ON THE PRODUCT PROJECTION OF NORM ONE IN THE DIRECT PRODUCT OF OPERATOR ALGEBRAS

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In the theory of operator algebras, the following result is known (cf. [2], [4], [9], [14]): If θ_1 and θ_2 are normal $*$-homomorphisms from W^*-algebras $\mathbf{M}_1$ and $\mathbf{M}_2$ onto the other W^*-algebras $\mathbf{N}_1$ and $\mathbf{N}_2$, then there exists the unique normal $*$-homomorphism θ from the W^*-direct product of $\mathbf{M}_1$ and $\mathbf{M}_2$, $\mathbf{M}_1 \otimes \mathbf{M}_2$ onto that of $\mathbf{N}_1$ and $\mathbf{N}_2$, $\mathbf{N}_1 \otimes \mathbf{N}_2$ such as $\theta(x \otimes y) = \theta_1(x) \otimes \theta_2(y)$. Moreover, combining this result with that of Takeda [8], it can be shown that if θ_1 and θ_2 are $*$-homomorphisms from C^*-algebras $\mathbf{M}_1$ and $\mathbf{M}_2$ onto the other C^*-algebras $\mathbf{N}_1$ and $\mathbf{N}_2$ there exists the unique $*$-homomorphism θ from the C^*-direct product of $\mathbf{M}_1$ and $\mathbf{M}_2$, $\mathbf{M}_1 \mathbin{\widehat{\otimes}_\alpha} \mathbf{M}_2$, onto that of $\mathbf{N}_1$ and $\mathbf{N}_2$, $\mathbf{N}_1 \mathbin{\widehat{\otimes}_\alpha} \mathbf{N}_2$ such as $\theta(x \otimes y) = \theta_1(x) \otimes \theta_2(y)$. In both cases, if θ_1 and θ_2 are $*$-isomorphisms θ is a $*$-isomorphism, too.

In the present paper we shall show that analogous results also hold for the projections of norm one in the operator algebras. Namely, for two W^*-algebras $\mathbf{M}_1$ and $\mathbf{M}_2$ and their W^*-subalgebras $\mathbf{N}_1$ and $\mathbf{N}_2$ if π_1 and π_2 are normal projections of norm one from $\mathbf{M}_1$ and $\mathbf{M}_2$ to $\mathbf{N}_1$ and $\mathbf{N}_2$ respectively, there exists the unique normal projection of norm one π from $\mathbf{M}_1 \otimes \mathbf{M}_2$ to its W^*-subalgebra $\mathbf{N}_1 \otimes \mathbf{N}_2$ such as $\pi(x \otimes y) = \pi_1(x) \otimes \pi_2(y)$ and similar result holds for the projections of norm one in C^*-algebras. After proving them we shall show some applications of these results in the next section.

Before going into discussions, the author recalls many valuable conversations with Mr. M. Takesaki in the presentation of this paper for which he must thank to him.

1. In the sequel, the algebraic direct product of two operator algebras $\mathbf{M}_1$ and $\mathbf{M}_2$ are always denoted by $\mathbf{M}_1 \odot \mathbf{M}_2$. For two C^*-algebras $\mathbf{M}_1$ and $\mathbf{M}_2$, the α-norm of an element $\sum_{i=1}^{n} x_i \otimes y_i$ of $\mathbf{M}_1 \odot \mathbf{M}_2$ is defined as follows:

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|^2 =$$

$$= \sup \left\{ \frac{< \left(\sum_{j=1}^{m} a_j \otimes b_j\right)^* \left(\sum_{i=1}^{n} x_i \otimes y_i\right)^* \left(\sum_{i=1}^{n} x_i \otimes y_i\right) \left(\sum_{j=1}^{n} a_j \otimes b_j\right),\ \varphi \otimes \psi >}{< \left(\sum_{j=1}^{m} a_j \otimes b_j\right)^* \left(\sum_{j=1}^{m} a_j \otimes b_j\right),\ \varphi \otimes \psi >} \right\}$$

where supremum is taken over all elements $\sum_{j=1}^{m} a_j \otimes b_j$ of $\mathbf{M}_1 \odot \mathbf{M}_2$ and all positive linear functionals φ and ψ on $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively and $\varphi \otimes \psi$ means the product linear functional of φ and ψ on $\mathbf{M}_1 \odot \mathbf{M}_2$ which is given by

$$< \sum_{i=1}^{n} x_i \otimes y_i,\ \varphi \otimes \psi > = \sum_{i=1}^{n} < x_i,\ \varphi > < y_i,\ \psi >.$$

Notice that the product linear functional $\varphi \otimes \psi$ is continuous for the α-norm of $\mathbf{M}_1 \odot \mathbf{M}_2$ defined above, so that we can extend this to the uniform closure of $\mathbf{M}_1 \odot \mathbf{M}_2$, $\mathbf{M}_1 \underset{\alpha}{\widehat{\otimes}} \mathbf{M}_2$. The extension of $\varphi \otimes \psi$ on $\mathbf{M}_1 \underset{\alpha}{\widehat{\otimes}} \mathbf{M}_2$ is again denoted by $\varphi \otimes \psi$.

A projection of norm one from a C^*-algebra $\mathbf{M}$ to its C^*-subalgebra $\mathbf{N}$ is called faithful if it is faithful on each positive element of $\mathbf{M}$.

THEOREM 1. *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be C^*-algebras and $\mathbf{N}_1$ and $\mathbf{N}_2$ their C^*-subalgebras respectively. If π_1 and π_2 are projections of norm one from $\mathbf{M}_1$ and $\mathbf{M}_2$ to $\mathbf{N}_1$ and $\mathbf{N}_2$, then there exists the unique projection of norm one π from $\mathbf{M}_1 \underset{\alpha}{\widehat{\otimes}} \mathbf{M}_2$ to $\mathbf{N}_1 \underset{\alpha}{\widehat{\otimes}} \mathbf{N}_2$ such as $\pi(x \otimes y) = \pi_1(x) \otimes \pi_2(y)$, besides if π_1 and π_2 are faithful on $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively, then π is also a faithful projection of norm one.*

We call this projection π the product projection of π_1 and π_2 and denote by $\pi_1 \widehat{\otimes} \pi_2$.

PROOF. For $\sum_{i=1}^{n} x_i \otimes y_i \in \mathbf{M}_1 \odot \mathbf{M}_2$, we define

$$\pi\left(\sum_{i=1}^{n} x_i \otimes y_i\right) \equiv \sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i).$$

Since π_1 and π_2 are expectations to $\mathbf{N}_1$ and $\mathbf{N}_2$ by [10], π is the projection from $\mathbf{M}_1 \odot \mathbf{M}_2$ to $\mathbf{N}_1 \odot \mathbf{N}_2$ that satisfies

$$\pi[(a \otimes b)(x \otimes y)(c \otimes d)] = a \otimes b\ \pi(x \otimes y)\ c \otimes d$$

for all $x \otimes y \in \mathbf{M}_1 \odot \mathbf{M}_2$ and $a \otimes b,\ c \otimes d \in \mathbf{N}_1 \odot \mathbf{N}_2$.

Next, consider the product linear functional $\varphi \otimes \psi$ where φ and ψ are positive linear functionals on $\mathbf{N}_1$ and $\mathbf{N}_2$, respectively. We have

$$
\begin{aligned}
&<\pi\left[\left(\sum_{i=1}^{n} x_i \otimes y_i\right)^*\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right],\ \varphi \otimes \psi> \\
&= <\pi\left(\sum_{i,j=1}^{n} x_i^* x_j \otimes y_i^* y_j\right),\ \varphi \otimes \psi> \\
&= <\sum_{i,j=1}^{n} \pi_1(x_i^* x_j) \otimes \pi_2(y_i^* y_j),\ \varphi \otimes \psi> \\
&= \sum_{i,j=1}^{n} <\pi_1(x_i^* x_j),\ \varphi> <\pi_2(y_i^* y_j),\ \psi> \\
&= <\left(\sum_{i=1}^{n} x_i \otimes y_i\right)^*\left(\sum_{i=1}^{n} x_i \otimes y_i\right),\ {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)> \geqq 0.
\end{aligned}
$$

The last inequality of positivity follows from the positivity of the product linear functional ${}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)$. Therefore,

$$
\begin{aligned}
<\pi\Bigg[\left(\sum_{i=1}^{n} x_i \otimes y_i - \sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i)\right)^*\Bigg(\sum_{i=1}^{n} x_i \otimes y_i - \\
\sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i)\Bigg)\Bigg],\ \varphi \otimes \psi> \geqq 0,
\end{aligned}
$$

where φ and ψ are positive linear functionals on $\mathbf{N}_1$ and $\mathbf{N}_2$, respectively. On the other hand, by the multiplicative property of π, one may easily verify the following identity,

$$
\begin{aligned}
&\pi\left[\left(\sum_{i=1}^{n} x_i \otimes y_i - \sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i)\right)^*\left(\sum_{i=1}^{n} x_i \otimes y_i - \sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i)\right)\right] \\
&= \pi\left[\left(\sum_{i=1}^{n} x_i \otimes y_i\right)^*\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right] - \pi\left(\sum_{i=1}^{n} x_i \otimes y_i\right)^* \pi\left(\sum_{i=1}^{n} x_i \otimes y_i\right).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&<\pi\left[\left(\sum_{i=1}^{n} x_i \otimes y_i\right)^*\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right],\ \varphi \otimes \psi> \\
&\qquad \geqq <\pi\left(\sum_{i=1}^{n} x_i \otimes y_i\right)^* \pi\left(\sum_{i=1}^{n} x_i \otimes y_i\right),\ \varphi \otimes \psi>.
\end{aligned}
$$

Now the α-norm of $\sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i)$ in $\mathbf{N}_1 \odot \mathbf{N}_2$ is given by

$$
\left\|\sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i)\right\|^2 =
$$

$$= \sup \left\{ \frac{< \left(\sum_{j=1}^{m} a_j \otimes b_j\right)^* \left(\sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i)\right)^* \left(\sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i)\right) \left(\sum_{j=1}^{m} a_j \otimes b_j\right), \varphi \otimes \psi >}{< \left(\sum_{j=1}^{m} a_j \otimes b_j\right)^* \left(\sum_{j=1}^{m} a_j \otimes b_j\right), \varphi \otimes \psi >} \right\}$$

where supremum is taken over all elements $\sum_{j=1}^{m} a_j \otimes b_j$ of $\mathbf{N}_1 \odot \mathbf{N}_2$ and all positive linear functionals φ and ψ on $\mathbf{N}_1$ and $\mathbf{N}_2$ respectively. From the above argument, one can get

$$< \left(\sum_{j=1}^{m} a_j \otimes b_j\right)^* \left(\sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i)\right)^* \left(\sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i)\right) \left(\sum_{j=1}^{m} a_j \otimes b_j\right), \varphi \otimes \psi >$$
$$= < \pi \left[\left(\sum_{i=1}^{n} x_i \otimes y_i\right) \left(\sum_{j=1}^{m} a_j \otimes b_j\right)\right]^* \pi \left[\left(\sum_{i=1}^{n} x_i \otimes y_i\right) \left(\sum_{j=1}^{m} a_j \otimes b_j\right)\right], \varphi \otimes \psi >$$
$$\leqq < \pi \left[\left(\sum_{j=1}^{m} a_j \otimes b_j\right)^* \left(\sum_{i=1}^{n} x_i \otimes y_i\right)^* \left(\sum_{i=1}^{n} x_i \otimes y_i\right) \left(\sum_{j=1}^{m} a_j \otimes b_j\right)\right], \varphi \otimes \psi >$$
$$= < \left(\sum_{j=1}^{m} a_j \otimes b_j\right)^* \left(\sum_{i=1}^{n} x_i \otimes y_i\right)^* \left(\sum_{i=1}^{n} x_i \otimes y_i\right) \left(\sum_{j=1}^{m} a_j \otimes b_j\right), {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi) > .$$

Besides, as π is the projection from $\mathbf{M}_1 \odot \mathbf{M}_2$ to $\mathbf{N}_1 \odot \mathbf{N}_2$ we have

$$< \left(\sum_{j=1}^{m} a_j \otimes b_j\right)^* \left(\sum_{j=1}^{m} a_j \otimes b_j\right), \varphi \otimes \psi >$$
$$= < \left(\sum_{j=1}^{m} a_j \otimes b_j\right)^* \left(\sum_{j=1}^{m} a_j \otimes b_j\right), {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi) > .$$

Therefore,

$$\left\| \pi \left(\sum_{i=1}^{n} x_i \otimes y_i\right) \right\|^2 = \left\| \sum_{i=1}^{n} \pi_1(x_i) \otimes \pi_2(y_i) \right\|^2$$
$$\leqq \sup \left\{ \frac{< \left(\sum_{j=1}^{m} a_j \otimes b_j\right)^* \left(\sum_{i=1}^{n} x_i \otimes y_i\right)^* \left(\sum_{i=1}^{n} x_i \otimes y_i\right) \left(\sum_{j=1}^{m} a_j \otimes b_j\right), {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi) >}{< \left(\sum_{j=1}^{m} a_j \otimes b_j\right)^* \left(\sum_{j=1}^{m} a_j \otimes b_j\right), {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi) >} \right\}$$
$$\leqq \left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|^2 .$$

The last inequality follows from the definition of the α-norm $\left\|\sum_{i=1}^{n} x_i \otimes y_i\right\|$ in $\mathbf{M}_1 \odot \mathbf{M}_2$. We have $\|\pi\| \leqq 1$ on $\mathbf{M}_1 \odot \mathbf{M}_2$. Hence π can be extended to the mapping π from the uniform closure of $\mathbf{M}_1 \odot \mathbf{M}_2$, $\mathbf{M}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{M}_2$, to that of $\mathbf{N}_1 \odot \mathbf{N}_2$, $\mathbf{N}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{N}_2$, which is easily seen to be the projection of norm one from $\mathbf{M}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{M}_2$ to $\mathbf{N}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{N}_2$ such as $\pi(x \otimes y) = \pi_1(x) \otimes \pi_2(y)$.

Since the unicity is obvious, this concludes the first half of our proof.

Next, suppose that π_1 and π_2 are faithful projections of norm one. From the assumption, the family $\{{}^t\pi_1(\varphi)\}$ where φ runs over all states on $\mathbf{N}_1$ is a faithful family of $\mathbf{M}_1$. The same result holds for the family $\{{}^t\pi_2(\psi)\}$ where ψ runs over all states on $\mathbf{N}_2$. Therefore we can conclude that the family $\{{}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)\}$ where φ and ψ run over all states on $\mathbf{N}_1$ and $\mathbf{N}_2$ respectively is also faithful on $\mathbf{M}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{M}_2$, which implies the faithfulness of π constructed above.

REMARK. A slight modification of this proof yields a simple and elementary alternative proof of the previously quoted results for homomorphisms of C^*-algebras and the method of deducing Theorem 2 from Theorem 1 is also applicable to get an another proof of those results for normal homomorphisms of W^*-algebras.

THEOREM 2. *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be W^*-algebras and $\mathbf{N}_1$ and $\mathbf{N}_2$ their W^*-subalgebras respectively. If π_1 and π_2 are normal projections of norm one from $\mathbf{M}_1$ and $\mathbf{M}_2$ to $\mathbf{N}_1$ and $\mathbf{N}_2$, then there exists the unique normal projection of norm one π from $\mathbf{M}_1 \otimes \mathbf{M}_2$ to $\mathbf{N}_1 \otimes \mathbf{N}_2$ such as $\pi(x \otimes y) = \pi_1(x) \otimes \pi_2(y)$. Besides, if the both projections are faithful, so is the projection of norm one constructed above.*

We call this projection π the weak product projection of π_1 and π_2 and denote by $\pi_1 \otimes \pi_2$.

PROOF. By Theorem 1, there exists uniquely a projection of norm one π from $\mathbf{M}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{M}_2$ to $\mathbf{N}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{N}_2$ such as $\pi(x \otimes y) = \pi_1(x) \otimes \pi_2(y)$. $\mathbf{M}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{M}_2$ and $\mathbf{N}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{N}_2$ are considered to be σ-weakly dense C^*-subalgebras of $\mathbf{M}_1 \otimes \mathbf{M}_2$ and $\mathbf{N}_1 \otimes \mathbf{N}_2$, respectively. Therefore, by density theorem, $(\mathbf{M}_1 \otimes \mathbf{M}_2)_*$, the space of all σ-weakly continuous linear functionals on $\mathbf{M}_1 \otimes \mathbf{M}_2$, and $(\mathbf{N}_1 \otimes \mathbf{N}_2)_*$, that of $\mathbf{N}_1 \otimes \mathbf{N}_2$, can be isometrically embedded into $(\mathbf{M}_1 \hat{\underset{\alpha}{\otimes}} \mathbf{M}_2)^*$ and

$(\mathbf{N}_1 \hat{\otimes}_\alpha \mathbf{N}_2)^*$.

Now, by [14], $(\mathbf{M}_1 \otimes \mathbf{M}_2)_* = \mathbf{M}_{1*} \hat{\otimes}_{\alpha'} \mathbf{M}_{2*}$ and $(\mathbf{N}_1 \otimes \mathbf{N}_2)_* = \mathbf{N}_{1*} \hat{\otimes}_{\alpha'} \mathbf{N}_{2*}$ where $\mathbf{M}_{1*} \hat{\otimes}_{\alpha'} \mathbf{M}_{2*}$ means the tensor product of two Banach spaces $\mathbf{M}_{1*}$ and $\mathbf{M}_{2*}$ with the associate norm α' for α in the sense of Schatten [7] and $\mathbf{N}_{1*} \hat{\otimes}_{\alpha'} \mathbf{N}_{2*}$ means the similar product of $\mathbf{N}_{1*}$ and $\mathbf{N}_{2*}$. Consider ${}^t\pi(\varphi \otimes \psi)$ in $(\mathbf{M}_1 \hat{\otimes}_\alpha \mathbf{M}_2)_*$ where φ and ψ are σ-weakly continuous linear functionals on $\mathbf{N}_1$ and $\mathbf{N}_2$ respectively. Then a usual computation shows that ${}^t\pi(\varphi \otimes \psi) = {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)$. Since π_1 and π_2 are σ-weakly continuous ${}^t\pi_1(\varphi)$ and ${}^t\pi_2(\psi)$ are σ-weakly continuous linear functionals, so that we can conclude ${}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi) \in \mathbf{M}_{1*} \hat{\otimes}_{\alpha'} \mathbf{M}_{2*}$. Hence ${}^t\pi(\mathbf{N}_{1*} \odot \mathbf{N}_{2*}) \subset \mathbf{M}_{1*} \hat{\otimes}_{\alpha'} \mathbf{M}_{2*}$ where $\mathbf{N}_{1*} \odot \mathbf{N}_{2*}$ means the algebraic tensor product of $\mathbf{N}_{1*}$ and $\mathbf{N}_{2*}$ with α'-norm. We have, therefore, ${}^t\pi(\mathbf{N}_{1*} \hat{\otimes}_{\alpha'} \mathbf{N}_{2*}) \subset \mathbf{M}_{1*} \hat{\otimes}_{\alpha'} \mathbf{M}_{2*}$; π is σ-weakly continuous on $\mathbf{M}_1 \hat{\otimes}_\alpha \mathbf{M}_2$. Thus we can extend π to the σ-weak closure of $\mathbf{M}_1 \hat{\otimes}_\alpha \mathbf{M}_2$, $\mathbf{M}_1 \otimes \mathbf{M}_2$ and one easily verifies that this extension, which is also denoted by π, is the normal projection of norm one from $\mathbf{M}_1 \otimes \mathbf{M}_2$ to $\mathbf{N}_1 \otimes \mathbf{N}_2$ such as $\pi(x \otimes y) = \pi_1(x) \otimes \pi_2(y)$.

The unicity is clear as in th case of Theorem 1.

Now, suppose that both π_1 and π_2 are faithful. From the assumption, the family $V_1 = \{{}^t\pi_1(\varphi) \mid \varphi \in \mathbf{N}_{1*}$ and $\varphi \geqq 0\}$ is a faithful family of positive normal linear functionals on $\mathbf{M}_1$ and $V_2 = \{{}^t\pi_2(\psi) \mid \psi \in \mathbf{N}_{2*}$ and $\psi \geqq 0\}$ is also a faithful family on $\mathbf{M}_2$.

Assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ are W^*-algebras on some Hilbert spaces $\mathbf{H}_1$ and $\mathbf{H}_2$ respectively. By [3], there exists a suitable family $\{\xi_i(\varphi)\}$ of vectors in $\mathbf{H}_1$ such as $<x, {}^t\pi_1(\varphi)> = \sum_{i=1}^{\infty} <x\xi_i(\varphi), \xi_i(\varphi)>$ for every element $x \in \mathbf{M}_1$. Put $K_1 = \{\xi_i(\varphi) \mid {}^t\pi_1(\varphi) \in V_1,\ i = 1, 2, 3, \ldots\ldots\}$. We see that the faithfulness of the family V_1 is equivalent to that the above family K_1 is a separating family of vectors for $\mathbf{M}_1$. Similar result holds for π_2, that is, the family $K_2 = \{\eta_j(\psi) \mid {}^t\pi_2(\psi) \in V_2,\ j = 1, 2, 3, \ldots\ldots\}$ is a separating family of vectors for $\mathbf{M}_2$ where $\{\eta_j(\psi) \mid j = 1, 2, \ldots\ldots\}$ is the family of vectors in $\mathbf{H}_2$ such as $<y, {}^t\pi_2(\psi)> = \sum_{j=1}^{\infty} <y\eta_j(\psi), \eta_j(\psi)>$ for $y \in \mathbf{M}_2$. Hence the family of all vectors $\{\xi_i(\varphi) \otimes \eta_j(\psi) \mid \xi_i(\varphi) \in K_1,\ \eta_j(\psi) \in K_2\}$ in $\mathbf{H}_1 \otimes \mathbf{H}_2$ is a separating family of vectors for $\mathbf{M}_1 \otimes \mathbf{M}_2$, too. Therefore one may conclude that the family $\{{}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi) \mid {}^t\pi_1(\varphi) \in V_1,\ {}^t\pi_2(\psi) \in V_2\}$ is a faithful family of positive normal linear functionals on $\mathbf{M}_1 \otimes \mathbf{M}_2$. Considering that ${}^t\pi(\varphi \otimes \psi)$

$= {}^t\pi_1(\varphi) \otimes {}^t\pi_2(\psi)$, this means the faithfulness of π on $\mathbf{M}_1 \otimes \mathbf{M}_2$. This completes the proof.

2. With help of the above theorems we get firstly the systematic explanation for the following results which are essentially proved in Sakai [6] (the case of W^*-algebras) and Takesaki [16] (the case of C^*-algebras).

THEOREM 3. *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be C^*-algebras, then there exist sufficiently many projections of norm one from $\mathbf{M}_1 \underset{\alpha}{\widehat{\otimes}} \mathbf{M}_2$ to $\mathbf{M}_1$ and $\mathbf{M}_2$ identifying $\mathbf{M}_1$ with $\mathbf{M}_1 \otimes 1$ and $\mathbf{M}_2$ with $1 \otimes \mathbf{M}_2$. On the other hand, if $\mathbf{M}_1$ and $\mathbf{M}_2$ are W^*-algebras there exist sufficiently many normal projections of norm one from $\mathbf{M}_1 \otimes \mathbf{M}_2$ to both $\mathbf{M}_1$ and $\mathbf{M}_2$ with the same identification in $\mathbf{M}_1 \otimes \mathbf{M}_2$.*

PROOF. At first, we notice that a state ψ of $\mathbf{M}_2$ induces a projection of norm one π_ψ from $\mathbf{M}_2$ to its trivial subalgebra $(\lambda 1)$ where λ is an arbitrary complex number and 1 is the identity of $\mathbf{M}_2$. π_ψ is nothing but the mapping $x \to \langle x, \psi \rangle 1$ for each $x \in \mathbf{M}_2$.

Then we get the family of product projections of norm one $\{\pi_0 \widehat{\otimes} \pi_\psi\}$ from $\mathbf{M}_1 \underset{\alpha}{\widehat{\otimes}} \mathbf{M}_2$ to $\mathbf{M}_1 \otimes (\lambda 1) = \mathbf{M}_1 \otimes 1 = \mathbf{M}_1$ where π_0 is the identity mapping of $\mathbf{M}_1$ and ψ runs over all states of $\mathbf{M}_2$.

Take an arbitrary non-zero element $x \in \mathbf{M}_1 \underset{\alpha}{\widehat{\otimes}} \mathbf{M}_2$. Since $\mathbf{M}_1^* \underset{\alpha'}{\widehat{\otimes}} \mathbf{M}_2^*$ is total on $\mathbf{M}_1 \underset{\alpha}{\widehat{\otimes}} \mathbf{M}_2$ and $\mathbf{M}_1^*$ and $\mathbf{M}_2^*$ are linearly spanned by their positive elements, we can fined a product linear functional $\varphi_0 \otimes \psi_0$ such as $\langle x, \varphi_0 \otimes \psi_0 \rangle \neq 0$ where φ_0 and ψ_0 are states on $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively. Then, consider the product projection $\pi_0 \widehat{\otimes} \pi_{\psi_0}$, we have

$$\langle \sum_{i=1}^n x_i \otimes y_i, \ {}^t(\pi_0 \widehat{\otimes} \pi_{\psi_0})(\varphi_0) \rangle = \sum_{i=1}^n \langle x_i \otimes \langle y_i, \psi_0 \rangle 1, \ \varphi_0 \rangle$$
$$= \sum_{i=1}^n \langle x_i, \varphi_0 \rangle \langle y_i, \psi_0 \rangle = \langle \sum_{i=1}^n x_i \otimes y_i, \ \varphi_0 \otimes \psi_0 \rangle$$

for every element $\sum_{i=1}^n x_i \otimes y_i \in \mathbf{M}_1 \odot \mathbf{M}_2$, which implies ${}^t(\pi_0 \widehat{\otimes} \pi_{\psi_0})(\varphi_0) = \varphi_0 \otimes \psi_0$. Therefore $\langle \pi_0 \widehat{\otimes} \pi_{\psi_0}(x), \varphi_0 \rangle = \langle x, {}^t(\pi_0 \widehat{\otimes} \pi_{\psi_0})(\varphi_0) \rangle = \langle x, \ \varphi_0 \otimes \psi_0 \rangle \neq 0$, that is, $\pi_0 \widehat{\otimes} \pi_{\psi_0}(x) \neq 0$. Thus, there exist sufficiently many projections of norm one $\{\pi_0 \widehat{\otimes} \pi_\psi\}$ from $\mathbf{M}_1 \underset{\alpha}{\widehat{\otimes}} \mathbf{M}_2$ to $\mathbf{M}_1$ where ψ runs over all states of $\mathbf{M}_2$.

The symmetric argument shows that the same result holds for $\mathbf{M}_2$, too.

In case that $\mathbf{M}_1$ and $\mathbf{M}_2$ are W^*-algebras, it suffices only to use a normal state ψ of $\mathbf{M}_2$, in place of a state ψ in the above arguments, recalling that $\mathbf{M}_{1*} \underset{\alpha'}{\widehat{\otimes}} \mathbf{M}_{2*}$ is total on $\mathbf{M}_1 \otimes \mathbf{M}_2$ by [14]. Hence, details are omitted.

REMARK. The statement of the last half of Theorem 3 lacks in many properties of mappings in [6] but it is the existence of the σ-weakly continuous projection of norm one that we need because other properties of mappings stated there, are all deduced from the property of the σ-weakly continuous projection of norm one by [10], [11].

The next application is an alternative proof of theorem by Misonou [4: Theorem 4 and Theorem 6] which we quote here for the convenience.

THEOREM 4. *Let* $\mathbf{M}$, $\mathbf{N}$ *be finite* W^**-algebras and* $\natural_1$ *and* $\natural_2$ *their* $\natural$*-applications. Then* $\mathbf{M} \otimes \mathbf{N}$ *is again a finite* W^**-algebra and its* $\natural$*-application coincides with the weak product projection of* $\natural_1$ *and* $\natural_2$, *i. e.* $(x \otimes y)^\natural = x^{\natural_1} \otimes y^{\natural_2}$.

PROOF. A $\natural$-application may be considered to be a faithful normal projection of norm one from a finite W^*-algebra to its center having special properties. Hence we can consider the weak product projection $\natural_1 \otimes \natural_2$ in $\mathbf{M} \otimes \mathbf{N}$. By Theorem 2, $\natural_1 \otimes \natural_2$ is a faithful normal projection of norm one from $\mathbf{M} \otimes \mathbf{N}$ to $Z_1 \otimes Z_2$ where Z_1 and Z_2 are centers of $\mathbf{M}$ and $\mathbf{N}$ respectively.

Now, $Z_1 \otimes Z_2$ coincides with the center of $\mathbf{M} \otimes \mathbf{N}$ besides a simple computation shows that $(xy)^{\natural_1 \otimes \natural_2} = (yx)^{\natural_1 \otimes \natural_2}$ for every $x, y \in \mathbf{M} \otimes \mathbf{N}$. Hence, by [1], $\mathbf{M} \otimes \mathbf{N}$ is a finite W^*-algebra and $\natural_1 \otimes \natural_2$ coincides with its $\natural$-application. This completes the proof.

REMARK. Using Theorem 4, we also get an another short proof of the fact that if $\mathbf{M}$ and $\mathbf{N}$ are semi-finite W^*-algebras then $\mathbf{M} \otimes \mathbf{N}$ is semi-finite, too. Thus, combining with the results of [6] and [12] we have succeeded to explain all problems of types of the direct product of W^*-algebras by the projection of norm one in them.

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