ON CONVERGENCE CRITERIA FOR FOURIER SERIES II

KENJI YANO

(Received March 10, 1958, Revised July 7, 1958)

1. Introduction. Let $\varphi(t)$ be integrable in $(0, \pi)$, even, periodic of period 2π, and let

$$\varphi(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt.$$

We write

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}[\varphi(u) - s]\, du \qquad (\alpha > 0),$$

and

$$s_n = \frac{1}{2} a_0 + \sum_{\nu=1}^{n} a_\nu.$$

F. T. Wang [1] has proved that if $\beta > \alpha > 0$, $\varphi(t) \in L$, $\Phi_\alpha(t) = o(t^\beta)$ as $t \to 0$ and if $a_n > -An^{-\alpha/\beta}$, $A > 0$, then $s_n \to s$ as $n \to \infty$. In order to prove this, Wang used the method of Riesz summability. In this paper we shall give an alternative proof by a method of generalized de la Vallée Poussin summability. In § 4 we shall refer to jump functions. This note is a continuation of K. Yano [5], but may be readed free from it.

DEFINITION 1. We define $g(x)$ such as
$1°$ $g(x) > 0$ for $x \geqq x_0 > 0$, $2°$ $g(x) \uparrow \infty$ as $x \uparrow \infty$, and $3°$ $H \leqq g(x^\delta)/g(x) \leqq 1$, $0 < \delta < 1$ for all $x \geqq x_0$, where $H = H(\delta)$ is a positive constant depending on δ only.

Then we see easily that $g(x) = o(x^\varepsilon)$ as $x \to \infty$ for every positive ε. In this definition we require no differentiability of $g(x)$. We may take for $g(x)$, e. g.,

$$\log x,\ (\log x)^\alpha \log\log x\ (\alpha \geqq 0) \text{ and } \log_p x,$$

where $\log_p$ denotes the p-times iterated logarithm. For the sake of simplicity we denote $(g(x))^\alpha$ by $g(x)^\alpha$ throughout this paper.

THEOREM 1.[1)] *Let $\beta \geqq \alpha > 0$ and let $g(x)$ be unity or defined by Definition* 1. *If*

$$(1.1) \qquad \int_0^t |\Phi_\alpha(u)|\, du = o\left(t^{\beta+1}/g\left(\frac{1}{t}\right)\right) \qquad (t \to 0),$$

and if for any assigned positive ε

1) $\varphi(t)$ requires no integrability in Lebesgue sense.

(1.2) $\qquad s_{n+\nu} - s_n > -\varepsilon$ *for* $\nu = 1, 2, \dots, m,$ $\qquad (n \geqq n_\varepsilon),$

where

(1.3) $\qquad m = [\eta n^{\alpha/\beta}/g(n)^{1/\beta}], \qquad \eta = \varepsilon^2,$

then $s_n \to s$ *as* $n \to \infty$, *i.e. the Fourier series of* $\varphi(t)$ *is* convergent to sum s at $t = 0$.

COROLLARY 1.1. *Theorem* 1 *holds when the condition* (1.2) *with* (1.3) *is replaced by*

(1.4) $\qquad a_n > -An^{-\alpha/\beta}g(n)^{1/\beta} \qquad (n \geqq n_0).$[2)]

This follows from Theorem 1 since (1.4) implies $a_n > -\varepsilon^{-1}n^{-\alpha/\beta}g(n)^{1/\beta}$ $(n \geqq n_\varepsilon)$, which also does (1.2) with (1.3). In the case $g(x) = 1$ this corollary coincides with the Wang's theorem stated above, and in the case $\beta = \alpha$ it does with a result from the theorem due to K. Kanno [4], the function $g(x)$ being slightly different from the original.

Letting $g(x) = (\log x)^{r\alpha}$ we have the following corollary:

COROLLARY 1.2. *If* $\beta \geqq \alpha > 0$, $r \geqq 0$ *and*

$$\Phi_\alpha(t) = o\left(t^\beta / \log\left(\frac{1}{t}\right)^{r\alpha}\right) \qquad (t \to 0),$$

and if

$$a_n > -An^{-\alpha/\beta}(\log n)^{r\alpha/\beta},\ A > 0, \qquad (n \geqq 2),$$

then $s_n \to s$ *as* $n \to \infty$.

In the case $\beta = \alpha$ this corollary is a theorem due to K. Kanno [3], and if in addition $r = 1/\alpha$ it is due to F. T. Wang [2].

2. Preliminary lemmas.

LEMMA 1. *Let* k *be any positive integer, and let* $m = m(n) < k^{-1}n$ *tend to infinity with* n *and be as same order as or lower order than* n. *If*

(2.1) $$\frac{2}{\pi}\int_0^\pi \varphi(t)\chi_n(t)dt = s + o(1) \qquad (n \to \infty),$$

where

(2.2) $$\chi_n(t) = \chi_n^{\pm}(m, k, t) = \frac{(2\sin(mt/2))^k}{m^k(2\sin(t/2))^{k+1}}\sin\left[n + \frac{1}{2} \pm \frac{1}{2}k(m+1)\right]t,$$

and if for any assigned positive ε

(2.3) $\qquad s_{n+\nu} - s_n > -\varepsilon$ *for* $\nu = 1, 2, \dots, m,$ $\qquad (n \geqq n_\varepsilon),$

then $s_n \to s$ *as* $n \to \infty$, *i.e. the Fourier series of* $\varphi(t)$ *is convergent to sum*

2) n_0 may be expressed by x_0 in Definition 1, and is an absolute constant depending on the function $g(x)$ only.

s at $t = 0$.

PROOF. We use the following identities which are analogous to those in L. S. Bosanquet [6]:

$$(2.4)\qquad s_n = I_n - \frac{1}{m^k}\sum_{\nu_1=1}^{m}\sum_{\nu_2=1}^{m}\cdots\sum_{\nu_k=1}^{m}(s_{n+\nu_1+\nu_2+\ldots+\nu_k} - s_n),$$

where
$$I_n = \frac{1}{m^k}\sum_{\nu_1=1}^{m}\sum_{\nu_2=1}^{m}\cdots\sum_{\nu_k=1}^{m} s_{n+\nu_1+\nu_2+\ldots+\nu_k},$$

and

$$(2.5)\qquad s_n = I'_n + \frac{1}{m^k}\sum_{\nu_1=1}^{m}\sum_{\nu_2=1}^{m}\cdots\sum_{\nu_k=1}^{m}(s_n - s_{n-\nu_1-\nu_2-\ldots-\nu_k}),$$

where
$$I'_n = \frac{1}{m^k}\sum_{\nu_1=1}^{m}\sum_{\nu_2=1}^{m}\cdots\sum_{\nu_k=1}^{m} s_{n-\nu_1-\nu_2-\ldots-\nu_k}.$$

Now if

$$(2.6)\qquad \lim_{n\to\infty} I_n = \lim_{n\to\infty} I'_n = s,$$

and if (2.3) holds, then from (2.4) and (2.5) we have

$$\limsup_{n\to\infty} s_n \leqq s \qquad \text{and} \qquad \liminf_{a\to\infty} s_n \geqq s$$

respectively, and then $\lim s_n = s$.

On the other hand, denoting by $D_n(t)$ the n-th Dirichlet kernel, clearly

$$s_{n+\nu_1+\nu_2+\ldots+\nu_k} = \frac{2}{\pi}\int_0^\pi \varphi(t) D_{n+\nu_1+\nu_2+\ldots+\nu_k}(t)dt$$
$$= \frac{2}{\pi}\int_0^\pi \varphi(t)\left(2\sin\frac{1}{2}t\right)^{-1}\sin\left(n+\frac{1}{2}+\nu_1+\nu_2+\ldots+\nu_k\right)t\,dt.$$

Add both sides from $\nu_1 = 1$ to m, from $\nu_2 = 1$ to $m, \ldots$, from $\nu_k = 1$ to m successively, and divide them by m^k, then we have

$$I_n = \frac{2}{\pi}\int_0^\pi \varphi(t)\frac{(2\sin(mt\ 2))^k}{m^k(2\sin(t/2))^{k+1}}\sin\left[n+\frac{1}{2}+\frac{1}{2}k(m+1)\right]tdt.$$

Similarly

$$I'_n = \frac{2}{\pi}\int_0^\pi \varphi(t)\frac{(2\sin(mt/2))^k}{m^k(2\sin(t/2))^{k+1}}\sin\left[n+\frac{1}{2}-\frac{1}{2}k(m+1)\right]tdt.$$

Hence the condition (2.6) coincides with (2.1), and we get the lemma since (2.6) and (2.3) imply $s_n \to s$.

LEMMA 2. *The kernel* $\chi_n(t) = \chi_n^{\pm}(m, k, t)$ *defined by* (2.2), m *being as same order as or lower order than* n, *possesses the following properties*:

$$(2.7)\qquad \frac{2}{\pi}\int_0^\pi \chi_n(t)dt = 1,$$

and for $\mu = 0, 1, \dots,$

$$(2.8)\qquad \left(\frac{d}{dt}\right)^{\mu} \chi_n(t) = \begin{cases} O(n^{\mu+1}) & (0 \leqq t \leqq \pi) \\ O(n^{\mu}/t) & (nt \geqq 1) \\ O(n^{\mu}/m^k t^{k+1}) & (mt \geqq 1), \end{cases}$$

as $n \to \infty$.

PROOF. By the definition of $\chi_n(t) = \chi_n^+(m, k, t)$ we have

$$(2.9)\qquad \chi_n(t) = \frac{1}{m^k} \sum_{\nu_1=1}^{m} \sum_{\nu_2=1}^{m} \cdots \sum_{\nu_k=1}^{m} D_{n+\nu_1+\nu_2+\dots+\nu_k}(t),$$

from which (2.7) follows immediately since $(2/\pi)\int_0^{\pi} D_n(t)dt = 1$. The first relation in (2.8) follows from (2.9) and $(d/dt)^{\mu} D_n(t) = O(n^{\mu+1})$ for $0 \leqq t \leqq \pi$. The second relation does from (2.9) and $(d/dt)^{\mu} D_n(t) = O(n^{\mu}/t)$ for $nt \geqq 1$. The third relation in (2.8) follows from (2.2) by developing the product

$$(2 \sin (mt/2))^k \sin (n + 1/2 + k(m + 1)/2)t$$

into a linear combination of sines or cosines, and differentiating term by term μ-times.

It is analogous to the kernel $\chi_n(t) = \chi_n^-(m, k, t)$, and we get the lemma.

3. Proof of Theorem 1. By Lemma 1, it is sufficient to show that

$$(3.1)\qquad \frac{2}{\pi}\int_0^{\pi} \varphi(t)\chi_n(t)dt = s + o(1) \qquad (n \to \infty),$$

where $\chi_n(t)$ is defined by (2.2), i.e.

$$\chi_n(t) = \frac{(2 \sin (mt/2))^k}{m^k (2 \sin (t/2))^{k+1}} \sin\left[n + \frac{1}{2} \pm \frac{1}{2} k(m + 1)\right] t,$$

k being determined in a moment. Here we may suppose that $s = 0$ with no loss of generality since $(2/\pi)\int_0^{\pi} \chi_n(t)dt = 1$ by (2.7). We take two integers l and k such as

$$(3.2)\qquad l - 1 \leqq \alpha < l \text{ and } k > l\beta/\alpha.$$

Then necessarily $l > 1$ since $\alpha > 0$. Observe that for $\mu = 0, 1, \dots,$ $\Phi_{\mu+1}(0) = 0$ and

$$\chi_n^{(\mu)}(t) = \left(\frac{d}{dt}\right)^{\mu} \chi_n(t) = \begin{cases} O(n^{\mu+1}) & (0 \leqq t \leqq \pi) \\ O(n^{\mu}/m^k) & (0 < \delta \leqq t \leqq \pi), \end{cases}$$

by Lemma 2, and then that

$$\begin{aligned} [\Phi_{\mu+1}(t)\chi_n^{(\mu)}(t)]_{t=0}^{\pi} &= \Phi_{\mu+1}(\pi)\chi_n^{(\mu)}(\pi) \\ &= O(n^{\mu}/m^k), \qquad m = [\eta n^{\alpha/\beta}/g(n)^{1/\beta}], \\ &= O(\eta^{-k} g(n)^{k/\beta}/n^{k\alpha/\beta-\mu}), \end{aligned}$$

which is $o(1)$ for $\mu = 0, 1, \dots, l-1$ since $g(n) = o(n^\varepsilon)$ for every $\varepsilon > 0$ and by (3.2) $k\alpha/\beta - \mu > l - (l-1) > 0$. Then, applying integration by parts l-times to the left hand side integral in (3.1) we have

$$\int_0^\pi \varphi(t)\chi_n(t)dt = (-1)^l \int_0^\pi \Phi_l(t)\chi_n^{(l)}(t)dt + o(1). \tag{3.3}$$

Further, since (1.1) and $l > \alpha$ imply

$$\int_0^t |\Phi_l(u)|\, du = o(t^{\beta+1+l-\alpha}) = o(t^{l+1}),$$

we have

$$\int_0^{n^{-1}} \Phi_l(t)\chi_n^{(l)}(t)dt = O\left(n^{l+1}\int_0^{n^{-1}} |\Phi_l(t)|\, dt\right) = o(1). \tag{3.4}$$

By (3.3) and (3.4), the proposition (3.1) is equivalent to

$$I \equiv \int_{n^{-1}}^\pi \Phi_l(t)\chi_n^{(l)}(t)dt = o(1) \qquad (n \to \infty). \tag{3.5}$$

Using then the identity

$$\Phi_l(t) = \frac{1}{\Gamma(l-\alpha)}\int_0^t (t-u)^{l-1-\alpha}\Phi_\alpha(u)du,$$

and neglecting the factor $1/\Gamma(l-\alpha)$,

$$\begin{aligned} I &= \int_{n^{-1}}^\pi \chi_n^{(l)}(t)dt \int_0^t (t-u)^{l-1-\alpha}\Phi_\alpha(u)du \\ &= \int_{n^{-1}}^\pi dt \int_0^{t-n^{-1}} du + \int_{n^{-1}}^\pi dt \int_{t-n^{-1}}^t du = I_1 + I_2. \end{aligned} \tag{3.6}$$

Exchanging the order of integration

$$\begin{aligned} I_1 &= \int_{n^{-1}}^\pi \chi_n^{(l)}(t)dt \int_0^{t-n^{-1}} (t-u)^{l-1-\alpha}\Phi_\alpha(u)du \\ &= \int_0^{m^{-1}-n^{-1}} du \int_{u+n^{-1}}^{m^{-1}} dt + \int_0^{m^{-1}-n^{-1}} du \int_{m^{-1}}^\pi dt \\ &\quad + \int_{m^{-1}-n^{-1}}^{\pi-n^{-1}} du \int_{u+n^{-1}}^\pi dt = J_1 + J_2 + J_3. \end{aligned} \tag{3.7}$$

Now, for the sake of simplicity we write

$$U(t, u) = (t-u)^{l-1-\alpha}\chi_n^{(l)}(t) \qquad (l - 1 - \alpha \leqq 0).$$

Then, when $n^{-1} \leqq u < u_1 < u_2 \leqq \pi$, by the second mean-value theorem

$$\left|\int_{u_1}^{u_2} U(t, u)dt\right| \leqq (u_1 - u)^{l-1-\alpha} \sup_{u_1<t<u_2} |\chi_n^{(l-1)}(t)|,$$

and so by (2.8) with $\mu = l - 1$

$$(3.8)\qquad \int_{u_1}^{u_2} U(t, u)dt = O((u_1 - u)^{l-1-\alpha}n^{l-1}/u_1) \qquad (nu_1 \geqq 1),$$

$$(3.9)\qquad \int_{u_1}^{u_2} U(t, u)dt = O((u_1 - u)^{l-1-\alpha}n^{l-1}/m^k u_1^{k+1}) \qquad (mu_1 \geqq 1).$$

Using (3.8) with $u_1 = u + n^{-1}$, i. e. $\int_{u+n^{-1}}^{m^{-1}} U(t, u)dt = O(n^\alpha/u)$,

$$J_1 = \int_0^{m^{-1}-n^{-1}} \Phi_\alpha(u)du \int_{u+n^{-1}}^{m^{-1}} U(t, u)dt = O\left(n^\alpha \int_0^{m^{-1}} |\Phi_\alpha(u)| u^{-1}du\right).$$

Hence, by the assumption

$$(1.1)\qquad \int_0^t |\Phi_\alpha(u)| du = o\left(t^{\beta+1}/g\left(\frac{1}{t}\right)\right),$$

and integrating by parts, and using the property of $g(x)$,

$$\begin{aligned} J_1 &= o(n^\alpha/m^\beta g(m)) + o\left(n^\alpha \int_0^{m^{-1}} u^{\beta-1} g\left(\frac{1}{u}\right)^{-1} du\right) \\ &= o(n^\alpha/m^\beta g(m)) = o(n^\alpha/m^\beta g(n)),\ m = [\eta n^{\alpha/\beta} g(n)^{-1/\beta}], \\ &= o(\eta^{-\beta}) = o(1). \end{aligned}$$

Next, by (1.1) and (3.9) with $u_1 = m^{-1} \geqq u + n^{-1}$, i. e. $\int_{m^{-1}}^{\pi} U(t, u)dt$ $= O((m^{-1} - u)^{l-1-\alpha}n^{l-1}m) = O(n^\alpha m)$,

$$\begin{aligned} J_2 &= \int_0^{m^{-1}-n^{-1}} \Phi_\alpha(u)du \int_{m^{-1}}^{\pi} U(t, u)dt = O\left(n^\alpha m \int_0^{m^{-1}} |\Phi_\alpha(u)| du\right) \\ &= o(n^\alpha/m^\beta g(m)) = o(1). \end{aligned}$$

Further, by (3.9) with $u_1 = u + n^{-1}$, i. e. $\int_{u+n^{-1}}^{\pi} U(t, u)dt = O(n^\alpha/m^k u^{k+1})$,

$$J_3 = \int_{m^{-1}-n^{-1}}^{\pi-n^{-1}} \Phi_\alpha(u)du \int_{u+n^{-1}}^{\pi} U(t, u)dt = O\left(n^\alpha m^{-k} \int_{m^{-1}}^{\pi} |\Phi_\alpha(u)| u^{-(k+1)} du\right).$$

Observing that $\beta < k$ by (3.2), using (1.1) and integrating by parts,

$$\begin{aligned} J_3 &= o(n^\alpha m^{-k}(m^{-1})^{\beta-k} g(m)^{-1}) \\ &\quad + o\left(n^\alpha m^{-k}\left(\int_{m^{-1}}^{m^{-\delta}} + \int_{m^{-\delta}}^{\pi}\right) u^{\beta-k-1} g\left(\frac{1}{u}\right)^{-1} du\right),\ 0 < \delta < 1, \\ &= o(n^\alpha/m^\beta g(m)) = o(1). \end{aligned}$$

Hence, J's are all $o(1)$, and then $I_1 = o(1)$ by (3.7).

Concerning I_2, exchanging the order of integration,

$$I_2 = \int_{n^{-1}}^{\pi} \chi_n^{(l)}(t)dt \int_{t-n^{-1}}^{t} (t - u)^{l-1-\alpha} \Phi_\alpha(u)du$$

$$= \int_0^{n^{-1}} du \int_{n^{-1}}^{u+n^{-1}} dt + \int_{n^{-1}}^{m^{-1}} du \int_u^{u+n^{-1}} dt + \int_{m^{-1}}^{\pi-n^{-1}} du \int_u^{u+n^{-1}} dt$$

$$+ \int_{\pi-n^{-1}}^{\pi} du \int_u^{\pi} dt = K_1 + K_2 + K_3 + K_4,$$

say. Then observing that

$$\chi_n^{(l)}(t) = \begin{cases} O(n^l/t) & (nt \geqq 1) \\ O(n^l/m^k t^{k+1}) & (mt \geqq 1), \end{cases}$$

by Lemma 2, we see that K's are all $o(1)$, and then $I_2 = o(1)$ by the same argument as above.

Thus $I_1 = o(1)$, $I_2 = o(1)$, and (3. 6) yields (3. 5) which completes the proof.

4. Jump Functions. Let $\psi(t)$ be integrable in $(0, \pi)$, odd, periodic of period 2π, and let

$$\psi(t) \sim \sum_{n=1}^{\infty} b_n \sin nt.$$

We write

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}[\psi(u) - l]du \qquad (\alpha < 0),$$

and

$$t_n = \sum_{\nu=1}^{n} \nu b_\nu.$$

Then, corresponding to Theorem 1 we have the following theorem:

THEOREM 2. *Let* $\beta \geqq \alpha > 0$ *and let* $g(x)$ *be unity or defined by Definition* 1 *in* § 1. 1° *If*

$$\int_0^t |\Psi_\alpha(u)|du = o\left(t^{\beta+1}/g\left(\frac{1}{t}\right)\right) \qquad (t \to 0),$$

and 2° *if for any assigned positive* ε

$$t_{n+\nu} - t_n > -\varepsilon n \text{ for } \nu = 1, 2, \dots, m, \qquad (n \geqq n_\varepsilon),$$

where $m = [\varepsilon^2 n^{\alpha/\beta}/g(n)^{1/\beta}]$, *or simply if*

$$b_n > -An^{-\alpha/\beta}/g(n)^{1/\beta}, \qquad A > 0, \qquad (n \geqq n_0),$$

then the sequence $\{nb_n\}$ *is summable* (C, 1) to $2l/\pi$.

REMARK. Observing that if $l = 0$ and the series Σb_n is summable in Abel sense then $t_n = o(n)$ implies the convergence of Σb_n, the above theorem may be easily transferred to a convergence theorem for the allied Fourier series of $\psi(t)$ at $t = 0$.

PROOF. Using the identity

$$\frac{t_n}{n+1} = I_n - \frac{1}{(n+1)m^k} \sum_{\nu_1=1}^{m} \sum_{\nu_2=1}^{m} \cdots \sum_{\nu_k=1}^{m} (t_{n+\nu_1+\nu_2+\ldots+\nu_k} - t_n),$$

where

$$I_n = \frac{1}{(n+1)m^k} \sum_{\nu_1=1}^{m} \sum_{\nu_2=1}^{m} \cdots \sum_{\nu_k=1}^{m} t_{n+\nu_1+\nu_2+\ldots+\nu_k}, \tag{4.1}$$

and its analogue, it is sufficient to show that

$$I_n = \frac{2l}{\pi} + o(1) \qquad (n \to \infty), \tag{4.2}$$

by Lemma 1. Here observe that

$$t_n = \sum_{\nu=1}^{n} \nu b_\nu = -\frac{2}{\pi} \int_0^\pi \psi(t) \frac{d}{dt} D_n(t) dt.$$

Substituting this replaced n by $n + \nu_1 + \nu_2 + \ldots + \nu_k$ into (4.1) we have

$$\begin{aligned} I_n &= -\frac{2}{\pi} \int_0^\pi \psi(t) \frac{1}{(n+1)m^k} \sum_{\nu_1=1}^{m} \sum_{\nu_2=1}^{m} \cdots \sum_{\nu_k=1}^{m} \frac{d}{dt} D_{n+\nu_1+\nu_2+\ldots+\nu_k}(t) dt \\ &= -\frac{2}{\pi} \int_0^\pi \psi(t) \frac{1}{n+1} \frac{d}{dt} \chi_n(t) dt, \end{aligned} \tag{4.3}$$

where $\chi_n(t) = \chi_n^+(m, k, t)$ coincides with that in (2.2), i.e.

$$\chi_n(t) = \frac{(2 \sin (mt/2))^k}{m^k (2 \sin (t/2))^{k+1}} \sin \left[n + \frac{1}{2} + \frac{1}{2} k(m+1) \right] t$$

And $(n+1)^{-1}(d/dt)\chi_n(t)$ has all the properties of $\chi_n(t)$ in (2.8), i.e.

$$\left(\frac{d}{dt}\right)^\mu \left(\frac{1}{n+1} \frac{d}{dt} \chi_n(t)\right) = \begin{cases} O(n^{\mu+1}) & (0 \leqq t \leqq \pi) \\ O(n^\mu / t) & (nt \geqq 1) \\ O(n^\mu / m^k t^{k+1}) & (mt \geqq 1), \end{cases} \tag{4.4}$$

for $\mu = 0, 1, \ldots$. Now, I_n in (4.3) is

$$\begin{aligned} I_n &= -\frac{2}{\pi} \int_0^\pi [\psi(u) - l] \frac{1}{n+1} \frac{d}{dt} \chi_n(t) dt \\ &\quad - \frac{2l}{\pi} \int_0^\pi \frac{1}{n+1} \frac{d}{dt} \chi_n(t) dt = K_1 + K_2, \end{aligned} \tag{4.5}$$

say. Then using (4.4) we see that $K_1 = o(1)$ under the assumptions in the theorem quite analogously as the proof of Theorem 1. Concerning K_2

$$\begin{aligned} K_2 &= -\frac{2l}{\pi} \frac{1}{n+1} [\chi_n(t)]_{t=0}^{\pi} = o(1) + \frac{2l}{\pi} \cdot \frac{1}{n+1} \chi_n(0) \\ &= o(1) + \frac{2l}{\pi} \cdot \frac{1}{n+1} \left[n + \frac{1}{2} + \frac{1}{2} k(m+1) \right] \\ &= \frac{2l}{\pi} + o(1). \end{aligned}$$

Hence, (4. 2) follows from (4. 5) and the theorem is proved.

REFERENCES

[1] F. T. WANG, On Riesz summability of Fourier serier (II), Journ. London Math. Soc. 17(1942), 98-107.
[2] F. T. WANG, On Riesz summability of Fourier series (III), Proc. London Math. Soc. (2), 51(1947), 215-231.
[3] K. KANNO, On the Riesz summability of Fourier series, Tôhoku Math. Journ. (2), 8 (1956), 223-234.
[4] K. KANNO, On the Riesz summability of Fourier series (II), Bull. of the Yamagata University, 4(1958), 323-331.
[5] K. YANO, On convergence criteria for Fourier series I, Proc. Japan Acad. 34(1958), 331-336.
[6] L. S. BOSANQUET, Notes on convexity theorems, Journ. London Math. Soc. 18(1943), 239-248.

MATHEMATICAL INSTITUTE, NARA WOMEN'S UNIVERSITY, NARA.