MEROMORPHIC FUNCTIONS WITH MAXIMUM DEFECT SUM

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1. Introduction. Pfluger [6] proved that if $f(z)$ be an entire function of finite order ρ with maximum defect sum 2, then ρ must be an integer. In this note we extend this theorem to meromorphic functions. We prove

THEOREM. *Let $f(z)$ be a meromorphic function of finite order ρ such that $\delta(a_1) = 1$, $\sum_2^\infty \delta(a_i) = 1$ where a_1, $a_2, \ldots$ are any constants (finite or infinite) different from each other. Then ρ must be a positive integer and $f(z)$ must be of regular growth order ρ.*

We show also by means of an example that $f(z)$ need not be of very regular growth order ρ or even proximate order $\rho(r)$.

2. Lemma. *Let $F(z)$ be a meromorphic function of non-integer order $\rho > 0$ and*

$$\limsup_{r\to\infty} \frac{N(r, a) + N(r, b)}{T(r)} = \chi(\rho)$$

where a and b are any two distinct numbers finite or infinite; then if $p = [\rho] > 0$,

$$\chi(\rho) \geqq (\rho - p)(p + 1 - \rho)/\{3e(2 + \log p)(1 + p)^2\}, \tag{1}$$

and if $p = 0$,

$$\chi(\rho) \geqq 1 - \rho \tag{2}$$

Two proofs of this lemma, with different constants,[3] on the right hand sides of (1) and (2), are known [4; pp. 51-54; 10, theorem 2 (a)]. We sketch a different proof depending on the proximate order $\rho(r)$.

Since $T\left(r, \dfrac{\alpha F + \beta}{\gamma F + \delta}\right) = T(r, F) + O(1)$, we may suppose $a = 0$, $b = \infty$. Then

$$F(z) = z^k e^{q(z)} \prod_1^\infty E\left(\frac{z}{a_i}, p_1\right) \Big/ \prod_1^\infty E\left(\frac{z}{b_j}, p_2\right) = z^k e^{q(z)} P_1/P_2 \qquad \text{(say)}$$

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3) The constants on the right hand sides of (1) and (2) are not "best possible."

where $q(z)$ is a polynomial of degree $q \leqq [\rho]$. Write max $(p_1, p_2) = p$. Then it is easily seen that $p = [\rho]$ and

$$T(r, F) \leqq O(\log r) + O(r^q) + \log M(r, P_1) + \log M(r, P_2).$$

Write $n(x, 0) + n(x, \infty) = n(x)$, $N(x, 0) + N(x, \infty) = N(x)$
and let $0 < \delta < \min(|a_1|, |b_1|)$. If $p = 0$ and $r > r_0(\varepsilon)$

$$T(r, F) \leqq O(\log r) + \int_\delta^\infty \frac{rn(t)dt}{t(t+r)} < (1+\varepsilon) r \int_r^\infty \frac{N(t)dt}{t^2}. \tag{3}$$

If $p > 0$ and $A = 3e(2 + \log p)(1 + p)$ then [5; pp. 225-6]

$$T(r, F) \leqq O(r^p) + A \int_\delta^\infty \frac{n(x)}{x^{1+p}} \frac{r^{1+p}}{(x+r)} dx. \tag{4}$$

Let $\rho(r)$ be the proximate order for $N(r)$. Then [8; 9, p. 321]

1) $\rho(r)$ is differentiable for $r > r_0$ except at isolated points at which $\rho'(r-0)$ and $\rho'(r+0)$ exist.
2) $\lim_{r\to\infty} \rho(r) = \rho$.
3) $\lim_{r\to\infty} r\rho'(r) \log r = 0$.
4) $N(r) \leqq r^{\rho(r)}$ for all $r > r_0$
   $= r^{\rho(r)}$ for a sequence of values of $r \uparrow \infty$.

Choose r_0 so large that $p < \rho(r) < p + 1$ for all $r \geqq r_0$. Then for all $r > R_0(\varepsilon) > r_0$, we have from (4) when $p > 0$

$$T(r, F) < (A+\varepsilon)\left\{ r^p \int_{r_0}^r \frac{N(x)dx}{x^{1+p}} + r^{p+1} \int_r^\infty \frac{N(x)dx}{x^{2+p}} \right\}(p+1)$$

$$< (A+\varepsilon)(p+1) r^{\rho(r)} \left\{ \frac{1}{\rho(r) - p} + \frac{1}{p+1-\rho(r)} \right\}$$

and (1) follows. If $p = 0$ we have from (3)

$$T(r, F) < (1+\varepsilon) r \int_r^\infty x^{\rho(x)-2} dx = (1+\varepsilon) \frac{r^{\rho(r)}}{1-\rho(r)}$$

and (2) follows.

3. Proof of Theorem. (a) Suppose first $a_1 = \infty$ so that $\delta(\infty) = 1$, $\sum_{i=2}^\infty \delta(a_i) = 1$.

Given $\varepsilon > 0$, choose $a_2, \ldots, a_{q+1}$ $(q \geqq 3)$ such that $\sum_{i=q+2}^\infty \delta(a_i) < \varepsilon$.
Since $f(z)$ has maximum defect sum, $f(z)$ can not reduce to a rational function and so $\log r = o(T(r, f))$. Now [11, p. 18]

$$N\left(r, \frac{1}{f'}\right) + \sum_2^{q+1} m(r, a_i) + S(r) \leqq T(r, f') \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and hence

$$1-\varepsilon<\sum_{2}^{q+1}\delta(a_i)<\liminf_{r\to\infty} T(r,f')/T(r,f).$$

Further [4; p. 104]

$$\limsup_{r\to\infty}\frac{T(r,f')}{T(r,f)}\leqq 2-\delta(\infty)-\mu(\infty)=1$$

and so

$$T(r,f')\sim T(r,f) \qquad \text{as } r\to\infty. \tag{6}$$

Hence from (5) we have

$$\limsup_{r\to\infty}\frac{N(r,1/f')}{T(r,f')}+\sum_{2}^{q+1}\delta(a_i)\leqq 1$$

and so

$$\limsup_{r\to\infty}\frac{N(r,1/f')}{T(r,f')}=0. \tag{7}$$

Further $N(r,f')\leqq 2N(r,f)$, and $\delta(\infty)=1$, and so from (6)

$$\limsup_{r\to\infty}\frac{N(r,f')}{T(r,f')}=0. \tag{8}$$

Write

$$g(r,f')=\frac{N(r,1/f')+N(r,f')}{T(r,f')}.$$

Then from (7) and (8), $\lim_{r\to\infty} g(r,f')=0$.

Now $f'(z)$ is meromorphic function of the same order ρ; if $\rho>0$ be non-integer then we should have from Lemma

$$\limsup_{r\to\infty} g(r,f')=\chi(\rho)>0,$$

and if $\rho=0$ then we have [7]

$$\limsup_{r\to\infty} g(r,f')\geqq 1.$$

Hence ρ must be an integer. To prove that $f'(z)$ is of regular growth we use the following theorem of Edrei and Fuchs[4].

THEOREM. *Let $F(z)$ be a meromorphic function of finite order ρ and lower order μ. Let p be the integer defined by $p-\frac{1}{2}\leqq\mu<p+\frac{1}{2}$.* If

4) A. Edrei and W. H. J. Fuchs "Deficient values and asymptotic values of a meromorphic function" in publication; see also, Notices Amer. Math. Soc. 6 (1958) pp. 496-7 abstract 548-71, 548-72 p. 606 abstract 549-26.

$$\limsup_{r\to\infty} \frac{N(r, F) + N(r, 1/F)}{T(r, F)} < \frac{\beta}{5e(1+p)}, \quad 0 < \beta \leqq \frac{1}{2},$$

then $p \geqq 1$ *and* $p - \beta \leqq \mu \leqq \rho \leqq p + \dfrac{\beta}{10}$.

Since $\lim_{r\to\infty} g(r, f') = 0$ and f' is of finite order ρ, we can choose β arbitrary small and so we have

$$\rho = \lim_{r\to\infty} \frac{\log T(r, f')}{\log r} = \mu.$$

Further $T(r, f') \sim T(r, f)$ and so $f(z)$ is of regular growth integer order $\rho \geqq 1$.

(b) Suppose now $a_1 \neq \infty$. We have $\delta(f, a_1) = 1$, $\sum_2^\infty \delta(f, a_i) = 1$.

Let $F(z) = 1/\{f(z) - a_1\}$. Then $T(r, F) = T(r, f) + O(1)$ (9)
and so $F(z)$ is of the same order ρ. Further $\delta(f, a_1) = \delta(F, \infty) = 1$, $\sum_2^\infty \delta(f, a_i) = \sum_2^\infty \delta(F, \alpha_i) = 1$ where $\alpha_i = 1/(a_i - a_1)$, $i = 2, 3, \ldots\ldots$ are finite constants different from each other. Hence by case (a) $F(z)$ is of regular growth integer order $\rho \geqq 1$ and by (9) the theorem is proved

4. Remarks. (i) We note that it is not possible to prove the theorem without some condition of the type $\delta(a_1) = 1$. In fact there are meromorphic functions of finite non-integer order ($(2k + 1)/2$ where k integer $\geqq 1$) with maximum defect sum 2 (see [2], [3]).

(ii) We can prove (6) under less restrictive hypothesis: $f(z)$ is of finite order and

$$\theta(\infty) = 1, \sum_{a \text{ finite}} \theta(a) = 1 \ldots\ldots \qquad (H_1)$$

where $\theta(a) = 1 - \limsup_{r\to\infty} \dfrac{N(r, a)}{T(r)}$.

However the conclusion that $f(z)$ is of integer order will not follow with (H_1). Consider for instance

$$f(z) = \exp(2\sqrt{z}) + \exp(-2\sqrt{z})$$

which is an entire function of order 1/2, and

$$\theta(\infty) = 1, \ \theta(2) + \theta(-2) = 1.$$

(iii) If $f(z)$ is meromorphic for $|z| < R < \infty$ and satisfies

(a) $\displaystyle\lim_{r\to R} \frac{-\log(R - r)}{T(r, f)} = 0$,

(b) $f(z)$ is of finite order ρ,

(c) $\delta(\infty) = 1, \quad \sum_{a\neq\infty} \delta(a) = 1$, then (6), (7), (8) where $r \to \infty$ is to be replaced by $r \to R$, follow by the same argument.

(iv) We construct an entire function $f(z)$ of integer order $\rho > 1$ for which $\delta(0) = 1 = \delta(\infty)$ and

$$\limsup_{r\to\infty} \frac{T(r)}{r^\rho} = \infty, \ \liminf_{r\to\infty} \frac{T(r)}{r^\rho} = a, \qquad 0 < a < \infty.$$

Let $\prod_1^\infty E\left(\frac{z}{z_n}, p\right)$ be a canonical product where $p = \rho$, $n(r) = o(r^\rho)$, z_n^ρ real, $\sum 1/|z_n|^{\rho+\epsilon}$ convergent, $\sum 1/z_n^\rho$ oscillating such that if $S(r) = \sum_{|z_n|\leqq r} 1/z_n^\rho$ then $\limsup_{r\to\infty} S(r) = \infty$, $\liminf_{r\to\infty} S(r) = b$ where $0 < b < \infty$.

For instance we can take $z_n = \{n \log(n+1)\}^{1/\rho}$ for a sequence of values of n and $e^{i\pi/\rho}\{n \log(n+1)\}^{1/\rho}$ for the remaining values of n in such a way that if $\varepsilon_n = \pm 1$,

$$\sum_{|z_n|\leqq r} \frac{1}{z_n^\rho} = \sum_{|z_n|\leqq r} \frac{\varepsilon_n}{n \log(n+1)} = S(r)$$

then $$\limsup_{r\to\infty} S(r) = \infty, \ \liminf_{r\to\infty} S(r) = b.$$

Further $n(r) = o(r^\rho)$, $p = \rho$, and $\sum 1/|z_n|^{\rho+\epsilon}$ is convergent. Consider the entire function $f(z) = z^k \exp(c_\rho z^\rho + c_{\rho-1}z^{\rho-1} + \dots\dots) \prod_1^\infty E\left(\frac{z}{z_n}, p\right)$, $\Re(c_\rho) \geqq 0$. It is an entire function of order and genus equal to ρ. If we write

$$c_\rho + \frac{1}{\rho} S(r) = A(r)e^{i\alpha(r)}$$

then $\liminf_{r\to\infty} A(r) = \left| c_\rho + \frac{b}{\rho} \right|$; $\limsup_{r\to\infty} A(r) = \infty$.

Further {cf. [1; pp. 27-29], [6; pp. 97-101]}

$$\log|f(re^{i\phi})| = \Re\{r^\rho A(r)e^{i(\alpha(r)+\rho\phi)}\} + o(r^\rho)$$

and so if $a\pi = |C_\rho + b/\rho|$ we have

$$\log M(r) = (A(r) + o(1))r^\rho \tag{10}$$

$$T(r) = (A(r)/\pi + o(1))r^\rho,$$

$$\limsup_{r\to\infty} T(r)/r^\rho = \infty, \ \liminf_{r\to\infty} T(r)/r^\rho = a, \ \delta(0) = \delta(\infty) = 1.$$

If we compare $\log M(r)$ with a proximate order $r^{\rho(r)}$ [6; p. 96; 8, pp. 326-7 (case A)], then from (10) we have

$$\limsup_{r\to\infty} \log M(r)/r^{\rho(r)} = 1, \ \limsup_{r\to\infty} T(r)/r^{\rho(r)} = \frac{1}{\pi}$$

$$\liminf_{r\to\infty}\frac{T(r)}{r^{\rho(r)}}=\liminf_{r\to\infty}\frac{\log M(r)}{r^{\rho(r)}}=0.$$

Further we note that by an appropriate choice of z_n and C_ρ we can construct an entire function $f(z)$ of mean type (with respect to comparison function r^ρ), with defect sum 2, for which

$$0<\liminf_{r\to\infty}\frac{T(r)}{r^\rho}<\limsup_{r\to\infty}\frac{T(r)}{r^\rho}<\infty.$$

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