RIEMANN-CESÀRO METHODS OF SUMMABILITY IV

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1. Introduction. Let $a(u)$ be bounded mesurable in every finite interval of $u \geqq 0$ and let

$$s_\alpha(u) = \frac{1}{\Gamma(\alpha+1)} \int_0^u (u-x)^\alpha a(x)\, dx, \ \alpha > -1,$$

$$\sigma_\alpha(u) = \Gamma(\alpha+1) s_\alpha(u)/u^\alpha, \ \alpha > -1,$$

and let

$$\sigma_{-1}(u) = u s_{-1}(u) = u a(u).$$

If $\sigma_\alpha(u) \to s$ as $u \to \infty$, then we say that the integral

$$\int_0^\infty a(u) du \tag{1.1}$$

is evaluable (C, α), $\alpha > -1$, to s and write

$$(C, \alpha) \int_0^\infty a(u) du = s.$$

If $\int_0^\infty |d\sigma_\alpha(u)|$ is finite and $\sigma_\alpha(u) \to s$ as $u \to \infty$, then we say that the integral (1.1) is evaluable $|C, \alpha,|$ $\alpha > -1$, to s and write

$$|C, \alpha| \int_0^\infty a(u) du = s.$$

Recently, Rajagopal [5] defined the Riemann-Cesàro methods of summability for integrals. In the following, let p be a positive integer and let α be a real number such that $\alpha \geqq -1$. The integral (1.1) is said to be evaluable (R, p, α) to s if the integral

$$C_{p,\alpha}^{-1} t^{\alpha+1} \int_0^\infty s_\alpha(u) \left(\frac{\sin ut}{ut} \right)^p du, \tag{1.2}$$

where

$$C_{p,\alpha} = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty u^{\alpha-p} \sin^p u\, du, & -1 < \alpha < p-1 \text{ or } \alpha = 0,\ p = 1, \\ 1, & \alpha = -1, \end{cases}$$

converges in some interval $0 < t < t_0$ and its limit tends to s as $t \to 0+$. The purpose of this paper is to establish the summability theorems for integrals analogous to those for series.

Rajagopal [5 ; Theorem I(A)] proved the following

THEOREM A. *Let* $0 \leqq \alpha+1<p$. *Suppose that the integral* (1.1) *is evaluable* (A) *to* s *and that*

$$\int_0^u |\sigma_\alpha(x)|\,dx = O(u).$$

Then the integral (1.1) *is evaluable* (R, p, α) *to* s.

Concerning this Theorem, we have the following two theorems.

THEOREM 1. *Let* $0 < r+1 \leqq \alpha+1 < p$. *Suppose that the integral* (1.1) *is evaluable* (A) *to* s *and that*

$$\int_0^u |\sigma_r(x)|\,dx = O(u) \tag{1.3}$$

Then the integral (1.1) *is evaluable* (R, p, α) *to* s.

If the integral (1.1) is evaluable (C, r) to s, then it satisfies the condition of Theorem 1. Hence we have

COROLLARY. *Let* $0 < r+1 \leqq \alpha+1 < p$. *If the integral* (1.1) *is evaluable* (C, r) *to* s, *then it is also evaluable* (R, p, α) *to* s.

THEOREM 2. *Let* $0 \leqq \alpha+1 < r+1 < p$. *Suppose that the integral* (1.1) *is evaluable* (A) *to* s *and that* (1.3) *holds. Then the integral* (1.1) *is evaluable* (R, p, α) *to* s, *provided that*

$$s_{\alpha+1}(u) = o(u^p). \tag{1.4}$$

The series analogues of Theorems 1 and 2 are Theorem 3 in the paper [3]. For the series, the condition corresponding to (1.4) is superflous, reasoning that the limitation theorem (Hardy [1 ; Theorem 56]) holds. But the limitation theorem of this kinds does not hold for integrals. In fact, in the section 6, we shall show that there exists a function $a(u)$ defined in $(0, \infty)$ such that the integral (1.1) is evaluable (C, β) to zero for some β, $0 < \beta < 1$, while it is not evaluable (R, p, α) for $p \geqq 2$ and $-1 \leqq \alpha < 0$. This example shows that the condition (1.4) is not dropped in Theorem 2 for $p \geqq 2$ and $-1 \leqq \alpha < 0$. On the other hand, we have the following

THEOREM 3. *Let* $0 \leqq \alpha+1 < p$. *If the integral* (1.1) *is evaluable* $|C, p|$ *to* s, *then it is evaluable* (R, p, α) *to* s, *provided that* (1.4) *holds. Further, if the integral* (1.1) *is evaluable* $|C, 1|$ *to* s, *then it is also evaluable* $(R, 1, 0)$ *to* s.

The series analogue of this Theorem is Theorem 4 in the paper [3]. Here, also, the supplementary remark to Theorem 2 is needed. In fact, we shall show, in the section 6, that there exists a function $a(u)$ defined in $(0, \infty)$ such that the integral (1.1) is evaluable $|C, p|$ to zero, while it is not evaluable (R, p, α) for $p \geqq 2$ and $-1 \leqq \alpha < 0$. Theorems 1, 2 and 3 are proved in the sections 3, 4 and 5, respectively. We shall prove, in the section 6, the following

THEOREM 4. *Let $p \geqq 2$. Then, there exists a function $a(u)$ defined in $(0, \infty)$ such that the integral* (1.1) *is not evaluable (R, p, α) for any α, $-1 \leqq \alpha < 0$, but it is evaluable (R, p, β) to zero for some β, $0 < \beta < 1$.*

In the last section 7, we shall prove the following

THEOREM 5. *Let $p \geqq 3$. Then, there exists a function $a(u)$ defined in $(0, \infty)$ such that the integral* (1.1) *is not evaluable (R, p, α) for any α, $-1 \leqq \alpha < 1$, but it is evaluable (R, p, β) to zero for some $\beta, 0 < \beta < 2$.*

Concluding this section, we shall state the following two theorems without the proofs. The methods of the proofs are quite similar to those of series analogues which are found in the paper [2].

THEOREM 6. *Let $-1 \leqq \alpha \leqq 0$ and let p be a positive odd integer. Suppose that*

$$s_\beta(u) = o(u^\gamma),\ 0 < \gamma < \beta,$$

and

$$\int_u^\infty \frac{|a(x)|}{x} dx = O(u^{-(1-\delta)}),$$

where

$$0 < \delta = p(\beta - \gamma)/(\beta + 1 - p) < 1.$$

Then the integral (1.1) *is evaluable (R, p, α) to zero.*

THEOREM 7. *Let $-1 \leqq \alpha \leqq 0$ and let p be a positive odd integer. Suppose that*

$$\int_0^u |s_{p-1}(x)|\, dx = o(u^p/\log u)$$

and

$$\int_u^\infty \frac{|a(x)|}{x^p} dx = O(u^{-\delta}),\ 0 < \delta < 1.$$

Then the integral (1.1) *is evaluable (R, p, α) to zero.*

2. Preliminary Lemmas.

LEMMA 1. *Let $0 \leqq \alpha + 1 < p$ and let*

$$b(u) = a(u) - s \quad (0 \leqq u < 1)$$
$$= a(u) \quad (1 \leqq u < \infty).$$

If the integral (1.1) *is evaluable* (R, p, α) *to* s, *then the integral*

$$\int_0^\infty b(u)\, du$$

is evaluable (R, p, α) *to zero, and conversely.*

PROOF. For $\alpha = -1$, Lemma is obvious. For $\alpha > -1$, let

$$T_\alpha(u) = \frac{1}{\Gamma(\alpha+1)}\int_0^u (u-x)^\alpha b(x)dx.$$

Then, if $u < 1$,

$$T_\alpha(u) = \frac{1}{\Gamma(\alpha+1)}\int_0^u (u-x)^\alpha a(x)dx - \frac{su^{\alpha+1}}{\Gamma(\alpha+2)} = s_\alpha(u) - \frac{su^{\alpha+1}}{\Gamma(\alpha+2)},$$

and, if $u \geqq 1$,

$$T_\alpha(u) = s_\alpha(u) - \frac{s}{\Gamma(\alpha+2)}\{u^{\alpha+1} - (u-1)^{\alpha+1}\}.$$

Hence we have formally, putting $\varphi(x) = (x^{-1}\sin x)^p$,

$$t^{\alpha+1}\int_0^\infty T_\alpha(u)\varphi(ut)du = t^{\ +1}\int_0^\infty s_\alpha(u)\varphi(ut)du - \frac{st^{\alpha+1}}{\Gamma(\alpha+2)}\int_0^1 u^{\alpha+1}\varphi(ut)du$$
$$- \frac{st^{\alpha+1}}{\Gamma(\alpha+2)}\int_1^\infty \{u^{\alpha+1} - (u-1)^{\alpha+1}\}\varphi(ut)du.$$

where

$$\lim_{t\to 0}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\int_0^1 u^{\alpha+1}\varphi(ut)du = 0.$$

Therefore, for the proof of Lemma, it is sufficient to prove that

$$\lim_{t\to 0}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\int_1^\infty \{u^{\alpha+1} - (u-1)^{\alpha+1}\}\varphi(ut)du = C_{p,\alpha} = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty u^{\alpha-p}\sin^p u\, du.$$

This is easily proved by an elementary calculation.

LEMMA 2. *Let* $\varphi(x) = (x^{-1}\sin x)^p$. *Then*

$$\frac{d^m}{dx^m}\varphi(x) = O(1) \tag{2.1}$$

and

$$\frac{d^m}{dx^m}\varphi(x) = O(x^{-p}). \tag{2.2}$$

This is due to Obreschkoff [4].

LEMMA 3. *Let* $0 < \delta < 1$ *and let*

(2. 3) $$G(x, t) = \int_x^\infty (u-x)^{-\delta} \frac{d}{du}\varphi(ut)du.$$

where $\varphi(u) = (\sin u/u)^p$. *Then, for* $x > 0$ *and* $t > 0$,

(2. 3) $$G(x, t) = O(t^\delta)$$

and,

(2. 4) $$G(x, t) = O(t^{\delta-p}x^{-p}).$$

PROOF. Since

$$\int_x^\infty \frac{d}{du}\varphi(ut)du = -\varphi(xt),$$

we have, using (2. 1),

$$\begin{aligned} G(x, t) &= \int_x^{x+1/t} (u-x)^{-\delta}\frac{d}{du}\varphi(ut)du + [(u-x)^{-\delta}\varphi(ut)]_{u=x+1/t}^\infty \\ &\quad + \int_{x+1/t}^\infty \varphi(u-x)^{-\delta-1}\varphi(ut)du \\ &= O\Big(\int_x^{x+1/t}(u-x)^{-\delta}du\Big) + O(t^\delta) + O\Big(\int_{x+1/t}^\infty (u-x)^{-\delta-1}du\Big) \\ &= O(t^\delta), \end{aligned}$$

which is the required (2. 3). On the other hand, using (2,2),

$$\begin{aligned} G(x, t) &= \int_x^{x+1/t} (u-x)^{-\delta}\frac{d}{du}\varphi(ut)du + [(u-x)^{-\delta}\varphi(ut)]_{u=x+1/t}^\infty \\ &\quad + \delta\int_{x+1/t}^\infty (u-x)^{-\delta-1}\varphi(ut)du \\ &= O\Big(t^{1-q}\int_x^{x+1/t}(u-x)^{-\delta}u^{-p}du\Big) + O(t^{\delta-p}(x+1/t)^{-p}) \\ &\quad + O\Big(t^{-p}\int_{x+1/t}^\infty (u-x)^{-\delta-1}u^{-p}du\Big) \\ &= O\Big(t^{1-p}x^{-p}\int_x^{x+1/t}(u-x)^{-\delta}du\Big) + O(t^{\delta-p}x^{-p}) \\ &\quad + O\Big(t^{-p}x^{-p}\int_{x+1/t}^\infty (u-x)^{-\delta-1}du\Big) \\ &= O(t^{\delta-p}x^{-p}) \end{aligned}$$

which is the repuired (2. 5).

LEMMA 4. *Let* $0 < \delta < 1$ *and let* $m = 1, 2, 3, \ldots\ldots\ldots$ *Then, for* $x > 0$ *and* $t > 0$, *the function* $G(x, t)$, *defined in Lemma* 3, *has the following properties:*

$$\frac{d^m}{dx^m} G(x, t) = \int_x^\infty (u - x)^{-\delta} \frac{d^{m+1}}{du^{m+1}} \varphi(ut) du, \tag{2.5}$$

$$\frac{d^m}{dx^m} G(x, t) = O(t^{m+\delta-1}) \tag{2.6}$$

and

$$\frac{d^m}{dx^m} G(x, t) = O(t^{m+\delta-p-1} x^{-p}). \tag{2.7}$$

PROOF. The property (2, 5) is easily proved by an elementary calculation. Further, for $m = 1, 2, 3, \ldots\ldots\ldots,$

$$\int_x^\infty \frac{d^m}{du^m} \varphi(ut) du = -\frac{d^{m-1}}{dx^{m-1}} \varphi(xt).$$

Therefore, by the method analogous to one which we obtained Lemma 3, the properties (2. 6) and (2.7) are proved.

LEMMA 5. *Suppose that the integral* (1. 1) *is evaluable* (A) *to* s *and* (1. 3) *holds. Then*

$$\sigma_{r+1}(u) = O(1), \qquad \text{as } u \to \infty, \tag{2.8}$$

and, for $\varepsilon > 0$,

$$\sigma_{r+1+\varepsilon}(u) \to s, \qquad \text{as } u \to \infty. \tag{2.9}$$

PROOF. Rajagopal [5; Lemma 6] proved (2. 8) and

$$\sigma_{r+2}(u) \to s, \qquad \text{as } u \to \infty. \tag{2.10}$$

Then, by the well-known theorem, (2. 8) and (2. 10) imply (2. 9).

3. Proof of Theorem 1. Since Theorem in which $r = \alpha$ is Theorem A, we shall consider the case $r < \alpha$. Let (1. 3) hold. Then, K denoting a constant, depending only α and r, which is not necessarily the same on any two occurrences, for $0 < r < \alpha < r + 1$,

$$\begin{aligned}
\int_0^u |\sigma_\alpha(x)| dx &= K \int_0^u \frac{|s_\alpha(x)|}{x^\alpha} dx \leqq K \int_0^u \frac{dx}{x^\alpha} \int_0^x (x - y)^{\alpha-r-1} |s_r(y)| \, dy \\
&\leqq K \int_0^u |s_r(y)| \, dy \int_y^u \frac{(x - y)^{\alpha-r-1}}{x^\alpha} dx \leqq K \int_0^u \frac{|s_r(y)|}{y^\alpha} (u - y)^{\alpha-r} dy \\
&\leqq K\, u^{\alpha-r} \left[\left(\int_0^y |\sigma_r(x)| \, dx \right) y^{r-\alpha} \right]_{y=0}^u \\
&\qquad - K\, u^{\alpha-r} \int_0^u \left(\int_0^y |\sigma_r(x)| \, dx \right) y^{r-\alpha-1} \, dy \\
&= O(u) + O(u^{\alpha+r} \int_0^u y^{r-\alpha} \, dy) = O(u).
\end{aligned}$$

For $\alpha < 0$, we see easily

$$\int_0^u |\sigma_\alpha(x)|\,dx = O(u).$$

Therefore, if (1.3) holds, we see that, for α, $\alpha > r$, (1.3) holds when r replaced by α. Then Theorem follows obviously.

4. Proof of Theorem 2. Since $r+1<p$, we take ε such that

$$r+1<r+1+\varepsilon<p.$$

Then, by Lemma 5,

$$\sigma_{r+1+\varepsilon}(u) \to s, \qquad \text{as } u \to \infty.$$

On the other hand, by the method of the proof of Theorem 1, we see that (1.3) implies, for $\varepsilon > 0$,

$$\int_0^u |\sigma_{r+\varepsilon}(u)|\,du = O(u).$$

Therefore, in our Theorem, the condition that the integral (1.1) is evaluable (A) to s may be replaced by

$$\sigma_{r+1}(u) \to s, \qquad \text{as } u \to \infty. \tag{4.1}$$

For the proof, we may suppose, without loss of generality, that $s=0$, by Lemma 1, and

$$r+1<p<r+2. \tag{4.2}$$

We take an integer q such that $r \leqq \alpha+q<r+1$. Then $q \geqq 1$. Now, by the Riesz theorem [6], (1.4) and (4.1) imply

$$s_{\alpha+\nu+1}(u) = o(u^{(p(r-\alpha-\nu)+(r+1)\nu)/(r-\alpha)}),$$

for $0 \leqq \nu \leqq r-\alpha$. Then we have, as $u \to \infty$ for a fixed $t>0$,

$$\begin{aligned} s_{\alpha+\nu}(u)\frac{d^{\nu-1}}{du^{\nu-1}}\varphi(ut) &= o(u^{(p(r-\alpha-\nu+1)+(r+1)(\nu-1))/(r-\alpha)-p}) \\ &= o(u^{(\nu-1)(r-p+1)/(r-\alpha)}) \\ &= o(1), \end{aligned}$$

for $1 \leqq \nu \leqq q$. Hence, by the integration by parts,

$$\begin{aligned} \int_0^\infty s_\alpha(u)\varphi(ut)\,du &= \left[\sum_{\nu=1}^{q}(-1)^{\nu+1}s_{\alpha+\nu}(u)\frac{d^{\nu-1}}{du^{\nu-1}}\varphi(ut)\right]_{u=0}^{\infty} \\ &\quad + (-1)^q\int_0^\infty s_{\alpha+q}(u)\frac{d^q}{du^q}\varphi(ut)\,du. \\ &= (-1)^q\int_0^\infty s_{\alpha+q}(u)\frac{d^q}{du^q}\varphi(ut)\,du. \end{aligned} \tag{4.3}$$

We shall now consider the case $r<\alpha+q<r+1$, since the case $r=\alpha+q$ may be easily deduced by the following argument. By the well-known

formula

$$s_{\alpha+q}(u)=\frac{1}{\Gamma(\alpha+q-r)}\int_0^u (u-x)^{\alpha+q-r-1}\ s_r(x)\ dx,$$

we have

$$\int_0^\infty s_{\alpha+q}(u)\frac{d^q}{du^q}\varphi(ut)du=\frac{1}{\Gamma(\alpha+q-r)}\int_0^\infty \frac{d^q}{du^q}\varphi(ut)du\int_0^u (u-x)^{\alpha+q-r-1}s_r(x)dx$$

$$=\frac{1}{\Gamma(\alpha+q-r)}\int_0^\infty s_r(x)dx\int_x^\infty (u-x)^{\alpha+q-r-1}\frac{d^q}{du^q}\varphi(ut)du.$$

Here, we shall prove that this interchange is legitimate. For this purpose, it is suffcient to prove that, for a fixed $t>0$,

$$I(N)\equiv\int_0^N s_r(x)dx\int_N^\infty (u-x)^{\alpha+q-r-1}\frac{d^q}{du^q}\varphi(ut)du=o(1),\ \text{as}\ N\to\infty.$$

Let us write

$$I(N)=\int_0^N dx\int_N^{N+1}du+\int_0^N dx\int_{N+1}^\infty du=I_1(N)+I_2(N).$$

Then, by (1.3) and (2.2),

$$\begin{aligned}I_1(N)&=O\Big(\int_0^N |s_r(x)|\ dx\int_N^{N+1}(u-x)^{\alpha+q-r-1}u^{-p}\ du\Big)\\
&=O\Big(\int_N^{N+1}u^{-p}\ du\int_0^N |s_r(x)|(u-x)^{\alpha+q-r-1}dx\Big)\\
&=O\Big(\int_N^{N+1}u^{-p}(u-N)^{\alpha+q-r-1}\ du\int_0^N |s_r(x)|\ dx\Big)\\
&=O\Big(N^{r+1-p}\int_N^{N+1}(u-N)^{\alpha+q-r-1}du\Big)\\
&=O(N^{r+1-p}).\end{aligned}$$

Since, for x, $0\leqq x\leqq N$

$$\begin{aligned}&\int_{N+1}^\infty (u-x)^{\alpha+q-r-1}\frac{d^q}{du^q}\varphi(ut)du\\
&\quad=\Big[-(u-x)^{\alpha+q-r-1}\frac{d^{q-1}}{du^{q-1}}\varphi(ut)\Big]_{u=N+1}^\infty\\
&\qquad-(\alpha+q-r-1)\int_{N+1}^\infty (u-x)^{\alpha+q-r-2}\frac{d^{q-1}}{du^{q-1}}\varphi(ut)du\\
&\quad=O(N^{-p})+O\Big(\int_{N+1}^\infty (u-x)^{\alpha+q-r-2}u^{-p}\ du\Big)\\
&\quad=O(N^{-p})+O\Big(N^{-p}\int_{N+1}^\infty (u-x)^{\alpha+q-r-2}\ du\Big)\end{aligned}$$

$$= O(N^{-p}) + O(N^{-p})$$
$$= O(N^{-p}),$$

we have

$$I_2(N) = O\left(N^{-p}\int_0^N |s_r(x)|\,dx\right) = O(N^{r+1-p}),$$

and then

$$I(N) = O(N^{r+1-p}) = o(1), \qquad \text{as } N \to \infty,$$

by (4. 2), which is the required. Therefore, for the proof of Theorem, it is sufficient to prove that

$$t^{\alpha+1}\int_0^\infty s_r(x)\,dx\int_x^\infty (u-x)^{\alpha+q-r-1}\frac{d^q}{du^q}\varphi(ut)du$$

converges in $0 < t < t_0$ and its limit tends to zero as $t \to 0+$. Let us write

$$t^{\alpha+1}\int_0^\infty s_r(x)dx\int_x^\infty (u-x)^{\alpha+q-r-1}\frac{d^q}{du^q}\varphi(ut)du = \int_0^\infty s_r(x)H(x,t)dx$$
$$= \left(\int_0^M dx + \int_M^\infty dx\right) = U(t) + V(t),$$

where $M = N/t$, N being an arbitrary fixed positive number, and

$$H(x, t) = t^{\alpha+1}\int_x^\infty (u-x)^{\alpha+q-r-1}\frac{d^q}{du^q}\varphi(ut)du.$$

Since $0 < r+1-\alpha-q < 1$, by Lemmas 3 and 4, we get, for $n = 0, 1, 2, \ldots\ldots$,

$$\text{(4. 4)} \qquad \frac{d^n}{dx^n}H(x, t) = O(t^{n+r+1})$$

and

$$\text{(4. 5)} \qquad \frac{d^n}{dx^n}H(x, t) = O(t^{n+r+1-p}x^{-p}).$$

Hence, using (4. 1) and (4. 4),

$$U(t) = \int_0^M s_r(x)H(x, t)dx$$
$$= [s_{r+1}(x)H(x, t)]_{x=0}^M - \int_0^M s_{r+1}(x)\frac{d}{dx}H(x, t)dx$$
$$= o(M^{r+1}t^{r+1}) + o\left(t^{r+2-p}\int_0^M x^{r+1-p}dx\right)$$
$$= o(N^{r+1}) + o(N^{r+2-p}) = o(1),$$

since $r + 2 - p > 0$, by (4. 2). On the other hand, by (1. 3) and (4. 5),

$$
\begin{aligned}
V(t) &= O\left(t^{r+1-p}\int_M^\infty |s_r(x)|\, x^{-p}\, dx\right) \\
&= O\left(t^{r+1-p}\int_M^\infty |\sigma_r(x)|\, x^{r-p}\, dx\right) \\
&= O\left(t^{r+1-p}\left[x^{r-p}\int_0^x |\sigma_r(u)|\, du\right]_{x=M}^\infty\right) + O\left(t^{r+1-p}\int_M^\infty x^{r-p}\, dx\right) \\
&= O(t^{r+1-p}M^{r+1-p}) \\
&= O(N^{r+1-p}).
\end{aligned}
$$

Therefore we get

$$
\int_0^\infty s_r(x)H(x, t)dx = o(1) + O(N^{r+1-p}),
$$

as $t \to 0+$. Since N is arbitrary, we have

$$
\lim_{t\to 0+}\int_0^\infty s_r(x)H(x, t)dx = 0,
$$

and Theorem is completely proved.

5. Proof of Theorem 3. The method of the proof is analogous to one of Theorem 3 in the paper [3], so that we shall sketch the outline of the proof. We shall consider the case in which α is not an integer, the case in which α is an integer being easily proved by the method analogous to the following argument. For the proof, we may suppose, without loss of generality, that $s = 0$ by Lemma 1. Let β be the greatest integer less than α. Then, by the integration by parts, putting $\varphi(x) = (x^{-1} \sin x)^p$, for a fixed $t > 0$,

$$
\begin{aligned}
\int_0^x s_\alpha(u)\varphi(ut)\, du = \sum_{\nu=1}^{p-\beta-1} (-1)^{\nu+1} s_{\alpha+\nu}(x)\frac{d^{\nu-1}}{dx^{\nu-1}}\varphi(xt) \\
+ (-1)^{p-\beta}\int_0^x s_{p+\alpha-\beta-1}(u)\frac{d^{p-\beta-1}}{du^{p-\beta-1}}\varphi(ut)du.
\end{aligned}
$$

By the summability $|C, p|$ of the integral (1, 1), $s_p(x) = o(x^p)$ and then, using Riesz's convexity theorem [6], by (1, 4), for ν such that $\alpha + 1 \leqq \nu \leqq p$,

$$
s_\nu(x) = o(x^p).
$$

Hence, by (2. 2),

$$
\sum_{\nu=1}^{p-\beta-1} (-1)^{\nu+1} s_{\alpha+\nu}(x)\frac{d^{\nu-1}}{dx^{\nu-1}}\varphi(xt) = \sum_{\nu=1}^{p-\beta-1} o(x^p x^{-p}) = o(1).
$$

Therefore, for the proof, it is sufficient to prove that the integral

$$
t^{\alpha+1}\int_0^\infty s_{p+\alpha-\beta-1}(u)\frac{d^{p-\beta-1}}{du^{p-\beta-1}}\varphi(ut)du
$$

converges in some interval $0 < t < t_0$ and its limit tends to zero as $t \to 0+$. This is proved by the quite same method to one of the proof of Theorem 3 in the paper [3]. Thus Theorem is proved.

6. Proof of Theorem 4. For the proof, we need the following

LEMMA 6. *Let*

$$0 < \xi < \eta, \quad 0 \leqq \alpha < \frac{\xi}{p+\eta} \quad \text{and} \quad \frac{p+\xi+1}{p+\eta+1} < \beta < 1. \tag{6.1}$$

Then, there exists a function $a(u)$ defined in $(0, \infty)$ such that

$$\Gamma(\alpha+1)s_\alpha(u) = \int_0^u (u-x)^\alpha a(x)dx \geqq \frac{1}{2} u^{p+\epsilon}, \ (n+2n^{-p-\eta}<u<n+3n^{-p-\eta}), \tag{6.2}$$

where $n = 4, 5, 6, \ldots\ldots$ and ε is a positive constant depending only α, p, ξ and η,

$$(C, \beta)\int_0^\infty a(u)du = 0 \tag{6.3}$$

and

$$|C, 1|\int_0^\infty a(u)du = 0. \tag{6.4}$$

PROOF. Let

$$\begin{aligned} b(u) &= 1 \ (0 \leqq u < 1) \\ &= -2^{-n} \ (n \leqq u < n+1, \ n = 1, 2, 3, \ldots\ldots). \end{aligned} \tag{6.5}$$

Then

$$\int_0^\infty b(u)du = 0 \quad \text{and} \quad \int_0^\infty |b(u)|du = 2.$$

Therefore, by the well-known theorem, for $\gamma > 0$,

$$(C, \gamma)\int_0^\infty b(u)du = 0 \quad \text{and} \quad |C, \gamma|\int_0^\infty b(u)du = 0. \tag{6.6}$$

Thus

$$T_\gamma(u) = \frac{1}{\Gamma(\gamma+1)}\int_0^u (u-x)^\gamma b(x)dx = o(u^\gamma). \tag{6.7}$$

On the other hand, for $n \leqq u < n+1$, $n = 1, 2, 3, \ldots\ldots$,

$$T_0(u) = T(u) = 2^{1-n} - 2^{-n}(u-n) > 0.$$

We shall now define the function $a(u)$. Let, for $n = 4, 5, 6, \ldots\ldots$,

$$\begin{aligned} a(u) &= b(u), \ (0 \leqq u < 4, \ n + 4n^{-p-\eta} \leqq u < n+1), \\ &= \frac{1}{2} n^{p+\eta}(n^{p+\xi} - 2^{1-n}), \ (n \leqq u < n + 2n^{-p-\eta}), \end{aligned} \tag{6.8}$$

$$= -\frac{1}{2}n^{p+\eta}(n^{p+\xi}-T(n+4n^{-p-\eta})),\ (n+2n^{-p-\eta}\leqq u<n+4n^{-p-\eta}).$$

Then, for $n = 4, 5, 6, \ldots\ldots$,

$$s_0(u) = s(u) = T(u)\ (0 \leqq u < 4,\ n + 4n^{-p-\eta} \leqq u < n + 1)$$

$$(6.9) \qquad \geqq \frac{1}{2}n^{p+\xi}\ (n + n^{-p-\eta} < u < n + 3n^{-p-\eta})$$

$$\leqq n^{p+\xi}\ (n \leqq u \leqq n + 4n^{-p-\eta}).$$

Thus (6.2) for $\alpha = 0$ holds. For $\alpha > 0$, by the integration by parts,

$$s_\alpha(u) = \frac{1}{\Gamma(\alpha)}\int_0^u (u - x)^{\alpha-1}s(x)\,dx.$$

Since $s(x) > 0$ for all x, when $n + 2n^{-p-\eta} \leqq u \leqq n + 3n^{-p-\eta}$,

$$\Gamma(\alpha+1)s_\alpha(u) \geqq \alpha\int_{n+n^{-p-\eta}}^u (u - x)^{\alpha-1}s(x)\,dx$$

$$= \frac{\alpha}{2}n^{p+\xi}\int_{n+n^{-p-\eta}}^u (u - x)^{\alpha-1}dx$$

$$= \frac{1}{2}n^{p+\xi-\alpha(p+\eta)},$$

by (6.9). Thus, when $n + 2n^{-p-\eta} \leqq u \leqq n + 3n^{-p-\eta}$,

$$\Gamma(\alpha+1)s_\alpha(u) \geqq \frac{1}{2}n^{p+\varepsilon},$$

where $\varepsilon = \xi - \alpha(p + \eta) > 0$ by (6.1). This proves (6.2) for $\alpha > 0$. On the other hand, using (6.9) and putting $\rho = [u]$, for $0 < \beta < 1$,

$$\Gamma(\beta+1)(s_\beta(u) - T_\beta(u)) = \beta\int_0^u (u-x)^{\beta-1}(s(x)-T(x))dx$$

$$\leqq \beta\sum_{n=4}^{\rho-1}\int_n^{n+4n^{-p-\eta}} (u - x)^{\beta-1}s(x)dx$$

$$+ \beta\int_\rho^u (u - x)^{\beta-1}(s(x) - T(x))dx$$

$$\leqq \sum_{n=4}^{\rho-1} n^{p+\xi}\{(u-n)^\beta-(u-n-4n^{-p-\eta})^\beta\} + 4^\beta\rho^{p+\xi-\beta(p+\eta)}.$$

$$= O\Big(\sum_{n=4}^{\rho} n^{p+\xi-\beta(p+\eta)}\Big) = o(u^\beta),$$

by (6.1). Then, using (6.6),

$$s_\beta(u) = o(u^\beta),$$

which proves (6.3). Lastly we shall prove (6.4). Let

$$\tau(u) = \frac{1}{u}\int_0^u T(x)dx.$$

Then, by (6.6),

$$\int_0^\infty |d\tau(u)| = \int_0^\infty |\tau'(u)|\,du = O(1), \tag{6.10}$$

where

$$\tau'(u) = -\frac{1}{u^2}\int_0^u T(x)dx - \frac{1}{u}T(u).$$

Let

$$J \equiv \left|\int_0^\infty |\sigma'(x)|\,dx - \int_0^\infty |\tau'(x)|\,dx\right| \leqq \int_0^\infty |\sigma'(x) - \tau'(x)|\,dx$$

$$\leqq \int_0^\infty \frac{1}{u^2}\left(\int_0^u (s(x) - T(x))dx\right)du + \int_0^\infty \frac{1}{u}(s(u) - T(u))du$$

$$= J_1 + J_2,$$

say. Then, by (6.9),

$$J_2 = \sum_{n=0}^\infty \int_n^{n+1} \frac{1}{u}(s(u) - \mathrm{T}(u))du \leqq \sum_{n=4}^\infty \frac{1}{n}\int_n^{n+4n^{-p-\eta}} n^{p+\xi}du$$

$$\leqq \sum_{n=4}^\infty 4n^{-(1+\eta-\xi)} = O(1),$$

since $\eta > \xi$. Concerning J_1, when $4 \leqq n \leqq u < n+1$,

$$\frac{1}{u^2}\int_0^u (s(x) - T(x))dx \leqq \frac{1}{u^2}\sum_{m=4}^n \int_m^{m+4m^{-p-\eta}} m^{p+\xi}dx \leqq \frac{4}{n^2}\sum_{m=4}^n m^{\xi-\eta} = O(n^{-1-\delta}),$$

where δ is a positive constant depending only ξ and η. Hence

$$J_1 = \sum_{n=4}^\infty \int_n^{n+1}\left(\frac{1}{u^2}\int_0^u (s(x) - T(x))dx\right)du = O\left(\sum_{n=4}^\infty n^{-1-\delta}\right) = O(1).$$

Thus

$$J = J_1 + J_2 = O(1).$$

Therefore, using (6.10),

$$\int_0^\infty |\sigma'(x)|\,dx = \int_0^\infty |d\sigma(x)| = O(1),$$

which proves (6.4).

PROOF OF THEOREM 4. For any α, $-1 \leqq \alpha < 0$, we take ξ and η such that

$$0 < \xi < \eta \quad\text{and}\quad 0 \leqq \alpha + 1 < \frac{\xi}{p+\eta}.$$

We shall now define the function $a(u)$ by (6.8). Then, by Lemma 6,

$$\Gamma(\alpha+2)s_{\alpha+1}(u) \geqq \frac{1}{2} u^{p+\varepsilon}, \quad (n + 2n^{-p-\eta} < u < n + 3n^{-p-\eta}) \tag{6.11}$$

where $n = 4, 5, 6, \dots\dots$ and ε is a positive constant depending only on α, p, ξ and η, and, when $\dfrac{p+\xi+1}{p+\eta+1} < \beta < 1$,

$$(C, \beta)\int_0^\infty a(u)du = 0. \tag{6.3}$$

Hence, by Corollary of Theorem 1, the integral (1.1) is evaluable (R, p, β) to zero, when $p \geqq 2$ and $\dfrac{p+\xi+1}{p+\eta+1} < \beta < 1$. But the integral (1.1) is not evaluable (R, p, α). In fact, putting $\varphi(u) = (u^{-1}\sin u)^p$, by the integration by parts,

$$t^{\alpha+1}\int_0^N s_\alpha(u)\varphi(ut)du = t^{\alpha+1}s_{\alpha+1}(N)\varphi(tN) - t^{\alpha+1}\int_0^N s_{\alpha+1}(u) \frac{d}{du}\varphi(ut)du,$$

where, as was shown in the proof of Theorem 2, the last term

$$t^{\alpha+1}\int_0^N s_{\alpha+1}(u)\frac{d}{du}\varphi(ut)du$$

converges as $N \to \infty$, in some interval $0 < t < t_0$, while, by (6.11),

$$t^{\alpha+1}s_{\alpha+1}(N)\varphi(tN)$$

is divergent as $N \to \infty$, in an arbitrary interval $0 < t < t_0$. Therefore the integral (1.1) is not evaluable (R, p, α). Thus the proof is complete.

REMARK. As was shown in the proof of Theorem, the integral (1.1) in which $a(u)$ is defined by (6.8) is evaluable (C, β) for some β, $0 < \beta < 1$, and is evaluable $|C, p|$ to zero by (6.4), while, for any α, $-1 \leqq \alpha < 0$, the integral (1.1) is not evaluable (R, p, α). Thus, the condition (1.4) is not dropped in Theorems 2 and 3 when $-1 \leqq \alpha < 0$ and $p \geqq 2$.

7. Proof of Theorem 5. For the proof, we need the following

LEMMA 7. *Let*

$$0 < \xi < \eta < 2\xi,\ 1 \leqq \alpha < 1 + \frac{\eta-\xi}{p+\xi} \text{ and } \frac{2p+\eta}{p+\xi} < \beta < 2. \tag{7.1}$$

Then, there exists a function $a(u)$ defined in $(0, \infty)$ such that

$$s_\alpha(u) > u^{p+\varepsilon}, \quad \left(n + \frac{3}{2}n^{-p-\xi} < u < n + 2n^{-p-\xi}\right), \tag{7.2}$$

where $n = 4, 5, 6, \dots\dots$ and ε is a positive constant depending only on α, p, ξ and η, and

$$(7.3)\qquad (C, \beta)\int_0^\infty a(u)du = 0.$$

PROOF. Let the function $b(u)$ be defined by (6.5) and let the function $a(u)$ be defined by

$$(7.4)\qquad \begin{aligned} a(u) &= 2n^{p+\xi}(n^{2p+\eta} - 2^{-n}), && (n \leqq u < n + n^{-p-\xi})\\ &= -2n^{3p+\xi+\eta}, && (n + n^{-p-\xi} \leqq u < n + 2n^{-p-\xi})\\ &= -n^{p+\xi}(2n^{2p+\eta} - 3\cdot 2^{1-n} + 3\cdot 2^{1-n}n^{-p-\xi}) && (n + 2n^{-p-\xi} \leqq u < n + 3n^{-p-\xi})\\ &= n^{p+\xi}(2n^{2p+\eta} - 2^{2-n} + 2^{1-n}n^{-p-\xi}), && (n + 3n^{-p-\xi} \leqq u < n + 4n^{-p-\xi})\\ &= b(u), && \text{elsewhere,} \end{aligned}$$

where $n = 4, 5, 6, \ldots\ldots\ldots$ Then, when $n + n^{-p-\xi} < u < n + 2n^{-p-\xi}$,

$$(7.5)\qquad s_1(u) = \int_0^u s(x)dx > \int_n^u s(x)dx > \int_n^{n+n^{-p-\xi}} s(x)dx = n^{p+\eta-\xi},$$

which proves (7.2) for $\alpha = 1$. When $n \leqq u \leqq n + 4n^{-p-\xi}$,

$$(7.6)\qquad 0 < s_1(u) = \int_0^n s(x)dx + \int_n^u s(x)dx \leqq 2 - \frac{3}{2^n} + \frac{1}{2}n^{-p-\xi}(2n^{2p+\eta} + 2^{1-n}) + n^{p+\eta-\xi} = O(n^{p+\eta-\xi}),$$

and, when $0 \leqq u < 4$ or $n + 4n^{-p-\xi} \leqq u < n + 1$, $n = 4, 5, 6, \ldots\ldots\ldots$,

$$(7.7)\qquad s_1(u) = T_1(u).$$

We shall now prove (7.2) for $\alpha > 1$. When $n + \dfrac{3}{2}n^{-p-\xi} < u < n + 2n^{-p-\xi}$, using (7.5),

$$\begin{aligned} \Gamma(\alpha + 1)s_\alpha(u) &= \alpha(\alpha - 1)\int_0^u (u - x)^{\alpha-2}s_1(x)dx\\ &> \alpha(\alpha - 1)\int_{n+n^{-p-\xi}}^u (u - x)^{\alpha-2}s_1(x)dx\\ &> \alpha n^{p+\eta-\xi}(u - n - n^{-p-\xi})^{\alpha-1}\\ &> \alpha n^{p+\eta-\xi}\left(\frac{1}{2}n^{-p-\xi}\right)^{\alpha-1}\\ &= 2^{1-\alpha}\alpha n^{p+\eta-\xi-(\alpha-1)(p+\xi)}\\ &= 2^{1-\alpha}\alpha n^{p+\varepsilon}, \end{aligned}$$

where $\varepsilon = \eta - \xi - (\alpha - 1)(p + \xi) > 0$ by (7.1), which proves (7.2). Next we shall prove (7.3). Using (7.6) and (7.7), putting $\rho = [u]$,

$$\Gamma(\beta + 1)(s_\beta(u) - T_\beta(u)) = \beta(\beta - 1)\int_0^u (u - x)^{\beta-2}(s_1(x) - T_1(x))dx$$

$$
\begin{aligned}
&\leqq \beta(\beta-1)\sum_{4=n}^{\rho-1}\int_n^{n+4\iota^{-p-\xi}}(u-x)^{\beta-2}s_1(x)dx \\
&\qquad + \beta(\beta-1)\int_\rho^u (u-x)^{\beta-2}\Big(s_1(x)-T_1(x)\Big)dx \\
&= O\Big(\sum_{n=4}^{\rho-1} n^{p+\eta-\xi}\{(u-n)^{\beta-1}-(u-n-4n^{-p-\xi})^{\beta-1}\} \\
&\qquad + O(\rho^{p+\eta-\xi-(\beta-1)(p+\xi)})\Big) \\
&= O\Big(\sum_{n=4}^{\rho-1} n^{p+\eta-\xi-(\beta-1)(p+\xi)}\Big) + O(u^{p+\eta-\xi-(\beta-1)(p+\xi)}) \\
&= O(u^\beta),
\end{aligned}
$$

since, by (7. 1),

$$p+\eta-\xi+1-(\beta-1)(p+\xi)<\beta.$$

Since, by (6. 7), $T_\beta(u)=o(u^\beta)$, we have $s_\beta(u)=o(u^\beta)$, that is, (7. 3) holds.

PROOF OF THEOREM 5. The case in which $-1\leqq\alpha<0$ is proved by Theorem 4. Hence we shall prove for the case in which $0\leqq\alpha<1$. For any α, we take ξ and η such that

$$0<\xi<\eta<2\xi \quad \text{and} \quad 1\leqq\alpha+1<1+\frac{\eta-\xi}{p+\xi}.$$

We shall now define the function $a(u)$ by (7. 4). Then, using Lemma 7, the proof runs similarly to one of Theorem 4.

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