ON SOME PROBLEMS OF C^*-ALGEBRAS

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1. Introduction. In this note, we shall consider two problems concerning C^*-algebras as follows:

(α). Putnam [5], using of the device of Fuglede [2], proved that if S is a normal operator on Hilbert space commuting with $ST - TS$, then S commutes with T.

Using Fuglede's theorem itself and a theorem on derivations, Kaplansky [4] has proved a generalization of Putnam's theorem: if, relative to a continuous derivation $'$ of a C^*-algebra, a normal element x commutes with x', then $x' = 0$. Further he has shown a generalization of Fuglede's theorem: if x is normal and $x' = 0$ relative to a continuous derivation, then $(x^*)' = 0$. Finally he asks whether these results are still valid without assuming the derivation to be continuous.

We shall show this problem is affirmative [Theorem 1]. The proof is obtained by a slight modification of Kaplansky's method.

(β). One of the classical results of the commutative C^*-algebras states that their pure state spaces are compact. This theorem has many important applications in the theory of the algebra.

Therefore it is natural to ask whether it is exact in non-commtative case—in fact an analogous problem in group algebras is raised by R. Godement [3]. We shall notice that there is a C^*-algebra whose pure state space is not compact [Theorem 2].

2. Theorems. We shall state

THEOREM 1. *If, relative to a derivation* $'$ *of a* C^**-algebra* A, *a normal element* x *commutes with* x', *then* $x' = 0$.

PROOF. By Fuglede's theorem, $x^*x' = x'x^*$, and so differentiating $xx^* = x^*x$, we obtain that x commutes with $(x^*)'$. Put $x = a + ib\,(a, b$ self-adjoint), then $x' = a' + ib'$ and $(x^*)' = a' - ib'$; hence a commutes with a' and b commutes with b'. Let B be the C^*-subalgebra generated by a and I, C the C^*-subalgebra of all elements of A commuting with a, then B is contained in the center of C. Let P be a primitive ideal in C, then $B \cap P$ is a maximal ideal in B. Therefore, $a - \lambda I \in B \cap P$ for some real λ, in which case it can be written $a - \lambda I = h^2 - k^2$ with $h, k \in B \cap P$ and $hk =$

0; hence $a - \lambda I = (h + ik)^2$.

Put $y = h + ik$, then

$$a' = (a - \lambda I)' = (y^2)' = yy' + y'y,$$

therefore

$$ya' = y^2y' + yy'y = (a - \lambda I)y' + yy'y,$$
$$a'y = yy'y + y'y^2 = yy'y + y'(a - \lambda I).$$

Since $y \in B$ and $a' \in C$, $ya' = a'y$; hence, by the above equalities, $(a - \lambda I)y' = y'(a - \lambda I)$ and so $ay' = y'a$; hence $y' \in C$. $y \in B \cap P$ and $y' \in C$ imply: $a' = (a - \lambda I)' = (y^2)' = yy' + y'y \in P$. Therefore, a' belongs to all primitive ideals in C; hence $a' = 0$, and analogously $b' = 0$, so that $x' = a' + ib' = 0$. This completes the proof.

COROLLARY. *If x is normal and $x' = 0$ relative to a derivation $'$ of a C^*-algebra A, then $(x^*)' = 0$.*

PROOF. Differentiating $xx^* = x^*x$ and using $x' = 0$, we obtain that x commutes with $(x^*)'$. By Fuglede's theorem, so does x^*; hence by Theorem 1, $(x^*)' = 0$. This completes the proof.

Next, we shall state Theorem 2. Let H be an infinite dimensional Hilbert space, C the C^*-algebra of all completely continuous operators on H, B the C^*-algebra of all bounded operators on H and $\widetilde{B}$ the dual space of B. Then we state

THEOREM 2. *The pure state space Ω of B is not $\sigma(\widetilde{B}, B)$-compact.*

PROOF. Let $(\psi_n | n = 0, 1, 2, \ldots\ldots)$ be a normalized orthogonal system of elements of H, then a sequence $\left(\frac{\psi_0 + \psi_n}{\sqrt{2}} \,\middle|\, n = 1, 2, \ldots\ldots\right)$ converges weakly to $\frac{\psi_0}{\sqrt{2}}$ and $\left\|\frac{\psi_0 + \psi_n}{\sqrt{2}}\right\| = 1$. Put $f_n(x) = < x\xi_n, \xi_n >$ for all $x \in B$, where $\xi_n = \frac{\psi_0 + \psi_n}{\sqrt{2}}$ and $< \cdot, \cdot >$ is the inner product of H, then f_n are pure states on B. Let f be an adherent point in B to $(f_n | n = 1, 2, \ldots\ldots)$, then for an arbitrary $x \in B$ there is a subsequence (n_j) of (n) depending on x such that $\lim_j f_{n_j}(x) = f(x)$

Since $(a\,\xi_n)$ for $a \in C$ is strongly convergent to $\frac{a\psi_0}{\sqrt{2}}$, therefore

$$\lim_n < a^*a \;\xi_n, \xi_n > = \lim_n \|a\xi_n\|^2 = \left\| a\left(\frac{\psi_0}{\sqrt{2}}\right) \right\|^2$$

$$= \frac{1}{2} < a^*a\ \psi_0, \psi_0 >;$$

hence $f(a^*a) = \frac{1}{2} < a^*a\ \psi_0, \psi_0 >$ for $a \in C$.

If Ω is $\sigma(\widetilde{B}, B)$-compact, f is also contained in Ω, so that a corresponding *-representation of B is irreducible; hence $f(C) = 0$ or $f(x) = < x\psi, \psi >$ for $x \in B$, where $\psi \in H$ and $||\psi||^2 = 1$ [cf. 1].

Since $f(C) \neq 0$, $f(x) = < x\psi, \psi >$ and so $f(a^*a) = ||a\psi||^2 = \frac{1}{2} ||a\,\psi_0||^2$ for $a \in C$. Now we take a one-dimensional projection on $(\alpha\psi)$ as the a, where α are complex numbers, then

$$1 = ||\psi||^2 = \frac{1}{2} ||a_0\psi_0||^2 \leqq \frac{1}{2}$$

This completes the proof, since we shows a contradiction.

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