ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES

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1. Introduction. Let us consider a series Σa_n. Denote by $\sigma_n^{(\alpha)}$ and $\tau_n^{(\alpha)}$ the n-th Cesàro means of order $\alpha\,(\alpha > -1)$ of the series Σa_n and of the sequence $\{na_n\}$ respectively.

Following T. M. Flett [1], the series Σa_n is called summable $|C,\alpha|_k$ $(k \geqq 1)$ if the following series, which are equiconvergent with each other (see e. g. [1]),

$$\sum_n n^{k-1}|\sigma_n^{(\alpha)} - \sigma_{n-1}^{(\alpha)}|^k, \quad \sum_n n^{-1}|\tau_n^{(\alpha)}|^k \tag{1}$$

and

$$\sum_n n^{-1}|\sigma_n^{(\alpha)} - \sigma_n^{(\alpha-1)}|^k$$

are convergent.

The series Σa_n is called strongly summable $(C,\alpha)_k$ $(\alpha > -1,\ k \geqq 1)$ if there exists a constant s such that

$$\sum_{j=1}^{n} |\sigma_j^{(\alpha-1)} - s|^k = o(n) \qquad \text{as } n \to \infty. \tag{2}$$

If the series Σa_n is strongly summable $(C,\alpha)_k$ and is summable (C,α), that is, if the relation (2) holds and $\sigma_n^{(\alpha)}$ tends to a finite limit as $n \to \infty$, then the relation (2) is equivalent to:

$$\sum_{j=1}^{n} |\sigma_n^{(\alpha-1)} - \sigma_n^{(\alpha)}|^k = o(n) \qquad \text{as } n \to \infty, \tag{3}$$

as we see easily by the Minkowski inequality. In the case of Fourier series the strong summability is often discussed in the form (3) by the reason of its (C,α) summability almost everywhere for $\alpha > 0$. We shall say in the sequel that the series $\Sigma\, a_n$ is summable $[C,\alpha]_k$ if the relation (3) holds.

We note that the relation (3) is equivalent to

$$\sum_{j=1}^{n} |\tau_j^{(\alpha)}|^k = o(n) \qquad \text{as } n \to \infty. \tag{3 }$$

By the Kronecker lemma the convergence of the series (1) implies the relation (3), but not necessarily the converse. We shall here introduce a generalization of the absolute summability. If the series

$$\sum_n \left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |\sigma_j^{(\alpha-1)} - \sigma_j^{(\alpha)}|^p\right)^{k/p} \tag{4}$$

or equivalently if the series

$$\sum_n \left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |\tau_j^{(\alpha)}|^p\right)^{k/p} \tag{4'}$$

is convergent, we shall say that the series Σa_n is summable $\{C, \alpha\}_{k,p}$ where $k > 0$ and $p \geqq 1$.

We shall note some elementary relations of the three summabilities mentioned above:

THEOREM 1. (1) *For* $0 < k \leqq p$, *if* Σa_n *is summable* $\{C, \alpha\}_{k,p}$ *then it is summable* $|C, \alpha|_k$
(2) *For* $1 \leqq p \leqq k$, *if* Σa_n *is summable* $|C, \alpha|_k$, *then it is summable* $\{C, \alpha\}_{k,p}$. *In the case* $0 < k = p$ *the two summabilities* $\{C, \alpha\}_{k,p}$ *and* $|C, \alpha|_k$ *are equivalent.*
(3) *For* $k > 0$ *and* $p \geqq 1$, *if* Σa_n *is summable* $\{C, \alpha\}_{k,p}$, *then it is summable* $[C, \alpha]_p$.

PROOF. (1) By the Hölder inequality the sum (4) is not smaller than

$$\sum_n \left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}} |\sigma_j^{(\alpha-1)} - \sigma_j^{(\alpha)}|^k\right) \geqq \sum_n \sum_{j=2^n}^{2^{n+1}-1} j^{-1}|\sigma_j^{(\alpha-1)} - \sigma_j^{(\alpha)}|$$
$$= \sum_j j^{-1}|\sigma_j^{(\alpha-1)} - \sigma_j^{(\alpha)}|.$$

(2) By the similar reason we have

$$\sum_n n^{-1}|\sigma_n^{(\alpha-1)} - \sigma_n^{(\alpha)}|^k = \sum_n \left(\sum_{j=2^n}^{2^{n+1}-1} j^{-1}|\sigma_j^{(\alpha-1)} - \sigma_j^{(\alpha)}|^k\right)$$
$$\geqq \frac{1}{2}\sum_n \left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |\sigma_n^{(\alpha-1)} - \sigma_n^{(\alpha)}|^k\right)$$
$$\geqq \frac{1}{2}\sum_n \left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |\sigma_n^{(\alpha-1)} - \sigma_n^{(\alpha)}|^p\right)^{k/p}.$$

(3) From the convergence of the series (4) we see evidently that

$$\frac{1}{2^n}\sum_{j=2^n}^{2^n+1} |\sigma_j^{(\alpha-1)} - \sigma_j^{(\alpha)}|^p = o(1) \qquad \text{as } n \to \infty,$$

from which we get easily

$$\sum_{j=1}^{n} |\sigma_j^{(\alpha-1)} - \sigma_j^{(\alpha)}|^p = o(n) \qquad \text{as } n \to \infty.$$

Thus we complete the proof of Theorem 1.

The main purpose of this paper is to mention some results of the summability $\{C, \alpha\}_{k,p}$ of Fourier series. The discussion will be done refering to the T. M. Flett paper [1].

2. Notations. We suppose throughout that $f(\theta)$ is of period 2π and integrable $(-\pi, \pi)$. We write

$$\varphi(t) = f(\theta + t) + f(\theta - t),$$
$$\psi(t) = f(\theta + t) - f(\theta - t).$$

For $\alpha < 0$ and $t \geqq 0$, denote by $\Phi_\alpha(t)$ the Riemann-Liouville α-th integral of $\varphi(t)$ with origin 0, that is,

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}\varphi(u)du,$$
$$\Phi_\alpha(t+0) = 0$$

and let $\Phi_0(t) = \varphi(t)$.
Similarly, let $\Psi_\alpha(t)$ be the α-th integral of $\psi(t)$, and we write

$$\varphi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t),$$
$$\psi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Psi_\alpha(t).$$

Let the Fourier series of $f(\theta)$ be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) = \sum_{n=0}^{\infty} A_n(\theta)$$

and let $B_n(\theta) = a_n \sin n\theta - b_n \cos n\theta$ so that the conjugate series of $f(\theta)$ is $\sum_{n=1}^{\infty} B_n(\theta)$. Hence we have

$$\varphi(t) \sim 2\sum_{n=0}^{\infty} A_n(\theta)\cos nt$$

and

$$\psi(t) \sim -2\sum_{n=1}^{\infty} B_n(\theta)\sin nt.$$

We denote by $t_n^{(\beta)} = t_n^{(\beta)}(\theta)$ and $\bar{t}_n^{(\beta)} = \bar{t}_n^{(\beta)}(\theta)$ be the n-th Cesàro means of order β of the sequences $\{nA_n(\theta)\}$ and $\{nB_n(\theta)\}$ respectively.

We use $A = A(\alpha, \beta, \ldots)$ to denote a positive constant depending on the parameters $\alpha, \beta, \ldots\ldots$, but it will be different in each occurrence.

The inequality of the form

$$L \leqq A \cdot R$$

is to be interpreted as: if the value of the expression R is finite, so is the expression L and the inequality mentioned holds.

REMARK. As an integral analogue of the summability $\{C, \alpha\}_{k,p}$, we may, e. g., consider the convergence of the series which appears in the first term of the right of the inequality in Theorem 2 below, and the analogue of Theorem 1 will be shown, but we do not treat it here.

3. One of the present authors obtained the following theorem [4].

THEOREM T. *If* $1 < p \leqq 2$ *and* $\beta > 1/p$, *then for* $\delta > p - 1$,

$$\sum_{1}^{\infty} \frac{|t_n^{(\beta)}|}{n} \leqq A \int_0^{\pi} \frac{|\varphi(t)|^p}{t} \left| \log \frac{1}{t} \right|^{\delta} dt.$$

Generalizing this theorem to the form of summability $\{C, \alpha\}_{k,p}$, we shall establish the following theorems.

THEOREM 2. *If* $1 \leqq p \leqq 2$, k 1 *and* $\beta > \alpha + \sup(1/p, 1/k')$ $(k' = k/(k-1))$, *then we have*

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |\bar{t}_j^{(\beta)}|^p\right)^{k/p} \leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j} \frac{|\psi_\alpha(u)|^p}{u} du\right)^{k/p} + A\left(\int_0^{\pi} |\psi(u)| du\right)^k.$$

For the case $0 \leqq \alpha \leqq 1$, *the second term on the right may be suppressed.*

THEOREM 3. *If* $1 < p \leqq 2$, $k \geqq 1$, $\beta > \alpha + \sup(1/p, 1/k')$ *and either* $\alpha = 0$ *or* $\alpha \geqq 1 - \sup(1/p, 1/k')$ *then*

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |t_j^{(\beta)}|^p\right)^{k/p} \leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j} \frac{|\varphi_\alpha(u)|^p}{u} du\right)^{k/p} + A\left(\int_0^{\pi} |\varphi(u)| du\right)^k.$$

When $\alpha = 0$ *the second term on the right may be suppressed. If* $p = 1$ *the inequality holds when* $k \geqq 1$, $\alpha \geqq 0$ *and* $\beta > \alpha + 1$.

Theorem T is an easy consequence of Theorem 3 with $\alpha = 0$, $k = 1$. As a remaining case of Theorem 2 we shall prove the

THEOREM 4. *If* $1 < p \leqq 2$, $k \geqq 1$ *and* $0 < \alpha < \inf(1/p', 1/k)$ $(p' = p/(p-1))$, *then we have*

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |t_j^{(1)}|^p\right)^{k/p} \leqq \sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j} \frac{|\varphi_\alpha(u)|^p}{u} du\right)^{k/p}.$$

Theorem 2 and 3 correspond to Theorems 1 and 7 of the Flett paper [2] respectively, but they do not mutually coincide.

4. The proof of Theorem 3 is similar to that of Cases I–III in Theorem 2, and we shall give the proof of Theorem 2 and 4.

We need some preliminary lemmas.

LEMMA 1. *If* $g(u)$ *is integrable and* $\alpha \geqq 1$, *then*

$$|g_\alpha(u)| \leqq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t |g(u)| du,$$

LEMMA 2. *Let $\delta > 0$, $p \geqq 0$ and let*

$$P_n(t) = P_n(p, \delta, t) = \sum_{j=0}^{n}{}' E_{n-j}^{(\delta-1)} j^p e^{ijt}$$

where the dash signifies that when $p = 0$, the term corresponding to $j = 0$ is $\frac{1}{2} E_n^{(\delta-1)}$, and generally we write

$$E_n^{(\alpha)} = \binom{\alpha+n}{n} \sim n^{\alpha} \qquad (\alpha > -1).$$

For all t we have

$$P_n(t) = O(n^{p+\delta}),$$

and for $\pi/n \leqq t \leqq \pi$ we have

$$P_n(t) = E_n^{(\delta-1)} Q(p, t) + \frac{n^p e^{nit}}{(1-e^{-it})^{\delta}} + O(n^{p-1} t^{-\delta-1}) + O(n^{\delta-2} t^{-p-2}),$$

where $Q(p, t)$ depends only on p and t and satisfies the relation

$$Q(p, t) = \Gamma(p+1) e^{(p+1)\pi i/2} t^{-p-1} + O(1).$$

If in addition $p \geqq 1$, then for $\pi/n \leqq t \leqq \pi$, we have

$$P_n(t) = E_n^{(\delta-1)} Q(p, t) + \frac{n^p e^{nit}}{(1-e^{-it})^{\delta}} - \frac{p\delta\, n^{p-1} e^{(n-1)it}}{(1-e^{-it})^{\delta+1}} + R_n(t),$$

where $$R_n(t) = O(n^{p-2} t^{-\delta-2}) + O(n^{\delta-2} t^{-p-2}).$$

For $\pi/n \leqq t \leqq \pi$, all O's are uniform and

$$(1-e^{-it})^{\delta} = \left(2 \sin \frac{t}{2}\right)^{\delta} e^{\delta(\pi-t)i/2}.$$

LEMMA 3. *Let $0 \leqq l < 1$, $p \geqq 1$, $\delta > 0$ and let $P_n(p, \delta, t)$ be defined in the preceding Lemma. If we write for $0 < u \leqq \pi$,*

$$K_n(u) = K_n(l, p, \delta, u) = \frac{1}{\Gamma(1-l)} \int_u^{\pi} (t-u)^{-l} P_n(t) dt,$$

then $$K_n(u) = O(n^{l+p+\delta-1})$$

and $$K_n'(u) = O(n^{l+p+\delta})$$

uniformly in $0 < u \leqq \pi/2$, and

$$K_n(u) = O\{(n^p + n^{\delta-1})(\pi-u)^{1-l}\}$$
$$= O\{(n^p + n^{\delta-1}) n^{l-1}\}$$

uniformly in $\pi - \pi/n \leqq u \leqq \pi$. Further for $\pi/n \leqq u < \pi$

$$K_n(u) = L_n(u) - M_n(u),$$

where

$$L_n(u) = \frac{n^{l+p-1}e^{niu-(l-1)\pi i/2}}{(1-e^{-iu})^\delta} + O(n^{l+p-2}\,u^{-\delta-1}) + O(n^{\delta-1}\,u^{-l-p}),$$

$$L_n'(u) = -\frac{n^{l+p}e^{niu-l\pi i/2}}{(1-e^{-iu})^\delta} + O(n^{l+p-1}\,u^{-\delta-1}) + O(n^{\delta-1}\,u^{-l-p-1})$$
$$+ O\{(n^{p-2}+n^{\delta-1})(\pi-u)^{-l}\},$$
$$M_n(u) = O\{n^{p-1}(\pi-u)^{-l}\},$$
$$\mathrm{Re}\{M_n(u)\} = O\{n^{p-2}(\pi-u)^{-l-1}\},$$

and

$$M_n'(u) = O\{n^{p-\epsilon}(\pi-u)^{-l-\epsilon}\}$$

uniformly in $\pi/n \leqq u < \pi$. *Here* ε *is any fixed number such that* $0 < \varepsilon \leqq 1$.

These Lemmas 1–3 are all due to T. M. Flett [2].

LEMMA 4. *Suppose that* $F(t)$ *is of period* 2π *and integrable* $(-\pi, \pi)$ *and that*

$$F(t) \sim \sum_{-\infty}^{\infty} c_n e^{nit}$$

If $1 < k \leqq r < \infty$, $0 \leqq \sigma < 1/k'$, $\lambda = 1/k - 1/r + \sigma - 1 \geqq 0$ *and* $1/k + 1/k' = 1$, *then*

$$\left\{\sum_{-\infty}^{\infty}(|n|+1)^{-\lambda r}|c_n|^r\right\}^{1/r} \leqq A\left\{\int_{-\pi}^{\pi}|F(t)|^k|t|^{k\sigma}dt\right\}^{1/k}.$$

This is due to H. R. Pitt [3]. Lemm 4 reduces to the Hausdorff-Young theorem if $\sigma = \lambda = 0$ and $r = k'$, and to the Hardy-Littlewood theorem if $r = k$ and $\sigma = 0$.

5. PROOF OF THEOREM 2. Since

$$B_n(\theta) = -\frac{1}{\pi}\int_0^\pi \psi(t)\sin nt\; dt \tag{5.0.1}$$

we get

$$\bar{t}_n^{(\beta)} = \frac{1}{E_n^{(\beta)}}\sum_{i=1}^{n} E_{n-j}^{(\beta-1)}\, jB_j(\theta)$$
$$= -\frac{1}{\pi E_n^{(\beta)}}\int_0^\pi \psi(t)\sum_{j=1}^{n} E_{n-j}^{(\beta-1)}\, j\sin jt\; dt$$
$$= -\frac{1}{\pi}\int_0^\pi \psi(t)\; S_n(t)\; dt \tag{5.0.2}$$

where

$$S_n(t) = \frac{1}{E_n^{(\beta)}} \sum_{j=1}^{n} E_{n-j}^{(\beta-1)} j \sin jt$$

$$= \frac{1}{E_n^{(\beta)}} \operatorname{Im} \{P_n(1, \beta, t)\}. \tag{5.0.3}$$

Integrating (5.2) by parts q times, we have

$$\bar{t}_n^{(\beta)} = -\frac{1}{\pi}\Big[\sum_{m=0}^{q-1}(-1)^m \Psi_{m+1}(t)\, S_n^{(m)}(t)\Big]_0^{\pi}$$

$$+ \frac{(-1)^{q+1}}{\pi}\int_0^{\pi} \Psi_q(t)\, S_n^{(q)}(t)\, dt\,. \tag{5.0.4}$$

We have now to distinguish five cases.

Case I.	$q = \alpha \geqq 0,$	$1 < p \leqq 2,$
Case II.	$1 \leqq q < \alpha,$	$1 < p \leqq 2,$
Case III.	$q > \alpha$	$1 < p \leqq 2,$
Case IV.	$\alpha > 0,\ \beta \leqq 1,$	$1 < p \leqq 2,$
Case V.	$p = 1.$	

In the first three of them we take q to be the greatest integer such that $q < \beta$. Since

$$S_n^{(m)}(\pi) = \frac{1}{E_n^{(\beta)}} \operatorname{Im} \{P_n^{(m)}(1, \beta, \pi)\}$$

$$= \frac{1}{E_n^{(\beta)}} \operatorname{Im} \{i^m P_n(m+1, \beta, \pi)\}$$

$$= O(n^{m+1-\beta} + n^{-2}),$$

it follows from (5.0.4) and Lemma 1 that in these three cases we have

$$\bar{t}_n^{(\beta)} = -\frac{1}{\pi E_n^{(\beta)}}\int_0^{\pi} \Psi_q(t) \operatorname{Im} \{(-i)^q P_n(q+1, \beta, t)\}\ dt$$

$$+ O(n^{q-\beta})\int_0^{\pi} |\psi(t)|\, dt. \tag{5.0.5}$$

In Case IV we take $q = 1$. Since $S(\pi) = 0$ we have

$$\bar{t}_n^{(\beta)} = \frac{1}{\pi E_n^{(\beta)}}\int_0^{\pi} \Psi_1(t) \operatorname{Re} \{P_n(2, \beta, t)\}\ dt \tag{5.0.6}$$

5.1. CASE I. $q = \alpha \geqq 0,\ 1 < p \leqq 2.$

1°. We first consider the case $k \leqq p$. Using (5.5) we get

$$\sum_{n=0}^{\infty}\Big(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |\bar{t}_j^{(\beta)}|^p\Big)^{k/p}$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\left[\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_0^\pi \Psi_\alpha^{(t)}\operatorname{Im}\{(-i)^\alpha P_n(t)\}dt+j^{\alpha-\beta}\int_0^\pi|\psi(t)|dt\right|^p\right]^{k/p}$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\left[\sum_{j=2^n}^{2^{n+1}-1}\left\{\left|\frac{1}{j^\beta}\int_0^{\pi/2^n}\right|^p+\left|\frac{1}{j^\beta}\int_{\pi/2^n}^{\pi}\right|^p+\left(j^{\alpha-\beta}\int_0^\pi|\psi(t)|\,dt\right)^p\right\}\right]^{k/p}$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\left(\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_0^{\pi/2^n}\Psi_\alpha(t)\operatorname{Im}\{(-i)^\alpha P_n(t)\}\;dt\right|^p\right)^{k/p}$$

$$+A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\left(\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_{\pi/2^n}^{\pi}\Psi_\alpha(t)\operatorname{Im}\{(-i)^\alpha P_n(t)\}\;dt\right|^p\right)^{k/p}$$

$$+A\left(\int_0^\pi|\psi(t)|\,dt\right)^k\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\left(\sum_{j=2^n}^{2^{n+1}-1}j^{(\alpha-\beta)p}\right)^{k/p}$$

$$=I_1+I_2+I_3 \tag{5.1.1}$$

say. Since $\beta>\alpha$ it is obvious that

$$I_3\leqq A\left(\int_0^\pi|\psi(t)|\,dt\right)^k\sum_{n=0}^{\infty}\frac{1}{2^{(\beta-\alpha)nk}}$$

$$\leqq A\left(\int_0^\pi|\psi(t)|\,dt\right)^k. \tag{5.1.2}$$

By Lemma 2 we have

$$P_n(\alpha+1,\beta,t)=O(n^{\alpha+\beta+1})=O(n^{\beta+1}t^{-\alpha}) \tag{5.1.3}$$

uniformly in $0<t\leqq\pi/n$. Hence

$$I_1\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left(j\int_0^{\pi/2^n}|\Psi_\alpha(t)|\,t^{-\alpha}\;dt\right)^p\right\}^{k/p}$$

$$\leqq A\sum_{n=0}^{\infty}\frac{2^{nk/p}\,2^{nk}}{2^{nk/p}}\left(\int_0^{\pi/2^n}|\Psi_\alpha(t)|t^{-\alpha}\,dt\right)^k$$

$$\leqq A\sum_{n=0}^{\infty}2^{nk}\left(\int_0^{\pi/2^n}|\Psi_\alpha(t)|^p t^{-\alpha p}\,dt\right)^{k/p}\left(\int_0^{\pi/2^n}dt\right)^{k/p'}\quad (p'=p/(p-1))$$

$$=A\sum_{n=0}^{\infty}2^{nk/p}\left(\sum_{j=n}^{\infty}\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{-\alpha p}\,dt\right)^{k/p}$$

$$\leqq A\sum_{n=0}^{\infty}2^{nk/p}\sum_{j=n}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{-\alpha p}\,dt\right)^{k/p}\qquad (\text{as } k\leqq p)$$

$$\leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{-\alpha p}\,dt\right)^{k/p}\sum_{n=0}^{j}2^{nk/p}$$

$$\leqq A\sum_{j=0}^{\infty}2^{jk/p}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{-\alpha p}\,dt\right)^{k/p}$$

$$\leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(t)|^p}{t}\,dt\right)^{k/p}. \tag{5.1.4}$$

We can suppose $\beta < \alpha + 1$, since for any $\gamma > \beta$ we have

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}|\bar{t}_j^{(\gamma)}|^p\right)^{k/p} \leqq \sum_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}|\bar{t}_j^{(\beta)}|^p\right)^{k/p},$$

which is an analogue of the inequality between the two summabilities $|C, \beta|_k$ and $|C,\gamma|_k$ (Flett [1]), and whose proof is omitted here. Under this condition, we have from Lemma 2

$$(-i)^\alpha P_n(\alpha+1,\beta,t) = \frac{n^{\alpha+1}e^{i\{(n+\beta/2)t-(\alpha+\beta)\pi/2\}}}{\left(2\sin\frac{t}{2}\right)^\beta} + O(n^\alpha t^{-\beta-1}) \tag{5.1.5}$$

uniformly in $\pi/n \leqq t \leqq \pi$. We get therefore

$$\begin{aligned}
I_2 &\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_{\pi/2^n}^{\pi}\Psi_\alpha(t)\left(\frac{j^{\alpha+1}\sin\{(j+\beta/2)t-(\alpha+\beta)\pi/2\}}{\left(2\sin\frac{t}{2}\right)^\beta}\right.\right.\right.\\
&\qquad\qquad\left.\left.\left. + O(j^\alpha t^{-\beta-1})\right)dt\right|^p\right\}^{k/p}\\
&\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^{\beta-\alpha-1}}\int_{\pi/2^n}^{\pi}\Psi_\alpha(t)\frac{\sin\{(j+\beta/2)t-(\alpha+\beta)\pi/2\}}{\left(\sin\frac{t}{2}\right)^\beta}\,dt\right|^p\right\}^{k/p}\\
&\qquad + A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left(\frac{1}{j^{\beta-\alpha}}\int_{\pi/2^n}^{\pi}|\Psi_\alpha(t)|t^{-\beta-1}\,dt\right)^p\right\}^{k/p}\\
&= I_2' + I_2''
\end{aligned} \tag{5.1.6}$$

say. As easily seen we have

$$I_2' \leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1/p)}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\frac{1}{j^{2-p}}\left|\int_{\pi/2^n}^{\pi}\Psi_\alpha(t)\frac{\sin\{(j+\beta/2)t-(\alpha+\beta)\pi/2\}}{\left(\sin\frac{t}{2}\right)^\beta}\,dt\right|^p\right\}^{k/p}.$$

Applying the Hardy-Littlewood theorem (Lemma 4) to the inner sum, we get

$$\begin{aligned}
I_2' &\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1/p)}}\left(\int_{\pi/2^n}^{\pi}|\Psi_\alpha(t)|^p t^{-\beta p}\,dt\right)^{k/p}\\
&\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1/p)}}\left(\sum_{j=0}^{n-1}\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{-\beta p}\,dt\right)^{k/p}\\
&\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1/p)}}\sum_{j=0}^{n-1}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{-\beta p}\,dt\right)^{k/p}
\end{aligned}$$

$$\leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{-\beta p}\,dt\right)^{k/p}\sum_{n=j+1}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1/p)}}$$

$$\leqq A\sum_{j=0}^{\infty}\frac{1}{2^{jk(\beta-\alpha-1/p)}}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{-\beta p}\,dt\right)^{k/p}\qquad\left(\text{as }\beta>\alpha+\frac{1}{p}\right)$$

$$\leqq A\sum_{j=0}^{\infty}\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p\, t^{-\beta p}\, t^{p(\beta-\alpha-1/p}\,dt\Big)^{k/p}$$

$$= A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(t)|^p}{t}\,dt\right)^{k/p}. \tag{5.1.7}$$

We take δ so that $\alpha+1/p<\delta<\alpha+1/p+\sup(1/p,\ 1/k)$, then by the Hölder inequality we get

$$I_2''\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\,2^{nk(\alpha-\beta)}\,2^{nk/p}\left(\int_{\pi/2^n}^{\pi}|\Psi_\alpha(t)\mid t^{-\beta-1}\,dt\right)^{k}$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha)}}\left(\int_{\pi/2^n}^{\pi}|\Psi_\alpha(t)|^p t^{-\delta p}\,dt\right)^{k/p}\left(\int_{\pi/2^n}^{\pi}t^{(\delta-\beta-1)p'}\,dt\right)^{k/p'}$$

$$= A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha)}}\left(\sum_{j=0}^{n-1}\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{-\delta p}\,dt\right)^{k/p}2^{-nk(\delta-\beta-1+1/p')},$$

since

$$(\alpha-\beta-1)p'+1<\left\{\alpha+\frac{1}{p}+\sup\left(\frac{1}{p},\ \frac{1}{k}\right)-\beta-1+\frac{1}{p'}\right\}p'$$

$$=\alpha+\sup\left(\frac{1}{p},\ \frac{1}{k'}\right)-\beta<0.$$

Hence

$$I_2''\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\delta-\alpha-1/p)}}\sum_{j=0}^{n-1}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{-\delta p}\,dt\right)^{k/p}$$

$$\leqq A\sum_{j=0}^{\infty}\frac{1}{2^{jk(\delta-\alpha-1/p)}}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{-\delta p}\,dt\right)^{k/p}$$

$$\leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(t)|^p}{t}\,dt\right)^{k/p} \tag{5.1.8}$$

From (5.1.1), (5.1.2), (5.1.4), (5.1.6), (5.1.7) and (5.1.8) we get the required result for $k\leqq p$ in Case I.

2^0. Now we suppose $k>p$. We get from (5.0.5), applying the Hölder inequality

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}|\bar{t}_j^{(\beta)}|^p\right)^{k/p}$$

$$\leqq \sum_{n=0}^{\infty} \frac{1}{2^{nk/p}} \Big(\sum_{j=2^n}^{2^{n+1}-1} |\bar{t}_j^{(\beta)}|^p \Big) \, 2^{n(k/p-1)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{j=2^n}^{2^{n+1}-1} |\bar{t}_j^{(\beta)}|^k$$

$$\leqq A \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{j=2^n}^{2^{n+1}-1} \left| \frac{1}{j^\beta} \int_0^\pi \Psi_\alpha(t) \operatorname{Im} \{(-i)^\alpha P_n(t)\} \, dt \right|^k$$

$$+ \sum_{n=0}^{\infty} \frac{1}{2^n} \Big(\frac{1}{j^{\beta-\alpha}} \int_0^\pi |\psi(t)| \, dt \Big)^k$$

$$\leqq A \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{j=2^n}^{2^{n+1}-1} \left| \frac{1}{j^\beta} \int_0^{\pi/2^n} \right|^k + A \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{j=2^n}^{2^{n+1}-1} \left| \frac{1}{j^\beta} \int_{\pi/2^n}^{\pi} \right|^k$$

$$+ A \Big(\int_0^\pi |\psi(t)| \, dt \Big)^k \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{j=2^n}^{2^{n+1}-1} \frac{1}{j^{k(\beta-\alpha)}}$$

$$= J_1 + J_2 + J_3 \tag{5.1.9}$$

say. Since $\beta > \alpha$ we have

$$J_3 \leqq A \Big(\int_0^\pi |\psi(t)| \, dt \Big)^k \sum_{n=0}^{\infty} \frac{1}{2^{n(\beta-\alpha)}}$$

$$\leqq A \Big(\int_0^\pi |\psi(t)| \, dt \Big)^k. \tag{5.1.10}$$

By (5.1.3) and the Hölder inequality we get

$$J_1 \leqq A \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{j=2^n}^{2^{n+1}-1} \Big(j \int_0^{\pi/2^n} |\Psi_\alpha(t)| \, t^{-\alpha} \, dt \Big)^k$$

$$\leqq A \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot 2^{n(k+1)} \Big(\int_0^{\pi/2^n} |\Psi_\alpha(t)|^p t^{-\alpha p} \Big)^{k/p} \Big(\int_0^{\pi/2^n} dt \Big)^{k/p'}$$

$$= A \sum_{n=0}^{\infty} 2^{nk/p} \Big(\sum_{j=n}^{\infty} \int_{\pi/2^{j+1}}^{\pi/2^j} |\psi_\alpha(t)|^p dt \Big)^{k/p}.$$

If we take a constant δ, $0 < \delta < 1$, then

$$J_1 \leqq A \sum_{n=0}^{\infty} 2^{nk/p} \Big(\sum_{j=n}^{\infty} \frac{1}{2^{j\delta}} \int_{\pi/2^{j+1}}^{\pi/2^j} |\psi_\alpha(t)|^p t^{-\delta} \, dt \Big)^{k/p}$$

$$\leqq A \sum_{n=0}^{\infty} 2^{nk/p} \Big\{ \sum_{j=n}^{\infty} \Big(\int_{\pi/2^{j+1}}^{\pi/2^j} |\psi_\alpha(t)|^p t^{-\delta} \, dt \Big)^{k/p} \Big\} \Big(\sum_{j=n}^{\infty} \frac{1}{2^{j\delta k/(k-p)}} \Big)^{k/p-1}$$

$$\leqq A \sum_{n=0}^{\infty} 2^{nk(1-\delta)/p} \sum_{j=n}^{\infty} \Big(\int_{\pi/2^{j+1}}^{\pi/2^j} |\psi_\alpha(t)|^p t^{-\delta} \, dt \Big)^{k/p}$$

$$= A \sum_{j=0}^{\infty} \Big(\int_{\pi/2^{j+1}}^{\pi/2^j} |\psi_\alpha(t)|^p t^{-\delta} \, dt \Big)^{k/p} \sum_{n=0}^{j} 2^{nk(1-\delta)/p}$$

$$\leqq A\sum_{j=0}^{\infty} 2^{j(1-\delta)k/p}\left(\int_{\pi/2^{j+1}}^{\pi/2^j} |\Psi_\alpha(t)|^p t^{-\delta}\, dt\right)^{k/p}$$

$$\leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j} \frac{|\Psi_\alpha(t)|^p}{t}\, dt\right)^{k/p}. \tag{5.1.11}$$

In order to estimate J_2, we use the estimation (5.1.5), we have

$$J_2 \leqq A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^{\beta-(\alpha+1)}}\int_{\pi/2^n}^{\pi}\Psi_\alpha(t)\frac{\sin\{(j+\beta/2)t-(\alpha+\beta)\pi/2\}}{\left(2\sin\frac{t}{2}\right)^\beta}\, dt\right|^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left(\frac{1}{j^{\beta-\alpha}}\int_{\pi/2^n}^{\pi}|\Psi_\alpha(t)|\, t^{-\beta-1}\, dt\right)^k$$

$$= J_2' + J_2'' \tag{5.1.12}$$

say. Suppose first that $p' \geqq k$. Applying the Hölder inequality to the inner sum of J_2'. we get

$$J_2' \leqq A\sum_{n=0}^{\infty}\frac{1}{2^{n(\beta-\alpha-1/p)k}}\left(\sum_{j=2^n}^{2^{n+1}-1}\left|\int_{\pi/2^n}^{\pi}\Psi_\alpha(t)\frac{\sin\{(j+\beta/2)t-(\alpha+\beta)\pi/2\}}{\left(2\sin\frac{t}{2}\right)^\beta}\, dt\right|^{p'}\right)^{k/p'}.$$

Hence by the Hausdorff-Young theorem we have

$$J_2' \leqq A\sum_{n=0}^{\infty}\frac{1}{2^{n(\beta-\alpha-1/p)k}}\left(\int_{\pi/2^n}^{\pi}|\Psi_\alpha(t)|^p t^{-\beta p}\, dt\right)^{k/p} \tag{5.1.13}$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{n(\beta-\alpha-1/p)k}}\left(\sum_{j=0}^{n-1}2^{j\eta}\int_{\pi/2^{j+1}}^{\pi/2^j} t^{\eta-\beta p}|\Psi_\alpha(t)|^p\, dt\right)^{k/p}$$

where η is a positive constant such that

$$\eta < p(\beta-\alpha-1/p).$$

Then,

$$J_2' \leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1/p)}}\left(\sum_{j=0}^{n-1}2^{j\eta k/(k-p)}\right)^{k/p-1}\left\{\sum_{j=0}^{n-1}\left(\int_{\pi/2^{j+1}}^{\pi/2^j} t^{\eta-\beta p}|\Psi_\alpha(t)|^p\, dt\right)^{k/p}\right\}$$

$$\leqq A\sum_{n=0}^{\infty}\frac{2^{nk\eta/p}}{2^{nk(\beta-\alpha-1/p)}}\sum_{j=0}^{n}\left(\int_{\pi/2^{j+1}}^{\pi/2^j} t^{\eta-\beta p}|\Psi_\alpha(t)|^p\, dt\right)^{k/p}$$

$$= A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j} t^{\eta-\beta p}|\Psi_\alpha(t)|^p\, dt\right)^{k/p}\sum_{n=j}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1/p-\eta/p)}}$$

The last series is convergent by the condition of η and has the sum $O(2^{-jk(\beta-\alpha-1/p-\eta/p)})$, we get easily

$$J_2' \leqq A\sum_{j=0}^{\infty}\frac{1}{2^{jk(\beta-\alpha-1/p-\eta/p)}}\left(\int_{\pi/2^{j+1}}^{\pi/2^j} t^{\eta-\beta p}|\Psi_\beta(t)|^p\, dt\right)^{k/p}$$

$$\leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(t)|^p}{t}\,dt\right)^{k/p}. \tag{5.1.14}$$

Now, suppose that $p' < k$. As $1 < k' < 2$ we can apply the Hausdorff-Young theorem, and we have

$$J_2' \leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1+1/k)}}\sum_{j=2^n}^{2^{n+1}-1}\left|\int_{\pi/2^n}^{\pi}\Psi_\alpha(t)\frac{\sin\{(j+\beta/2)t-(\alpha+\beta)\pi/2\}}{\left(2\sin\frac{t}{2}\right)^\beta}\,dt\right|^k$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1/k')}}\left(\int_{\pi/2^n}^{\pi}|\Psi_\alpha(t)|^{k'}t^{-\beta k'}\,dt\right)^{k-1}.$$

Employing the same argument as in the preceding case, we get

$$J_2' \leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(t)|^{k'}}{t}\,dt\right)^{k-1}$$

$$\leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(t)|^p}{t}\,dt\right)^{k/p}$$

since $k' < p$. (5.1.15)

We estimate J_2''. Let δ be the constant appeared in the estimation of I_2'', and let τ be a positive constant such that

$$\delta - \alpha - 1/p > \tau/p.$$

We have, by the Hölder inequality,

$$J_2 \leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\delta-\alpha-1/p)}}\left(\sum_{j=0}^{n}2^{\tau j}\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{\tau-\delta p}\,dt\right)^{k/p}$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\delta-\alpha-1/p-\tau/p)}}\sum_{j=0}^{n}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{\tau-\delta p}\,dt\right)^{k/p}$$

$$\leqq A\sum_{j=0}^{n}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(t)|^p t^{\tau-\delta p}\,dt\right)^{k/p}2^{-jk(\delta-\alpha-1/p-\tau/p)}$$

$$\leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(t)|^p}{t}\,dt\right)^{k/p}. \tag{5.1.16}$$

From (5.1.9), (5.1.10), (5.1.11), (5.1.12), (5.1.14), (5.1.15) and (5.1.16), we complete the proof of Case I for $k > p$.

5.2. CASE II. $1 \leqq q < \alpha$, $1 < p \leqq 2$. Since

$$\int_0^\pi \Psi_q(t)\,\mathrm{Im}\,\{(-i)^q P_n(t)\}\,dt$$

$$= \frac{1}{\Gamma(q-\alpha+1)}\int_0^\pi \mathrm{Im}\,\{(-i)^q P_n(t)\}\,dt\int_0^t (t-u)^{q-\alpha}\Psi_{\alpha-1}(u)\,du$$

$$= \frac{1}{\Gamma(q-\alpha+1)}\int_0^\pi \Psi_{\alpha-1}(u)\int_u^\pi (t-u)^{q-\alpha}\,\mathrm{Im}\,\{(-i)^q P_n(t)\}\;dt$$

$$= A\int_0^\pi \Psi_{\alpha-1}(u)\,\mathrm{Im}\,\{(-i)^q K_n(\alpha-q, q+1, \beta, u)\}\;du,$$

integrating by parts and observing $\Psi_\alpha(0) = K_n(\pi) = 0$ we get

$$\int_0^\pi \Psi_q(t)\,\mathrm{Im}\,\{(-i)^q P_n(t)\}\,dt = -\int_0^\pi \Psi_\alpha(u)\,\mathrm{Im}\,\{(-i)^q K_n'(u)\}\;du$$

where $K_n(u)$ is defined in Lemma 3.

Therefore we can write, by (5. 0. 5),

$$\bar{t}_n^{(\beta)} = \frac{1}{\pi E_n^{(\beta)}}\int_0^\pi \Psi_\alpha(u)\,\mathrm{Im}\,\{(-i)^q K_n'(u)\}\;du + O(n^{q-\beta})\int_0^\pi |\psi(t)|\;dt. \qquad (5.2.1)$$

1°. As before we consider the case $k \leqq p$. We have

$$\sum_{n=0}^\infty \left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |\bar{t}_j^{(\beta)}|^p\right)^{k/p}$$

$$\leqq A\sum_{n=0}^\infty \frac{1}{2^{nk/p}}\left(\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_0^{\pi/2^n} \Psi_\alpha(u)\,\mathrm{Im}\,\{(-i)^q K_n'(u)\}\;du\right|^p\right)^{k/p}$$

$$+ A\sum_{n=0}^\infty \frac{1}{2^{nk/p}}\left(\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_{\pi/2^n}^{\pi} \Psi_\alpha(u)\,\mathrm{Im}\,\{(-i)^q K_n'(u)\}\;du\right|^p\right)^{k/p}$$

$$+ A\left(\int_0^\pi \psi(t)\,dt\right)^k \sum_{n=0}^\infty \frac{1}{2^{nk/p}}\left(\sum_{j=2^n}^{2^{n+1}-1} j^{(q-\beta)p}\right)^{k/p}$$

$$= K_1 + K_2 + K_3 \qquad (5.2.2)$$

say. By Lemma 3 we have

$$K_n'(\alpha - q, q + 1, \beta, u) = O(n^{\alpha+\beta+1}) = O(n^{\beta+1}u^{-\alpha}) \qquad (5.2.3)$$

uniformly for $0 < u \leqq \pi/n$. Hence

$$K_1 \leqq \sum_{n=0}^\infty \frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left(\frac{1}{j}\int_0^{\pi/2^n} \Psi_\alpha(u)u^{-\alpha}\,du\right)^p\right\}^{k/p}.$$

Thus the estimation of K_1 is quite similar to that of I_1, and so is K_3 to I_3. We may omit the detail calculation.

We may suppose $\beta < \alpha + 1$ as before, and then we may suppose

$$\beta + \varepsilon < q + 2 < \alpha + 2$$

where ε is a fixed constant such that $0 < \varepsilon < 1$. Under these restrictions we have Lemma 3,

$$(-i)^q K_n'(u) = \frac{-n^{\alpha+1}\,e^{i\{(n+\beta/2)u-(\alpha+\beta)\pi/2\}}}{\left(2\sin\frac{u}{2}\right)^\beta} + O(n^\alpha u^{-\beta-1})$$

$$+ O\{n^{q+1-\epsilon}(\pi - u)^{-\alpha-\epsilon+q}\} \tag{5.2.4}$$

uniformly in $\pi/n \leqq u < \pi$. we have

$$K_2 \leqq \sum_{n=0}^{\infty} \frac{1}{2^{nk/p}} \left\{ \sum_{j=2^n}^{2^{n+1}-1} \left| \frac{1}{j^\beta} \int_{\pi/2^n}^{\pi} \Psi_\alpha(u) \left(j^{\alpha+1} \frac{\sin\{(j+\beta/2)u-(\alpha+\beta)\pi/2\}}{\left(2\sin\frac{u}{2}\right)^\beta} \right) du \right|^p \right\}^{k/p}$$

$$+ A \sum_{n=0}^{\infty} \frac{1}{2^{nk/p}} \left\{ \sum_{j=2^n}^{2^{n+1}-1} \left(\frac{1}{j^{\beta-\alpha}} \int_{\pi/2^n}^{\pi} |\Psi_\alpha(u)| u^{-\beta-1}\, du \right)^p \right\}^{k/p}$$

$$+ A \sum_{n=0}^{\infty} \frac{1}{2^{nk/p}} \left\{ \sum_{j=2^n}^{2^{n+1}-1} \left(j^{q+1-\epsilon-\beta} \int_{\pi/2^n}^{\pi} |\Psi_\alpha(u)| (\pi-u)^{q-\alpha-\epsilon} du \right)^p \right\}^{k/p}$$

$$= K_2' + K_2'' + K_2'''$$

say. The estimation of K_2' and K_2'' will be done along the similar way to those of I_2' and I_2'', therefore it is sufficient to estimate K_2'''. We take η so small that $0 < \eta < \frac{1}{2}(\alpha - q)$ and that $0 < 1 + \alpha - \beta - \eta < 1$. We may suppose $\varepsilon = 1 + \alpha - \beta - \eta$.

Since $q < \alpha - \eta$, we have

$$\beta + \varepsilon - q - 1 = \alpha - \eta - q > \alpha - \eta - (\alpha - \eta) = 0.$$

From $q \geqq \beta - 1$, we get

$$q + 1 - \alpha - \varepsilon \geqq (\beta - 1) + 1 - \alpha - (1 + \alpha - \beta - \eta)$$
$$= 2(\beta - \alpha) - 1 + \eta \geqq \eta > 0,$$

since $\beta > \alpha + 1/p \geqq \alpha + 1/2$, or $2(\beta - \alpha) \geqq 1$.

Thus we get

$$\alpha + \beta < q + 1 < \beta + \varepsilon.$$

Considering these inequalities we get

$$K_2''' \leqq A \sum_{n=0}^{\infty} \frac{1}{2^{nk(\beta-\epsilon-q-1)}} \left(\int_0^{\pi} (\pi-u)^{q-\alpha-\epsilon} |\Psi_\alpha(u)|\, du \right)^k$$

$$\leqq A \left(\int_0^{\pi} (\pi-u)^{q-\alpha-\epsilon} du \int_0^u (u-v)^{\alpha-1} |\psi(v)|\, dv \right)^k$$

$$\leqq A \left(\int_0^{\pi} |\psi(v)|\, dv \int_v^{\pi} (\pi-u)^{q-\alpha-\epsilon} (u-v)^{\alpha-1}\, du \right)^k$$

$$\leqq A \left(\int_0^{\pi} (\pi-v)^{q-\epsilon} |\psi(v)|\, dv \right)^k$$

$$\leqq A \left(\int_0^{\pi} |\psi(v)|\, dv \right)^k.$$

Combining the above estimations, we obtain the desired result.

2° We consider next the case $k > p$. By (5.2.1), we have

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}|\bar{t}_j^{(\beta)}|^p\right)^{k/p} \leqq \sum_{n=,}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}|\bar{t}_j^{(\beta)}|^k$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_0^{\pi/2^n}\Psi_\alpha(u)\,\mathrm{Im}\,\{(-i)^q K_n'(u)\}\,du\right|^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_{\pi/2^n}^{\pi}\Psi_\alpha(u)\,\mathrm{Im}\,\{(-i)^q K_n'(u)\}\,du\right|^k$$

$$+ A\left(\int_0^\pi|\psi(t)|\,dt\right)^k, \tag{5.2.5}$$

where the last term is what obtained by the reason similar to the estimate of J_3.

In virtue of (5.2.3) the first term of (5.2.5) is inferior to

$$A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left(j\int_0^{\pi/2^n}|\Psi_\alpha(u)|\,u^{-\alpha}\,du\right)^k,$$

and the third term is inferior to

$$\mathrm{A}\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^{\beta-\alpha-1}}\int_{\pi/2^n}^{\pi}\Psi_\alpha(u)\frac{\sin\{(j+\beta/2)u-(\alpha+\beta)\pi/2\}}{\left(2\sin\dfrac{u}{2}\right)^\beta}\,du\right|^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left(\frac{1}{j^{\alpha-\beta}}\int_{\pi/2^n}^{\pi}|\Psi_\alpha(\mathrm{u})|\,u^{-\beta-1}\,du\right)^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left(\frac{j^{q+1-\epsilon}}{j^\beta}\int_{\pi/2^n}^{\pi}|\Psi_\alpha(u)|(\pi-u)^{-\alpha-\epsilon+q}du\right)^k.$$

Therefore we obtain the required inequalities by repeating the quite similar estimations to those of J_1, J_2', J_2'' and K_2''' respsctively.

5.3. CASE III. $q > \alpha$, $1 < p \leqq 2$. Integrating by parts, we have

$$\int_0^\pi \Psi_q(t)\,\mathrm{Im}\,\{(-i)^q P_n(t)\}\;dt$$

$$= \frac{1}{\Gamma(q-\alpha)}\int_0^\pi \mathrm{Im}\,\{(-i)^q P_n(t)\}\;dt\int_0^t (t-u)^{q-\alpha-1}\Psi_\alpha(u)\;du$$

$$= \frac{1}{\Gamma(q-\alpha)}\int_0^\pi \Psi_\alpha(u)\,du\int_u^\pi (t-u)^{q-\alpha-1}\,\mathrm{Im}\,\{(-\iota)^q P_n(t)\}\;dt$$

$$= \int_0^\pi \Psi_\alpha(u)\,\mathrm{Im}\,\{(-i)^q K_n(\alpha+1-q, q+1, \beta, u)\}du.$$

Hence from (5.0.5) we get

$$\bar{t}_n^{(\beta)} = -\frac{1}{\pi E_n^{(\beta)}}\int_0^\pi \Psi_\alpha(u)\,\mathrm{Im}\,\{(-i)^q K_n(u)\}\,du + O(n^{q-\beta})\int_0^\pi |\psi(t)|\,dt. \tag{5.3.1}$$

We distinguish as before the two cases $k \leqq p$ and $k > p$.

1° For the case $k \leqq p$, we have

$$\sum_{n=0}^\infty \left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |\bar{t}_j^{(\beta)}|^p\right)^{k/p}$$
$$\leqq A\sum_{n=0}^\infty \frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_0^{\pi/2^n}\Psi_\alpha(u)\,\mathrm{Im}\,\{(-i)^q K_n(u)\}\,du\right|^p\right\}^{k/p}$$
$$+ A\sum_{n=0}^\infty \frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_{\pi/2^n}^{\pi-\pi/2^n}\right|^p\right\}^{k/p}$$
$$+ A\sum_{n=0}^\infty \frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_{\pi-\pi/2^n}^{\pi}\right|^p\right\}^{k/p}$$
$$+ K_3.$$
$$= L_1 + L_2 + L_3 + K_3$$

say. By Lemma 3, the function $K_n(t)$ satisfies the relations:

$$K_n(u) = O(n^{\beta+1}u^{-\alpha}) \tag{5.3.2}$$

uniformly in $0 < u \leqq \pi/n$,

$$K_n(u) = O(n^{\alpha+1}) \tag{5.3.3}$$

uniformly in $\pi - \pi/n \leqq u \leqq \pi$; and for $\pi/n \leqq u \leqq \pi - \pi/n$

$$(-i)^q K_n(u) = \frac{n^{\alpha+1}e^{i\{(n+\beta/2)u-(\alpha+\beta)\pi/2\}}}{\left(2\sin\frac{u}{2}\right)^\beta} + O(n^\alpha u^{-\beta-1})$$
$$+ O\{n^q(\pi-u)^{q-\alpha-1}\}. \tag{5.3.4}$$

Using (5.3.2) and (5.3.4) we get

$$L_1 \leqq A\sum_{n=0}^\infty \frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left(j\int_0^{\pi/2^n}|\Psi_\alpha(u)|\,u^{-\alpha}\,du\right)^p\right\}^{k/p},$$

$$L_2 \leqq A\sum_{n=0}^\infty \frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^{\beta-\alpha-1}}\int_{\pi/2^n}^{\pi-\pi/2^n}\Psi_\alpha(u)\frac{\sin\{(j+\beta/2)u-(\alpha+\beta)\pi/2\}}{\left(2\sin\frac{u}{2}\right)^p}\,du\right|^p\right\}^{k/p}$$
$$+ A\sum_{n=0}^\infty \frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left(\frac{1}{j^{\beta-\alpha}}\int_{\pi/2^n}^{\pi-\pi/2^n}|\Psi_\alpha(u)|\,u^{-\beta-1}\,du\right)^p\right\}^{k/p}$$
$$+ A\sum_{n=0}^\infty \frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left(\frac{1}{j^{\beta-q}}\int_{\pi/2^n}^{\pi-\pi/2^n}|\Psi_\alpha(u)|\,(\pi-u)^{q-\alpha-1}du\right)^p\right\}^{k/p}.$$

The estimations of L_1 and the first two terms of the right of the last inequality are similar to those of I_1, I_2' and I_2'' respectively. We denote by L_2' the last term of the inequality for L_2.

If $q < \alpha + 1/p$, then

$$(q-\alpha-1)p' + 1 = p'(q-\alpha-1/p) < 0.$$

We get therefore

$$\begin{aligned} L_2' &\leqq A\sum_{n=0}^{\infty} 2^{n(q-\beta)k}\left(\int_{\pi/2^n}^{\pi-\pi/2^n} |\Psi_\alpha(u)|(\pi-u)^{q-\alpha-1}\,du\right)^k \\ &\leqq A\sum_{n=0}^{\infty} 2^{n(q-\beta)k}\left(\int_{\pi/2^n}^{\pi-\pi/2^n} |\Psi_\alpha(u)|^p\,du\right)^{k/p}\left(\int_{\pi/2^n}^{\pi-\pi/2^n}(\pi-u)^{p'(q-\alpha-1)}\,du\right)^{k/p'} \\ &\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{n(\beta-\alpha-1/p)}}\left(\int_{\pi/2^n}^{\pi}|\Psi_\alpha(u)|^p\,du\right)^{k/p} \qquad (5.3.5) \end{aligned}$$

which is majorated by the required quantity.

If $q = \alpha + 1/p$, that is, $(q-\alpha-1)p' = -1$, then we have

$$\begin{aligned} L_2' &\leqq A\sum_{n=0}^{\infty} 2^{n(q-\beta)k}\left(\int_{\pi/2^n}^{\pi-\pi/2^n} |\Psi_\alpha(u)|^p\,du\right)^{k/p}\left(\int_{\pi/2^n}^{\pi-\pi/2^n}(\pi-u)^{-1}\,du\right)^{k/p'} \\ &\leqq A\sum_{n=0}^{\infty}\frac{(\log 2^n)^{k/p'}}{2^{nk(\beta-q)}}\left(\int_{\pi/2^n}^{\pi}|\Psi_\alpha(u)|^p\,du\right)^{k/p} \qquad (5.3.6) \\ &\leqq A\sum_{n=0}^{\infty}\frac{(\log 2^n)^{k/p'}}{2^{nk(\beta-q)}}\sum_{j=0}^{u}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(u)|^p\,du\right)^{k/p} \\ &\leqq A\sum_{j=0}^{\infty}\frac{(\log 2^j)^{k/p'}}{2^{j(\beta-q)k}}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Psi_\alpha(u)|^p\,du\right)^{k/p} \\ &\leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\Psi_\alpha(u)|^p}{u^{pq}}\,du\right)^{k/p} \\ &\leqq A\sum_{j=0}^{\infty}\left(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(u)|^p}{u}\,du\right)^{k/p}. \end{aligned}$$

Using (5.3.3) we have

$$\begin{aligned} L_3 &\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\left\{\sum_{j=2^n}^{2^{n+1}-1}\left(\frac{1}{j^{\beta-\alpha-1}}\int_{\pi-\pi/2^n}^{\pi}|\Psi_\alpha(u)|\,du\right)^p\right\}^{k/p} \\ &\leqq A\sum_{n=0}^{\infty}2^{nk(\alpha+1-\beta)}\left(\int_{\pi-\pi/2^n}^{\pi}|\Psi_\alpha(u)|^p\,du\right)^{k/p} \\ &\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1/p)}}\left(\int_{\pi/2^n}^{\pi}|\Psi_\alpha(u)|^p u^{-\beta p}\,du\right)^{p/k} \qquad (5.3.7) \end{aligned}$$

which satisfies the inequality of the required type, as we see in the estimation of I_2'.

2°. The case $k > p$. From (5. 3. 1)

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}|\bar{t}_j^{(\beta)}|^p\right)^{k/p} \leqq \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}|\bar{t}_j^{(\beta)}|^k.$$

By the similar argument as before, this is majorated by the sum of M_1, M_2, M_3, and J_3 where

$$M_1 \leqq A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^\beta}\int_0^{\pi/2^n}\Psi_\alpha(u)\,\mathrm{Im}\,\{(-i)^q K_n(u)\}\,du\right|^k$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left(j\int_0^{\pi/2^n}|\Psi_\alpha(u)|\,u^{-\alpha}\,du\right)^k; \tag{5. 3. 8}$$

by (5. 3. 2),

$$M_2 \leqq A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left|\frac{1}{j^{\beta-\alpha+1}}\int_{\pi/2^n}^{\pi-\pi/2^n}\Psi_\alpha(u)\frac{\sin\{(j+\beta/2)u-(\alpha+\beta)\pi/2\}}{\left(2\sin\frac{u}{2}\right)^\beta}\,du\right|^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left(\frac{1}{j^{\beta-\alpha}}\int_{\pi/2^n}^{\pi-\pi/2^n}|\Psi_\alpha(u)|\,u^{-\beta-1}\,du\right)^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-q)}}\left(\int_{\pi/2^n}^{\pi-\pi/2^n}|\Psi_\alpha(u)|\,(\pi-u)^{q-\alpha-1}\,du\right)^k \tag{5. 3. 9}$$

by (5. 3. 4), and

$$M_3 \leqq A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left(\frac{1}{j^{\beta-\alpha-1}}\int_{\pi-\pi/2^n}^{\pi}|\Psi_\alpha(u)|\,du\right)^k \qquad \text{(by (5. 3. 3))}$$

$$\leqq A\sum_{n=0}^{\infty}2^{n(\alpha+1-\beta)}\left(\int_{\pi-\pi/2^n}^{\pi}|\Psi_\alpha(u)|\,du\right)^k. \tag{5. 3. 10}$$

We can continue the estimations of (5 3. 8) and the first two terms in the right of (5. 3. 9) by the same fashion as those of J_1 and J_2 respectively. From (5. 3. 4) – (5. 3. 6) it follows as in (3. 1. 13) that the last term in the right of (5. 3. 9) and M_3 both satisfy the required inequality.

5. 4. CASE IV. $\alpha > 0$, $\beta \leqq 1$, $1 < p \leqq 2$. From (5. 0. 6), we have as in Case III,

$$\bar{t}_n^{(\beta)} = \frac{1}{\pi E_n^{(\beta)}}\int_0^\pi \Psi_\alpha(u)\,\mathrm{Re}\,\{K_n(\alpha, 2, \beta, u)\}\,du.$$

By Lemma 3, the kernel $K_n(u)$ satisfies the relations:

$$K_n(u) = O(n^{\beta+1}\,u^{-\alpha})$$

uniformly in $0 < u \leqq \pi/n$,

$$K_n(u) = O(n^{\alpha+1})$$

uniformly in $\pi - \pi/n \leqq u \leqq \pi$, and in $\pi/n \leqq u \leqq \pi - \pi/n$,

$$\text{Re}\,\{K_n(u)\} = \text{Re}\left\{\frac{n^{\alpha+1}\, e^{\{nu-(\alpha-1)\pi/2\}i}}{(1-e^{-iu})^\beta}\right\} + O(n^\alpha\, u^{-\beta-1}) + O((\pi-u)^{-\alpha-1}).$$

In order to estimate $\bar{t}_n^{(\beta)}$ we follow the same way as in Case III, and it is sufficient to consider only the following two expressions N_1 and N_2:

$$N_1 = \sum_{n=0}^{\infty} \frac{1}{2^{nk/p}} \left\{ \sum_{j=2^n}^{2^{n+1}-1} \left(\frac{1}{j^\beta} \int_{\pi/2^n}^{\pi-\pi/2^n} |\Psi_\alpha(u)|\,(\pi-u)^{-\alpha-1}\, du \right)^p \right\}^{k/p}$$

and

$$N_2 = \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{j=2^n}^{2^{n+1}-1} \left(\frac{1}{j^\beta} \int_{\pi/2^n}^{\pi-\pi/2^n} |\Psi_\alpha(u)|\,(\pi-u)^{-\alpha-1}\, du \right)^k.$$

First we have

$$\begin{aligned} N_1 &\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{nk\beta}} \left(\int_{\pi/2^n}^{\pi-\pi/2^n} |\Psi_\alpha(u)|\,(\pi-u)^{-\alpha-1}\, du \right)^k \\ &\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{nk\beta}} \left(\int_{\pi/2^n}^{\pi-\pi/2^n} |\Psi_\alpha(u)|^p\, du \right)^{k/p} \left(\int_{\pi/2^n}^{\pi-\pi/2^n} (\pi-u)^{-(\alpha+1)p'}\, du \right)^{k/p'} \\ &\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{nk\beta}} \left(\left[(\pi-u)^{-(\alpha+1)p'+1} \right]_{\pi/2^n}^{\pi-\pi/2^n} \right)^{k/p'} \left(\int_{\pi/2^n}^{\pi} |\Psi_\alpha(u)|^p\, du \right)^{k/p} \end{aligned}$$

where $-(\alpha+1)p'+1 = -p'(\alpha+1/p) < 0$. Hence

$$N_1 \leqq A \sum_{n=0}^{\infty} \frac{1}{2^{n(\beta-\alpha-1/p)k}} \left(\int_{\pi/2^n}^{\pi} \frac{|\Psi_\alpha(u)|}{u^{\beta p}}\, du \right)^{k/p}.$$

For N_2 we have

$$\begin{aligned} N_2 &\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{nk\beta}} \left(\int_{\pi/2^n}^{\pi-\pi/2^n} |\Psi_\alpha(u)|\,(\pi-u)^{-\alpha-1}\, du \right)^k \\ &\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{nk(\beta-\alpha-1/p)}} \left(\int_{\pi/2^n}^{\pi} \frac{|\Psi_\alpha(u)|}{u^{\beta p}}\, du \right)^{k/p} \end{aligned}$$

as in N_1.

We can now adopt the same argument as in I_2' and J_2'.

5.5. CASE V. $p = 1$. Let q be the greatest integer such that $q \leqq \alpha + 1$. In this case the function $K_n(\alpha+1-q,\, q+1,\, \beta,\, u) = K_n(u)$ satisfies the relations:

$$K_n(u) = O(n^{\beta+1}\, u^{-\alpha}) \quad \text{and} \quad K_n(u) = O(n^{\alpha+1})$$

uniformly in $0 < u \leqq \pi/n$ and $\pi - \pi/n \leqq u \leqq \pi$ respectively, and in $\pi/n \leqq u \leqq \pi - \pi/n$,

$$K_n(u) = O(n^{\alpha+1} u^{-\beta}) + O(n^q(\pi - u)^{q-\alpha-1}).$$

Employing these relations, we have, as in Case III 2°,

$$\sum_{n=0}^{\infty}\Big(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |\bar{t}_j^{(\beta)}|\Big)^k \leqq \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} |\bar{t}_j^{(\beta)}|^k$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1} j^k\Big(\int_0^{\pi/2^n} |\Psi_\alpha(u)|\, u^{-\alpha}\, du\Big)^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\Big(\frac{1}{j^{\beta-\alpha-1}}\int_{\pi/2^n}^{\pi-\pi/2^n} |\Psi_\alpha(u)|\, u^{-\beta}\, du\Big)^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\frac{1}{j^{k(\beta-q)}}\Big(\int_{\pi/2^n}^{\pi-\pi/2^n} |\Psi_\alpha(u)|\,(\pi-u)^{q-\alpha-1} du\Big)$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\Big(\frac{1}{j^{\alpha+1}}\int_{\pi-\pi/2^n}^{\pi} |\Psi_\alpha(u)|\, du\Big)^k$$

$$+ AJ_3$$

$$= N_1 + N_2' + N_2'' + N_3 + AJ_3$$

say. We estimate N's as follows:

$$N_1 \leqq A\sum_{n=0}^{\infty}2^{nk}\Big(\int_0^{\pi/2^n} |\Psi_\alpha(u)|\, u^{-\alpha}\, du\Big)^k$$

$$\leqq A\sum_{n=0}^{\infty}2^{nk}\Big(\sum_{j=n}^{\infty}\frac{1}{2^{j\delta}}\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(u)|}{u^\delta}\, du\Big)^k$$

where we take $0 < \delta < 1$.

$$N_1 \leqq A\sum_{n=0}^{\infty}2^{nk}\sum_{j=n}^{\infty}\Big(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(u)|}{u^\delta}\, dt\Big)^k\Big(\sum_{j=n}^{\infty}2^{-j\delta k'}\Big)^{k/k'}$$

$$\leqq A\sum_{j=0}^{\infty}2^{j(1-\delta)k}\Big(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(u)|}{u^\delta}\, du\Big)^k$$

$$\leqq A\sum_{j=0}^{\infty}\Big(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\psi_\alpha(u)|}{u}\, du\Big)^k.$$

$$N_2' \leqq A\sum_{n=0}^{\infty}2^{-nk(\beta-\alpha-1)}\Big(\int_{\pi/2^n}^{\pi}\frac{|\Psi_\alpha(u)|}{u^\beta}\, du\Big)^k$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1)}}\Big(\sum_{j=0}^{n-1}\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\Psi_\alpha(u)|}{u^\beta}\, du\Big)^k$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk(\beta-\alpha-1)}}\Big(\sum_{j=0}^{n-1}2^{j\eta}\int_{\pi/2^{j+1}}^{\pi/2^j} u^{\eta-\beta}\,|\Psi_\alpha(u)|\, du\Big)^k$$

where η is so chosen that $0 < \eta < \beta - \alpha - 1$. Then,

$$N_2' \leqq A \sum_{n=0}^{\infty} \frac{1}{2^{nk(\beta-\alpha-1)}} \sum_{j=0}^{n} \left(\int_{\pi/2^{j+1}}^{\pi/2^j} u^{\eta-\beta} |\Psi_\alpha(u)| \, du \right)^k \left(\sum_{j=0}^{n} 2^{j\eta k'} \right)^{k/k'}$$

$$\leqq A \sum_{j=0}^{\infty} \frac{1}{2^{jk(\beta-\alpha-1)}} \left(\int_{\pi/2^{j+1}}^{\pi/2^j} u^{\eta-\beta} |\Psi_\alpha(u)| \, du \right)^k$$

$$\leqq A \sum_{j=0}^{\infty} \left(\int_{\pi/2^{j+1}}^{\pi/2^j} \frac{|\Psi_\alpha(u)|}{u} \, du \right)^k,$$

$$N_2'' \leqq A \sum_{n=0}^{\infty} \frac{1}{2^{nk(\beta-q)}} \left(\int_{\pi/2^n}^{\pi-\pi/2^n} |\Psi_\alpha(u)| (\pi-u)^{q-\alpha-1} \, du \right)^k$$

$$\leqq A \sum_{n=0}^{\infty} \frac{1}{2^{nk(\beta-\alpha-1)}} \left(\int_{\pi/2^n}^{\pi} \frac{|\Psi_\alpha(u)|}{u^\beta} \, du \right)^k,$$

and

$$N_3 \leqq A \sum_{n=0}^{\infty} \frac{1}{2^{nk(\beta-\alpha-1)}} \left(\int_{\pi/2^n}^{\pi} \frac{|\Psi_\alpha(u)|}{u^\beta} \, du \right)^k.$$

Thus the estimations of N_2'' and N_3 are exactly the same as of N_1, and we complete the proof in this case.

We proved Theorem 2 completely.

6. PROOF OF THEOREM 4. We have

$$t_n^{(1)} = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu A_\nu(\theta) = \frac{1}{\pi(n+1)} \int_0^\pi \varphi(t) \operatorname{Re} \{P_n(1,1,t)\} \, dt$$

$$= \frac{1}{\Gamma(1-\alpha)(n+1)\pi} \int_0^\pi \operatorname{Re} \{P_n(1,1,t)\} \, dt \int_0^t (t-u)^{n-\alpha} \, d\Phi_\alpha(u)$$

$$= - \frac{1}{(n+1)\pi} \int_0^\pi \Phi_\alpha(u) \operatorname{Re} \{K_n'(\alpha,1,1,u)\} \, du. \tag{6.1}$$

By Lemma 3 we get

$$K_n'(u) = O(n^2 u^{-\alpha}) \tag{6.2}$$

uniformly in $0 < u \leqq \pi/n$, and

$$K_n'(u) = - \frac{n^{\alpha+1} e^{(nu-\alpha\pi/2)i}}{1-e^{-iu}} + O(n^\alpha u^{-2}) + O(n^{1-\epsilon}(\pi-u)^{-\alpha-\epsilon}) \tag{6.3}$$

uniformly in $\pi/n \leqq u \leqq \pi$ where ε is any fixed number such as $0 < \varepsilon \leqq 1$. We distinguish two cases $k \leqq p$ and $k > p$.

1°. Case $k \leqq p$. Proceeding as in Case II, 1° in the proof of Theorem 2, we get by (6.2) and (6.3),

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^n} \sum_{j=2^n}^{2^{n+1}-1} |t_j^{(1)}|^p \right)^{k/p}$$

$$\leqq A\sum_{n=0}^{\infty} 2^{nk}\Big(\int_0^{\pi/2^n} |\Phi_\alpha(u)|\, u^{-\alpha}\, du\Big)^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\Big(\sum_{j=2^n}^{2^{n+1}-1}\Big| j^\alpha\int_{\pi/2^n}^{\pi}\Phi_\alpha(u)\frac{\cos\{(j+1/2)u-\alpha/2\}}{2\sin\dfrac{u}{2}}\,du\Big|^p\Big)^{k/p}$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\Big\{\sum_{j=2^n}^{2^{n+1}-1}\Big(j^{\alpha-1}\int_{\pi/2^n}^{\pi}|\Phi_\alpha(u)|\, u^{-2}\, du\Big)^p\Big\}^{k/p}$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^{nk/p}}\Big\{\sum_{j=2^n}^{2^{n+1}-1}\Big(j^{-\varepsilon}\int_{\pi/2^n}^{\pi}|\Phi_\alpha(u)|(\pi-u)^{-\alpha-\varepsilon}\, du\Big)^p\Big\}^{k/p}$$

$$= R_1 + R_2 + R_3 + R_4$$

say. Considering the condition $\alpha < 1/p'$, we can estimate the terms R_1, R_2 and R_3 in the same fashion as in I_1, I_2' and I_2'' respectively, and we get the required inequalities. Concerning the term R_4, choose ε so that $\alpha + \varepsilon < 1/p'$, then

$$R_4 \leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk\varepsilon}}\Big(\int_{\pi/2^n}^{\pi}|\Phi_\alpha(u)|(\pi-u)^{-\alpha-\varepsilon}\, du\Big)^k$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk\varepsilon}}\Big(\int_{\pi/2^n}^{\pi}|\Phi_\alpha(u)|^p\, du\Big)^{k/p}\Big(\int_{\pi/2^n}^{\pi}(\pi-u)^{-(\alpha+\varepsilon)p'}\, du\Big)^{k/p'}$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk\varepsilon}}\Big(\sum_{j=0}^{n}\int_{\pi/2^{j+1}}^{\pi/2^j}|\Phi_\alpha(u)|^p\, du\Big)^{k/p}$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^{nk\varepsilon}}\sum_{j=0}^{n}\Big(\int_{\pi/2^{j+1}}^{\pi/2^j}|\Phi_\alpha(u)|^p\, du\Big)^{k/p}$$

$$\leqq A\sum_{j=0}^{\infty}\Big(\int_{\pi/2^{j+1}}^{\pi/2^j}\frac{|\varphi_\alpha(u)|^p}{u}\, du\Big)^{k/p}.$$

In this case the proof is finished.

2°. Case $k > p$. By (6.1) – (6.3) and the Hölder inequality, we have

$$\sum_{n=0}^{\infty}\Big(\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}|t_j^{(1)}|^p\Big)^{k/p} \leqq \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}|t_j^{(1)}|^k$$

$$\leqq A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\Big(j\int_0^{\pi/2^n}|\Phi_\alpha(u)|\, u^{-\alpha}\, du\Big)^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=20}^{2^{n+1}-1} j^{\alpha k}\Big|\int_{\pi/2^n}^{\pi}\Phi_\alpha(u)\frac{\cos\{(j+1/2)u-\alpha/2\}}{2\sin\dfrac{u}{2}}\, du\Big|^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\Big(j^{\alpha-1}\int_{\pi/2^n}^{\pi}|\Phi_\alpha(u)|\, u^{-2}\, du\Big)^k$$

$$+ A\sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{j=2^n}^{2^{n+1}-1}\left(j^{-\epsilon}\int_{\pi/2^n}^{\pi}|\Phi_\alpha(u)|(\pi-u)^{-\alpha-\epsilon}\,du\right)^k$$

$$= S_1 + S_2 + S_3 + S_4$$

say. We can estimate S_1 and S_2 quite similarly to J_1 and J_2''. To estimate S_2 we have to distinguish two cases $k \leqq p$ and $k > p'$; and use the Hausdorff-Young inequality after the suitable use of the Hölder inequality, and we get, as in J_2' the desired result. For the estimation of S_4, we choose ε and δ such that $\delta/p < \varepsilon < 1/p' - \alpha$ and $0 < \delta < 1$, and proceed as in the estimation of I_1 to get the required inequality. Thut the proof of Theorem 4 is completed.

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