for which lim sup $|nu_n| < \infty$, then $\Sigma |u_n|^2 < \infty$. Since the converse is not true, the following lemma has more generality than we need.

LEMMA 3. 1. *If Σu_n is a series for which $\Sigma |u_n|^2 < \infty$ then the series in*

$$\sigma_2(p, t) = c_p t \sum_{k=0}^{\infty} \left(\frac{\sin kt}{kt} \right)^p s_k \tag{3.11}$$

converges when $0 < t < 2\pi$ and

$$\sigma_2(p, t) = c_p t \sum_{k=0}^{\infty} \left[\sum_{j=k}^{\infty} \left(\frac{\sin jt}{jt} \right)^p \right] u_k. \tag{3.12}$$

To prove this lemma, let Σu_n be a series for which $\Sigma |u_n|^2 = M^2 < \infty$. In case the series in

$$\sum_{j=1}^{\infty} \sum_{k=1}^{j} \left(\frac{\sin jt}{jt} \right)^p u_k = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \left(\frac{\sin jt}{jt} \right)^p u_k \tag{3.13}$$

are convergent and the equality holds, the conclusion of the lemma follows because the series in

$$\begin{aligned} \sigma_2(p, t) &= c_p t \left[\sum_{j=0}^{\infty} \left(\frac{\sin jt}{jt} \right)^p u_0 + \sum_{j=1}^{\infty} \left(\frac{\sin jt}{jt} \right)^p (s_j - u_0) \right] \\ &= c_p t \left[\sum_{j=0}^{\infty} \left(\frac{\sin jt}{jt} \right)^p u_0 + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \left(\frac{\sin jt}{jt} \right)^p u_k \right] \\ &= c_p t \left[\sum_{j=0}^{\infty} \left(\frac{\sin jt}{jt} \right)^p u_0 + \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \left(\frac{\sin jt}{jt} \right)^p u_k \right] \end{aligned} \tag{3.14}$$

$$= c_p t \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \left(\frac{\sin jt}{jt} \right)^p u_k$$

are then all convergent and the equality holds. It is therefore sufficient to prove (3.13). In case $p \geqq 2$, the series in (3.13) converge and the equality holds because the series in the left member converges absolutely, being dominated by the series in the inequalities.

$$\sum_{j=1}^{\infty} \sum_{k=1}^{j} \frac{1}{t^p j^p} |u_k| \leqq \sum_{j=1}^{\infty} \frac{M}{t^p j^p} j^{1/2} < \infty. \tag{3.2}$$

In (3.2) we used the elementary inequality

$$\sum_{k=1}^{n} |u_k| \leqq \left[\sum_{k=1}^{n} 1^2 \right]^{1/2} \left[\sum_{k=1}^{n} |u_k|^2 \right]^{1/2} \leqq M n^{1/2}. \tag{3.21}$$

Coming to the case $p = 1$, it remains to be proved that the series in

$$\sum_{j=1}^{\infty} \sum_{k=1}^{j} \frac{\sin jt}{j} u_k = \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \frac{\sin jt}{j} u_k \tag{3.3}$$

are convergent and the equality holds. To prove this, it is sufficient to prove that one of the two series in (3.3) is convergent and that

$$\lim_{n \to \infty} \sum_{k=1}^{n} \sum_{j=n}^{\infty} \frac{\sin jt}{j} u_k = 0. \tag{3.31}$$

Supposing that $0 < t \leqq \pi$ and putting

$$f_n(t) = \sum_{k=1}^{n} \frac{\sin kt}{k} \tag{3.32}$$

we find that $f_n(\pi) = 0$ and

$$f_n'(t) = \Re e^{it} \sum_{k=0}^{n-1} e^{ikt} = \frac{1}{2} \cot \frac{t}{2} \sin nt - \sin^2 \frac{nt}{2}. \tag{3.33}$$

Integration gives

$$\sum_{k=1}^{n} \frac{\sin kt}{k} = \frac{\pi - t}{2} + \frac{\sin nt}{2n} - \frac{1}{2} \int_t^{\pi} \cot \frac{x}{2} \sin nx \, dx. \tag{3.34}$$

Letting $n \to \infty$ gives the standard formula

$$\sum_{k=1}^{\infty} \frac{\sin kt}{k} = \frac{\pi - t}{2}. \tag{3.35}$$

This and (3.34) give the formula

$$\sum_{k=n}^{\infty} \frac{\sin kt}{k} = - \frac{\sin(n-1)t}{2(n-1)} + \frac{1}{2} \int_t^{\pi} \cot \frac{x}{2} \sin(n-1)x \, dx \tag{3.36}$$

which is valid when $n > 1$ and is, when $n > 1$, equivalent to the formula

$$\sum_{k=n}^{\infty} \frac{\sin kt}{k} = \frac{\sin nt}{2n} + \frac{1}{2}\int_{t}^{\pi} \cot\frac{x}{2} \sin nx \, dx \tag{3.37}$$

which is valid when $n \geqq 1$. Using (3.37), we see that the series in the right member of (3.3) is convergent if the series in

$$\frac{1}{2}\sum_{k=1}^{\infty} a_k u_k + \frac{1}{2}\sum_{k=1}^{\infty} b_k u_k \tag{3.4}$$

are convergent when

$$a_k = \frac{\sin kt}{k}, \quad b_k = \int_{t}^{\pi} \cot\frac{x}{2} \sin kx \, dx. \tag{3.41}$$

The first series in (3.4) converges because $\Sigma |a_k|^2 < \infty$ and $\Sigma |u_k|^2 < \infty$. Likewise the second series in (3.4) is convergent because $\Sigma |b_k|^2 < \infty$, the numbers b_k in (3.41) being the Fourier sine coefficients of the function $g_t(x)$ defined over $0 \leqq x \leqq \pi$ by the formulas $g_t(x) = 0$ when $0 \leqq x \leqq t$ and $g_t(x) = \cot(x/2)$ when $t < x \leqq \pi$. Therefore the series in the right member of (3.3) is convergent. To prove (3.31), we use (3.37) to see that (3.31) will hold if

$$\lim_{n\to\infty} (a_n + b_n) \sum_{k=1}^{n} u_k = 0 \tag{3.42}$$

when a_n and b_n are defined by (3.41). Since $\Sigma |u_k|^2 < \infty$, (3.26) shows that it is sufficient to prove that $a_n n^{1/2} \to 0$ and $b_n n^{1/2} \to 0$. Since $n|a_n|$ is bounded, we see that $a_n n^{1/2} \to 0$. Since b_n is a Fourier sine coefficient of a function $g_t(x)$ having bounded variation over $0 \leqq t \leqq \pi/2$, it follows that $n|b_n|$ is bounded and hence that $b_n n^{1/2} \to 0$. This proves (3.31) and completes the proof of Lemma 3.1. Without the assumption $\Sigma |u_k|^2 < \infty$, Szàsz [45, page 779] has proved that if the series in the left member of (3.3) converges when $0 < t < t_0$, then (3.3) holds and, conversely, if $s_n/n \to 0$ then convergence of the series in the right member of (3.3) implies (3.3).

LEMMA 3.5. *If* Σu_n *is a series for which* $\limsup |nu_n| < \infty$, *then the series in* (1.2) *and* (3.11) *for* $\sigma_2(p, t)$ *converges when* $0 < t < 2\pi$ *and, as* $t \to 0$,

$$\sigma_2(p, t) = o(1) + \sigma_2^*(p, t) \tag{3.51}$$

where

$$\sigma_2^*(p, t) = c_p \sum_{k=0}^{\infty} \left[\int_{kt}^{\infty} \left(\frac{\sin y}{y} \right)^p dy \right] u_k. \tag{3.52}$$

Supposing that $|ku_k| \leqq M$, we find from (3.12) and (3.52) that

$$|\sigma_2(p, t) - \sigma_2^*(p, t)| \leqq o(1) + c_p M g(t) \tag{3.53}$$

where

$$(3.54)\qquad g(t)=\sum_{k=1}^{\infty}\frac{1}{k}\left|t\sum_{j=k}^{\infty}\left(\frac{\sin jt}{jt}\right)^{p}-\int_{kt}^{\infty}\left(\frac{\sin y}{y}\right)^{p}dy\right|.$$

Hence it is sufficient to prove that $g(t)=o(1)$. To prove this we need estimates of the sums in (3.54) which we obtain by use of the Euler-Maclaurin summation formula

$$(3.6)\qquad \sum_{j=k}^{m}f(j)=\int_{k}^{m}f(x)dx+\frac{f(k)+f(m)}{2}+\int_{k}^{m}f'(x)B_1(x)dx$$

where $B_1(x)$ is the Bernoulli function of order 1 defined by

$$(3.61)\qquad B_1(x)=x-[x]-1/2.$$

The functions with which we have to deal seem to be such that there is no advantage in using variants of (3.6) involving Bernoulli functions of higher order. Putting

$$(3.7)\qquad f(x)=t\left(\frac{\sin xt}{xt}\right)^{p}$$

in (3.6) and letting $m\to\infty$ gives

$$(3.71)\qquad t\sum_{j=k}^{\infty}\left(\frac{\sin jt}{jt}\right)^{p}=\int_{kt}^{\infty}\left(\frac{\sin y}{y}\right)^{p}dy+H_1+H_2+H_3$$

where

$$(3.72)\qquad H_1=\frac{1}{2}t\left(\frac{\sin kt}{kt}\right)^{p}$$

$$(3.73)\qquad H_2=-p\int_{k}^{\infty}\left(\frac{\sin xt}{xt}\right)^{p-1}\frac{\sin xt}{x^2}B_1(x)dx$$

$$(3.74)\qquad H_3=pt\int_{k}^{\infty}\left(\frac{\sin xt}{xt}\right)^{p-1}\frac{\cos xt}{x}B_1(x)dx.$$

From (3.54) and (3.71) we obtain

$$g(t)=\sum_{k=1}^{\infty}\frac{1}{k}|H_1+H_2+H_3|.$$

We find that

$$(3.76)\qquad \sum_{k=1}^{\infty}\frac{1}{k}|H_1|\leqq\frac{t}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left|\frac{\sin kt}{kt}\right|=\frac{1}{2}\sum_{k=1}^{\infty}\frac{|\sin kt|}{k^2}=o(1).$$

For each $\varepsilon>0$ we can choose an integer N so great that

$$(3.77)\qquad \sum_{k=1}^{\infty}\frac{1}{k}|H_2|\leqq p\sum_{k=1}^{\infty}\frac{1}{k}\int_{k}^{\infty}\frac{|\sin xt|}{x^2}dx$$

$$= o(1) + p\sum_{k=N}^{\infty}\frac{1}{k}\int_k^{\infty}\frac{1}{x^2}dx = o(1) + p\sum_{k=N}^{\infty}\frac{1}{k^2} < \varepsilon$$

and it follows that the function H_2 contributes $o(1)$ to the right side of (3.75). In case $p \geqq 2$, we find that

$$\sum_{k=1}^{\infty}\frac{1}{k}|H_3| \leqq pt\sum_{k=1}^{\infty}\frac{1}{k}\int_k^{\infty}\left|\frac{\sin xt}{xt}\right|\frac{|\cos xt|}{x}dx \tag{3.78}$$

$$\leqq p\sum_{k=1}^{\infty}\frac{1}{k}\int_k^{\infty}\frac{|\sin xt|}{x^2}dx = o(1).$$

In case $p = 1$, we have to show that

$$\sum_{k=1}^{\infty}\frac{1}{k}|H_3| = t\sum_{k=1}^{\infty}\frac{1}{k}\left|\int_k^{\infty}\frac{\cos xt}{x}B_1(x)dx\right| = o(1). \tag{3.8}$$

Supposing for the moment that k and t are fixed such that $0 < t < 1$ and that $u > k$, let

$$a_k(u) = \int_k^u\frac{\cos xt}{x}B_1(x)dx,\quad a_k = \int_k^{\infty}\frac{\cos xt}{x}B_1(x)dx. \tag{3.81}$$

The well-known fact that the series in the right member of

$$B_1(x) = -\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n\pi} \tag{3.82}$$

is the Fourier series of the function $B_1(x)$ in (3.61) follows from (3.35). Since $B_1(x)$ and the function $(\cos xt)/x$ both belong to the Lebesgue class L_2 over the finite interval $k \leqq x \leqq u$, we can multiply (3.82) by $(\cos xt)/2$ and integrate termwise to obtain

$$a_k(u) = -\sum_{n=1}^{\infty}\frac{1}{n\pi}\int_k^u\frac{\sin 2n\pi x\,\cos xt}{x}dx \tag{3.83}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{2n\pi}\left[\int_k^u\frac{\sin(2n\pi + t)x}{x}dx + \int_k^u\frac{\sin(2n\pi - t)x}{x}dx\right]$$

$$= -\sum_{n=1}^{\infty}\frac{1}{2n\pi}\left[\int_{k(2n\pi+t)}^{u(2n\pi+t)}\frac{\sin y}{y}dy + \int_{k(2n\pi-t)}^{u(2n\pi-t)}\frac{\sin y}{y}dy\right].$$

When $0 < y_1 < y_2$,

$$\left|\int_{y_1}^{y_2}\frac{\sin y}{y}dy\right| = \left|\frac{\cos y_1}{y_1} - \frac{\cos y_2}{y_2} - \int_{y_1}^{y_2}\frac{\cos y}{y^2}dy\right| < \frac{3}{y_1}. \tag{3.84}$$

This and (3.43) imply that

$$|a_k(u)| \leqq \frac{1}{k}\sum_{n=1}^{\infty}\frac{3}{\pi}\frac{1}{n(2n\pi - 1)} = \frac{C}{K} \tag{3.85}$$

where C is an absolute constant. Since (3.81) implies that $a_k(u) \to a_k$ as $u \to \infty$, (3.45) implies that $|a_k| \leqq C/k$. This and (3.81) imply (3.8). The two relations (3.78) and (3.8) show that the function H_3 contributes $o(1)$ to the right side of (3.75). Hence (3.75) implies that $g(t) = o(1)$ and Lemma 3.5 is proved.

4. R_p transforms and partial sums. Let $n = n(\alpha)$ and $t = t(\alpha)$ be positive functions, defined for $\alpha > 0$, such that $n(\alpha)$ is an integer for each α and $n(\alpha) \to \infty$ and $t(\alpha) \to 0$ as $\alpha \to \infty$. It is the purpose of this section to determine the least constant B, which depends upon p and the functions $n(\alpha)$ and $t(\alpha)$ and which is either finite or $+\infty$, such that

(4.1) $$\limsup_{\alpha\to\infty} |\sigma_2(p, t) - s_n| \leqq B \limsup_{n\to\infty} |nu_n|$$

whenever Σu_n is a series for which $\limsup |nu_n| < \infty$. In particular, we characterize the pairs of functions $n(\alpha)$ and $t(\alpha)$ for which B is finite and those for which B attains its minimum value B_0.

When $\limsup |nu_n| < \infty$, it follows from Lemma 3.5 and (1.22) that the series in (1.2) for the R_p transform $\sigma_2(p, t)$ of Σu_n converges and

(4.11) $$\sigma_2(p, t) = o(1) + u_0 + \sum_{k=1}^{\infty} \left[c_p \int_{kt}^{\infty} \left(\frac{\sin y}{y} \right)^p dy \right] u_k.$$

This and the procedure of section 2 show that the least constant B for which (4.1) holds whenever $\limsup |nu_n| < \infty$ is the constant B defined by

(4.12) $$B = \limsup_{\alpha\to\infty} F_2(\alpha)$$

where

(4.13) $$F_2(\alpha) = \sum_{k=1}^{n} \frac{1}{k} \left| 1 - c_p \int_{tk}^{\infty} \left(\frac{\sin y}{y} \right)^p dy \right| + \sum_{k=n+1}^{\infty} \frac{1}{k} \left| c_p \int_{kt}^{\infty} \left(\frac{\sin y}{y} \right)^p dy \right|.$$

Use of (1.22) enables us to put (4.13) in the form

(4.14) $$F_2(\alpha) = c_p \sum_{k=1}^{n} \frac{1}{k} \left| \int_0^{kt} \left(\frac{\sin y}{y} \right)^p dy \right| + c_p \sum_{k=n}^{\infty} \frac{1}{k} \left| \int_{kt}^{\infty} \left(\frac{\sin y}{y} \right)^p dy \right|.$$

As was the case for $F_1(\alpha)$, the behavior of $F_2(\alpha)$ as $\alpha \to \infty$ depends solely upon the function $q(\alpha)$ defined by $q(\alpha) = n(\alpha)t(\alpha)$.

LEMMA 4.2. *If $F_2(\alpha)$ is given by* (4.14) *and the function $q(\alpha)$ defined by $q(\alpha) = n(\alpha)t(\alpha)$ is such that*

(4.21) $$0 < q_1 = \liminf_{\alpha\to\infty} q(\alpha) \leqq \limsup_{\alpha\to\infty} q(\alpha) = q_2 < \infty,$$

then, as $\alpha \to \infty$,

$$F_2(\alpha) = o(1) + G_2(\alpha) \tag{4.22}$$

where

$$G_2(\alpha) = c_p \int_0^q \frac{1}{x} \left| \int_0^x \left(\frac{\sin y}{y} \right)^p dy \right| dx + c_p \int_q^\infty \frac{1}{x} \left| \int_x^\infty \left(\frac{\sin y}{y} \right)^p dy \right| dx. \tag{4.23}$$

The last integral in (4.23) is convergent because if $p > 1$, then

$$\left| \int_x^\infty \left(\frac{\sin y}{y} \right)^p dy \right| \leqq \int_x^\infty \frac{1}{y^2} dy = \frac{1}{x} \tag{4.24}$$

and if $p = 1$, then

$$\left| \int_x^\infty \left(\frac{\sin y}{y} \right)^p dy \right| = \left| \frac{\cos x}{x} - \int_x^\infty \frac{\cos y}{y^2} dy \right| \leqq \frac{2}{x}. \tag{4.25}$$

Introducing the function $q(\alpha)$, we put (4.14) in the form

$$F_2(\alpha) = c_p \sum_{t \leqq k \leqq q} \frac{1}{kt} \left| \int_0^{kt} \left(\frac{\sin y}{y} \right)^p dy \right| t + c_p \sum_{kt > q} \frac{1}{kt} \left| \int_{kt}^\infty \left(\frac{\sin y}{y} \right)^p dy \right| t \tag{4.26}$$

and then, under the hypothesis (4.21), the conclusion of Lemma 4.2 follows from the elementary theory of Riemann integration because if $\varepsilon > 0$ and h is sufficiently great then we can see with the aid of (4.24) and (4.25) that

$$\sum_{kt > h} \frac{1}{kt} \left| \int_{kt}^\infty \left(\frac{\sin y}{y} \right)^p dy \right| t \leqq 2 \sum_{kt > h} \frac{1}{(kt)^2} t \leqq 2 \int_{h-t}^\infty \frac{1}{x^2} dx < \varepsilon \tag{4.27}$$

when $0 < t < 1$ and hence when α is sufficiently great.

A modification of the proof of Lemma 4.2 shows that if $n(\alpha)$ and $t(\alpha)$ are functions such that $\liminf q(\alpha) = 0$ or $\limsup q(\alpha) = \infty$, then $\limsup F_2(\alpha) = \infty$.

The formula (4.23) corresponds more closely to (2.41) when it is written in the form

$$G_2(q) = c_p \int_0^q \frac{1}{x} |I - \phi(x)| dx + c_p \int_q^\infty \frac{1}{x} |\phi(x)| dx \tag{4.3}$$

where

$$I = \int_0^\infty \left(\frac{\sin y}{y} \right)^p dy, \quad \phi(x) = \int_x^\infty \left(\frac{\sin y}{y} \right)^p dy. \tag{4.31}$$

It is easy to see that there is a unique positive number q_0 such that

$$\int_0^{q_0} \left(\frac{\sin y}{y} \right)^p dy = \frac{I}{2} = \int_{q_0}^\infty \left(\frac{\sin y}{y} \right)^p dy = \phi(q_0). \tag{4.32}$$

It then follows that $0 \leqq I - \phi(x) < I/2$ when $0 \leqq x < q_0$ and $|\phi(x)| < I/2$ when $x > q_0$. These facts and (4.3) imply that $G_2(q)$ is positive when $q > 0$ and attains its absolute minimum when and only when $q = q_0$. In fact

$G_2(q) \to \infty$ as $q \to 0$, $G_2(q) \to \infty$ as $q \to \infty$, $G_2(q)$ is decreasing over $0 < q < q_0$, and $G_2(q)$ is increasing over $q_0 < q < \infty$. These results and (4.12) imply that if the functions $n(\alpha)$ and $t(\alpha)$ are such that (4.21) holds, then the least constant B for which (4.1) is valid is finite and is the maximum of $G_2(q_1)$ and $G_2(q_2)$. This maximum is a minimum when and only when $q_1 = q_2 = q_0$.

A consequence of the results which we have obtained is set forth in the following theorem.

THEOREM 4.4 *The constant* $G_2(q_0)$ *determined by* (4.23) *or* (4.3) *and* (4.32) *is the least constant with the following property. There exist functions* $n(\alpha)$ *and* $t(\alpha)$ *such that* $n(\alpha) \to \infty$, $t(\alpha) \to 0$, *and*

$$\limsup_{\alpha\to\infty} |\sigma_2(p, t) - s_n| \leqq G_2(q_0) \limsup_{n\to\infty} |nu_n| \tag{4.41}$$

whenever Σu_n *is a series for which* $\limsup |nu_n| < \infty$. *Moreover the functions* $n(\alpha)$ *and* $t(\alpha)$ *are such that* (4.41) *is valid if and only if*

$$\lim_{\alpha\to\infty} n(\alpha)t(\alpha) = q_0 \tag{4.42}$$

Our results also show that for each $q > 0$ the inequalities

$$\limsup_{n\to\infty} |\sigma_2(p, q/n) - s_n| \leqq G_2(q) \limsup_{n\to\infty} |nu_n| \tag{4.51}$$

and

$$\limsup_{t\to 0} |\sigma_2(p, t) - s_{[q/t]}| \leqq G_2(q) \limsup_{n\to\infty} |nu_n| \tag{4.52}$$

are optimal inequalities and that the right members are minimal when $q = q_0$. It is not difficult to modify the above work to obtain the following theorem.

THEOREM 4.6. *In order that functions* $n(\alpha)$ *and* $t(\alpha)$ *be such that*

$$\lim_{n\to\infty} |\sigma_2(p, t) - s_n| = 0 \tag{4.61}$$

whenever Σu_n *is a series for which* $\lim nu_n = 0$, *it is necessary and sufficient that*

$$0 < \liminf_{\alpha\to\infty} n(\alpha)t(\alpha) \leqq \limsup_{\alpha\to\infty} n(\alpha)t(\alpha) < \infty. \tag{4.62}$$

It is a corollary of Theorem 4.6 that if $\lim nu_n = 0$, then Σu_n is evaluable R_p to s if and only if it converges to s. This corollary is a Tauberian theorem of classical type, and numerous theorems of this type are given in papers cited in the bibliography.

5. Tauberian relations involving $R(p)$ and R_p. For each $q > 0$ the two relations (2.62) and (4.52) imply that the $R(p)$ transform $\sigma_1(p, t)$ and the R_p transform $\sigma_2(p, t)$ of a series Σu_n for which $\limsup |nu_n| < \infty$ are so related that the formula

$$(5.1) \qquad \limsup_{t\to 0} |\sigma_1(p, t) - \sigma_2(p, t)| \leqq C_q \limsup_{n\to\infty} |nu_n|$$

holds when

$$(5.11) \qquad C_q = G_1(q) + G_2(q).$$

This implies that if Σu_n is a series for which $\limsup |nu_n| < \infty$, then the $R(p)$ and R_p transforms are both bounded or both unbounded. It implies also that if Σu_n is a series for which $\lim nu_n = 0$, then the $R(p)$ and R_p transforms are (whether they be convergent or divergent) equiconvergent in the sense that

$$(5.12) \qquad \lim_{t\to 0} |\sigma_1(p, t) - \sigma_2(p,t)| = 0.$$

In particular, a series Σu_n for which $\lim nu_n = 0$ is evaluable $R(p)$ to s (finite or infinite) if and only if it is evaluable R_p to s.

For further study of matters relating to (5.1), we introduce two positive functions $t(\alpha)$ and $v(\alpha)$ such that $t(\alpha) \to 0$ and $v(\alpha) \to 0$ as $\alpha \to \infty$ and obtain information about the least constant C, which is finite or infinite and depends upon p and the functions $t(\alpha)$ and $v(\alpha)$, such that

$$(5.2) \qquad \limsup_{\alpha\to\infty} |\sigma_1(p, t) - \sigma_2(p, v)| \leqq C \limsup_{n\to\infty} |nu_n|$$

whenever Σu_n is a series for which $\limsup |nu_n| < \infty$. Use of (1.2), Lemma 3.5, and the procedure of section 2 shows that the least constant C for which (5.2) holds is the constant C defined by

$$(5.3) \qquad C = \limsup_{\alpha\to\infty} F_3(\alpha)$$

where

$$(5.31) \qquad F_3(\alpha) = \sum_{k=1}^{\infty} \frac{1}{k}\left| \left(\frac{\sin kt}{kt}\right)^p - c_p \int_{kv}^{\infty} \left(\frac{\sin y}{y}\right)^p dy \right|.$$

Letting $\lambda(\alpha)$ be the function defined by

$$(5.32) \qquad \lambda(\alpha) = v(\alpha)/t(\alpha),$$

we put (5.3 1) in the form

$$(5.32) \qquad F_3(\alpha) = \sum_{k=1}^{\infty} \frac{1}{kt}\left| \left(\frac{\sin kt}{kt}\right)^p - c_p \int_{\lambda kt}^{\infty} \left(\frac{\sin y}{y}\right)^p dy \right| t$$

which involves modified Riemann sums. In case

$$(5.33) \qquad 0 < \lambda_1 = \liminf_{\alpha\to\infty} \lambda(\alpha) \leqq \limsup_{\alpha\to\infty} \lambda(\alpha) = \lambda_2 < \infty,$$

it follows from (5.32) that

$$(5.34) \qquad F_3(\alpha) = o(1) + G_3(\lambda)$$

where

$$(5.4) \qquad G_3(\lambda) = \int_0^\infty \frac{1}{x}\left|\left(\frac{\sin x}{x}\right)^p - c_p\int_{\lambda x}^\infty \left(\frac{\sin y}{y}\right)^p dy\right| dx.$$

In case $\liminf \lambda(\alpha) = 0$ or $\limsup \lambda(\alpha) = \infty$, (5.32) shows that $\limsup |F_3(\alpha)| = \infty$. As a function of λ, $G_3(\lambda)$ is positive and continuous over $\lambda > 0$ and $G_3(\lambda) \to \infty$ as $\lambda \to 0$ and as $\lambda \to \infty$. Therefore $G_3(\lambda)$ has a positive absolute minimum attained for at least one positive value λ_0 of λ. This implies that the optimal constant C in (5.2) is always positive and $C \geqq G_3(\lambda_0)$. This constant C attains its minimum value $G_3(\lambda_0)$ when and only when the functions $t(\alpha)$ and $v(\alpha)$ are such that $v(\alpha)/t(\alpha) \to \lambda_0$. In particular for each $\lambda > 0$ the inequality

$$(5.5) \qquad \limsup_{t\to 0} |\sigma_1(p, t) - \sigma_2(p, \lambda t)| \leqq G_3(\lambda) \limsup_{n\to\infty} |nu_n|$$

is an optimal inequality and the right member is minimal when $\lambda = \lambda_0$.

6. Riemann and Cesàro methods. It is the purpose of this section to show that a simple Tauberian argument can be applied to prove that the product transformations $R(p)C_r$ and R_pC_r are regular when $r \geqq 1$, and to show how this result is related to others in the theory of Riemann and Cesàro methods. The $R(p)C_r$ transform of a given series is, if it exists, the $R(p)$ transform of the C_r transform of the series, and R_pC_r is similarly defined.

For each real (or even complex) number r which is not a negative integer, the Cesàro transform $\sigma_n^{(r)}$ of order r of a series Σu_n with partial sums s_n is defined in standard notation by

$$(6.1) \qquad \sigma_n^{(r)} = S_n^{(r)}/A_n^{(r)}$$

where

$$(6.11) \qquad A_n^{(r)} = \binom{n+r}{r}, \quad S_n^{(r)} = \sum_{k=0}^{n} A_{n-k}^{(r-1)} s_k = \sum_{k=0}^{n} A_{n-k}^{(r)} u_k.$$

The series is evaluable C_r to L if $\sigma_n^{(r)} \to L$ as $n \to \infty$. In order to obtain the transform of the C_r transform by a series to function transformation, we need the series to series version of C_r. To get this, we use (6.1) and (6.11) to obtain

$$(6.12) \qquad \sigma_n^{(r)} = \sum_{k=0}^{n} \frac{n!(n-k+r)!}{(n+r)!(n-k)!} u_k$$

and

$$(6.13) \qquad \sigma_{n-1}^{(r)} = \sum_{k=0}^{n} \frac{n!(n-k+r)!}{(n+r)!(n-k)!} \frac{(n+r)(n-k)}{n(n-k+r)} u_k$$

where in (6. 23) and the following formulas we suppose that $r \neq 0$. Letting $u_0^{(r)} + u_1^{(r)} + \dots\dots$ be the series with partial sums $\sigma_n^{(r)}$, we have $u_n^{(r)} = \sigma_n^{(r)} - \sigma_{n-1}^{(r)}$ and it follows from (6. 12) and (6. 13) that

$$u_n^{(r)} = \frac{r}{n} \sum_{k=0}^{n} \frac{n!(n-k+r)!}{(n+r)!(n-k)!} \frac{k}{n-k+r} u_k. \tag{6.14}$$

Thus (6. 14) is the series to series version of the Cesàro transformation, a series Σu_k being evaluable to L if the series $\Sigma u_n^{(r)}$ converges to L.

Using the identity

$$\frac{k}{n-k+r} = \frac{n+r}{n-k+r} - 1, \tag{6.15}$$

splitting the right member of (6. 24) into two sums, and then using (6. 22) gives

$$n u_n^{(r)} = r(\sigma_n^{(r-1)} - \sigma_n^{(r)}). \tag{6.16}$$

We are now in a position to give a simple proof of the following theorem in which the transformation T can be either $R(p)$ or R_p.

THEOREM 6. 2. *If T is a transformation such that Σu_n is evaluable T to L whenever Σu_n converges to L and* $\lim n u_n = 0$ *and if $r > 0$, then each series evaluable C_{r-1} to L is evaluable TC_r to L.*

The hypothesis that $r > 0$ and Σu_n is evaluable C_{r-1} to L implies that Σu_n is evaluable C_r to L. Thus $\sigma_n^{(r-1)} \to L$ and $\sigma_n^{(r)} \to L$ and (6. 16) implies that $n u_n^{(r)} \to 0$. Since $\Sigma u_n^{(r)}$ converges to L because $\sigma_n^{(r)} \to L$, it follows from the hypothesis on T that $\Sigma u_n^{(r)}$ is evaluable T to L. Thus Σu_n is evaluable TC_r to L and Theorem 6. 2. is proved. Using standard terminology we say that T_2 includes T_1 and write $T_2 \supset T_1$ and $T_1 \subset T_2$ if each series evaluable T_1 to L is also evaluable T_2 to L, and we say that T_2 and T_1 are equivalent and write $T_2 \sim T_1$ if $T_2 \supset T_1$ and $T_1 \supset T_2$. There are occasions upon which it is convenient to recognize that the usual formulas defining C_r have no meaning when $r = -1$ and to adopt a special definition under which Σu_n is said to be evaluable C_{-1} to L if Σu_n converges to L and $n u_n \to 0$. We do this now. Theorem 6. 2 then reduces to the simple fact that if $T \supset C_{-1}$, then $TC_r \supset C_{r-1}$ when $r > 0$. This shows the if $r \geqq 1$ and $T \supset C_{-1}$, then $TC_r \supset C_{r-1} \supset C_0$ and hence TC_r is regular. Thus $R(p)C_r$ and R_pC_r are both regular when $r \geqq 1$.

Beginning perhaps in 1904 when Fejér [6] proved that $C_1 \subset R(4)$, many authors have given relations among Riemann and Cesàro methods of various orders. Hardy [8, page93 and page 371] gives a brief account of the subject and some references. A more complete list of references is given in the bibliography at the end of this paper. In particular, Obreschkoff [30] and

Hirokawa [13] have obtained properties of transformations which are modifications of $R(p)C_r$ and R_pC_r. The papers noted above show that if T_p is one of $R(p)$ and R_p and p is a positive integer, then the first of the relations

(6.3) $$C_{p-1-\delta} \subset T_p, \quad T_p \subset C_{p+\delta}$$

holds when $0 < \delta < 1$. In case $T_p = R(p)$, it has been shown by Zygmund [53] and Kuttner [22] that the second relation in (6.3) is valid when $p = 1$ and when $p = 2$. However this second relation is certainly not valid when $T_p = R(p)$ and $p \geqq 3$ because Kutter [22] has shown that if $p \geqq 3$ then there exist series evaluable $R(p)$ but not evaluable by the Abel power series method which includes all Cesàro methods C_r of order $r > -1$.

If (6.3) holds, then

(6.31) $$C_{p-1-\delta}C_r \subset T_pC_r, \quad T_pC_r \subset C_{p+\delta}C_r$$

and conversely. Using the well known fact that $C_\alpha C_\beta \sim C_{\alpha+\beta}$ when $\alpha > -1$, $\beta > -1$, $\alpha + \beta > -1$, we see that if $p \geqq 1$, $r > -1$, $0 < \delta < p$, and $0 < \delta < p + r$, then $C_{p-1-\delta}C_r \sim C_{p+r-1-\delta}$ and $C_{p+\delta}C_r \sim C_{p+\delta+r}$. Therefore when $p \geqq 1$, $r > -1$, $0 < \delta < p$, and $0 < \delta < p + r$, the relations (6.3) and (6.31) hold if and only if

(6.32) $$C_{p+r-1-\delta} \subset T_pC_r, \quad T_pC_r \subset C_{p+r+\delta}.$$

Introducing inverses of the Cesàro methods, we see that if (6.3) holds, then

(6.33) $$C_{p-1-\delta}C_r^{-1} \subset T_pC_r^{-1}, \quad T_pC_r^{-1} \subset C_{p+\delta}C_r^{-1},$$

and conversely. Using the well-known fact that $C_\alpha C_\beta^{-1} \sim C_{\alpha-\beta}$ when $\alpha > -1$, $\beta > -1$, $\alpha - \beta > -1$, we see that if $p \geqq 1$, $r > -1$, $0 < \delta < p$, $0 < \delta < p - r$, then $C_{p-1-\delta}C_r^{-1} \sim C_{p-r-1-\delta}$ and $C_{p+\delta}C_r^{-1} \sim C_{p+r+\delta}$. Therefore when $p \geqq 1$, $r > -1$, $0 < \delta < p$, and $0 < \delta < p - r$ the relations (6.3) and (6.33) hold if and only if

(6.34) $$C_{p-r-1-\delta} \subset T_pC_r^{-1}, \quad T_pC_r^{-1} \subset C_{p-r+\delta}.$$

In particular, putting $r = p - 1$ shows that (6.3) holds if and only if

(6.35) $$C_{-\delta} \subset T_pC_{p-1}^{-1}, \quad T_pC_{p-1}^{-1} \subset C_{1+\delta}.$$

In connection with the questions whether the relations (6.3) and (6.35) are valid and are optimal relations, it would be of interest to know whether inclusion relations exist among the methods $T_pC_{p-1}^{-1}$. If it could be shown that

(6.4) $$T_1 = T_1C_0^{-1} \subset T_2C_1^{-1} \subset T_3C_2^{-1} \subset \ldots\ldots$$

or

(6.41) $$T_1 = T_1C_0^{-1} \supset T_2C_1^{-1} \supset T_3C_2^{-1} \supset \ldots\ldots$$

or

(6. 42) $$T_1 = T_1C_0^{-1} \sim T_2C_1^{-1} \sim T_3C_2^{-1} \sim \ldots\ldots$$

these results would be of intrinstic interest and would perhaps have significant consequences.

BIBLIOGRAPHY

In addition to papers to which the texi refers, this bibliography contains others involving $R(p)$ and R_p. It contains relatively few of the many works in which $R(p)$ and R_p are explicitly or implicitly applied in the theory of Fourier and more general trigonometric series. While there may be serious omissions, an attempt has been made to include a useful list of papers which relate $R(p)$ and R_p to each other and to other methods for evaluating series and sequences.

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Added in proof September 1959 :

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Section 76, pp. 158-160, of the book of Zeller gives, with references, a very informative summarly of fundamental properties of the Riemann methods.

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK.