SOME ASYMPTOTIC PROPERTIES OF POISSON PROCESS

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1. Poisson Process $X(t, \omega)$ $[\omega \in \Omega,\ 0 \leqq t < \infty]$ is a temporally and spatially homogeneous Markoff Process [Stationary increments (in the strict sense)] satisfying $X(0, \omega) = 0$ and $X(t, \omega) =$ integer greater than or equal to zero for every $\omega \in \Omega$ (ω denotes the probability parameter)

$$Pr[X(t, \omega) - X(t', \omega) = i] = \frac{[\lambda(t - t')]^i}{i!} e^{-\lambda(t-t')} \tag{1}$$

for $t > t'$, where i is a non-negative integer and λ is a positive constant.

2. **Definition of $L_m(\omega)$.**

$$L_m(\omega) = t_{m+1}(\omega) - t_m(\omega)$$

where

$$t_m(\omega) = \text{Min}\,[T,\ X(T, \omega) = m],$$

$t_m(\omega)$ exists almost certainly by the right continuity property of the Poisson Process. Further $t_m(\omega)$ is measurable. Thus $L_m(\omega)$ is a non-negative random variable.

3. **A known Theorem.** $L_0, L_1, \ldots, L_m, \ldots$ are mutually independent random variables, with a common distribution function $F(x)$, where

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geqq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Further
$$E(L_m) = 1/\lambda$$
$$V(L_m) = 1/\lambda^2, \qquad m = 0, 1, \ldots.$$

This theorem was suggested by P. Lévy [1] and a rigorous proof was given by T. Nishida [2].

4. **Summary.** From the sequence $L_0, L_1 \ldots$ we form a new sequence $y_1, y_2, \ldots$ where y_1 is the mean of the first n elements, y_2 is the mean of the next n elements and so on in the L-sequence.

We define $u_m = \text{Max}\,(y_1, y_2, \ldots y_m)$ and $l_m = \text{Min}\,(y_1, y_2, \ldots y_m)$ we have investigated the asymptotic behaviour of u_m and l_m. Takeyuki Hida [3] has defined

$$M_n = \text{Max}\,(L_0, L_1, \ldots L_{n-1})$$

and

$$Z_n = \frac{L_0 + L_1 + \ldots + L_{n-1}}{M_n}.$$

Using some of his results, we have obtained the asymptotic behaviour of M_n and Z_n. We have also investigated the asymptotic properties of

$$\frac{t_1 + t_2 + \ldots + t_m}{m^2},$$

which we have shown converges in probability to $\frac{1}{2\lambda}$,

5. Distribution of the arithmetic mean $y = \frac{L_0 + L_1 + \ldots + L_{n-1}}{n}$.

It follows immediately from the theorem in [3] that the characteristic function of L is

$$\phi_L(t) = \frac{\lambda}{\lambda - it}$$

Hence the characteristic function of y is

$$\phi_y(t) = \left(1 - \frac{it}{n\lambda}\right)^{-n}.$$

Hence the frequency function $p(x)$ of y is

$$p(x) = \frac{(n\lambda)^n}{\Gamma(n)} x^{n-1} e^{-n\lambda x} \quad \text{if } x > 0$$

$$= 0 \qquad \text{if } x \leqq 0.$$

6. Definition and distribution of u_m. Let us consider the sequence of independent random variables $L_0, L_1, L_2, \ldots$ we now form a new sequence as follows

$$y_1 = \frac{L_0 + L_1 + \ldots L_{n-1}}{n},$$

$$y_2 = \frac{L_n + \ldots\ldots + L_{2n-1}}{n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

So $y_1, y_2, \ldots\ldots$ form a sequence of independent and indentically distributed random variables.

Let $$u_m = \operatorname{Max}(y_1, y_2 \ldots\ldots, y_m),$$
we now obtain the distribution function of u_m

$$Pr[u_m < x] = Pr[y_1 < x,\ y_2 < x, \ldots, y_m < x]$$
$$= Pr[y_1 < x] \cdot Pr[y_2 < x] \ldots Pr[y_m < x]$$
$$= \left[\frac{(n\lambda)^n}{\Gamma(n)} \int_0^x y^{n-1} e^{-n\lambda y} dy\right]^m$$
$$= \left[1 - \frac{1}{2^n \Gamma(n)} \int_{2n\lambda x}^{\infty} v^{n-1} e^{-v/2} dv\right]^m \tag{6.1}$$

where $x \geqq 0$.

7. We now prove the following result which will be used in the next section.

$$\frac{1}{\Gamma(n)}\int_{\theta}^{\infty} t^{n-1} e^{-t}\, dt \geqq e^{-\theta} \quad (\theta > 0). \tag{7.1}$$

To show this it is enough, we prove

$$\frac{1}{\Gamma(n)}\int_{\theta}^{\infty} t^{n-1}\, e^{(\theta-t)}\, dt \geqq 1.$$

Put $t-\theta = w$. Then

$$\frac{1}{\Gamma(n)}\int_{\theta}^{\infty} t^{n-1}\, e^{(\theta-t)}\, dt = \frac{1}{\Gamma(n)}\int_{0}^{\infty} (w+\theta)^{n-1}\, e^{-w}\, dw$$

$$\geqq \frac{1}{\Gamma(n)}\int_{0}^{\infty} w^{n-1}\, e^{-w}\, dw \qquad (\theta \text{ being positive})$$

$$= 1.$$

Hence the result (7.1).

8. THEOREM 1. *If* $0 < \alpha < 1$, *Then*

$$Pr\left[\operatorname*{Lim\,Inf}_{m\to\infty} \frac{n\lambda u_m}{\alpha \log m} \geqq 1\right] = 1. \tag{8.1}$$

PROOF.

$$Pr(u_m < x) < (1-e^{-n\lambda x})^m, \text{if } x \geqq 0 \text{ from (7.1).}$$

So
$$Pr\left(u_m < \frac{\alpha \log m}{n\lambda}\right) < \left(1-\frac{1}{m^\alpha}\right)^m.$$

Therefore

$$\sum_{m=1}^{\infty} Pr\left[u_m < \frac{\alpha\log m}{n\lambda}\right] < \sum_{m=1}^{\infty}\left(1-\frac{1}{m^\alpha}\right)^m.$$

The series on the right side is convergent if $0<\alpha<1$. So if $0<\alpha<1$

$$\sum_{m=1}^{\infty} Pr\left[u_m < \frac{\alpha\log m}{n\lambda}\right] < \infty.$$

Let
$$S_m = \left(\omega,\ u_m < \frac{\alpha\log m}{n\lambda}\right).$$

Let Λ be the set of points in infinitely many S_m's. By Borel-Cantelli Lemma

$$Pr(\Lambda) = 0.$$

Therefore $Pr(\Lambda^c) = 1$, where Λ^c is the complement of Λ

i. e.
$$Pr\ (\operatorname*{Lim\,Inf}_{m\to\infty} S_m^c) = 1$$

i. e.
$$Pr\left[\operatorname*{Lim\,Inf}_{m\to\infty} \frac{n\lambda\ u_m}{\alpha\log m} \geqq 1\right] = 1.$$

9. Definition and distribution of l_m. We define

$$l_m = \text{Min}(y_1, y_2, \ldots . y_m),$$

$$\begin{aligned} Pr(l_m > x) &= Pr(y_1 > x, \ldots, y_2 > x, y_m > x) \\ &= Pr(y_1 > x)\, Pr(y_2 > x) \ldots Pr(y_m > x) \\ &= \left[\frac{(n\lambda)^n}{\Gamma(n)} \int_x^\infty u^{n-1} e^{-n\lambda u}\, du \right]^m \\ &= \left[\frac{1}{\Gamma(n)} \int_{n\lambda x}^\infty v^{n-1} e^{-v}\, dv \right]^m \\ &= \left[1 - \frac{1}{\Gamma(n)} \int_0^{n\lambda x} v^{n-1} e^{-v}\, dv \right]^m \end{aligned} \tag{9.1}$$

where $x \geqq 0$.

10. THEOREM 2. *If $\beta > 1$, then*

$$Pr\left[\operatorname*{Lim\,Sup}_{m \to \infty} \frac{l_m}{\beta \left[\frac{n!}{(n\lambda)^n} \frac{\log m}{m} \right]^{1/n}} \leqq 1 \right] = 1. \tag{10.1}$$

PROOF. By (9.1) we get

$$Pr[l_m > x] = \left[1 - \frac{1}{\Gamma(n)} \int_0^{n\lambda x} v^{n-1} e^{-v}\, dv \right]^m.$$

If $0 \leqq \theta \leqq 1$

$$\frac{1}{\Gamma(n)} \int_0^\theta e^{-v} v^{n-1}\, dv = \frac{\theta^n}{n!} [1 + O(\theta)].$$

Assuming $n\lambda x \leqq 1$, we get

$$Pr[l_m > x] = \left[1 - \frac{(n\lambda x)^n}{n!} \{1 + O(n\lambda x)\} \right]^m.$$

Write

$$n\lambda x = n\lambda\phi(m), \quad \text{where} \quad \phi(m) \to 0 \text{ as } m \to \infty.$$

Then

$$Pr[l_m > \phi(m)] = \exp\left\{ -m \frac{[n\lambda\phi(m)]^n}{n!} [1 + o(1)] \right\}.$$

Now take

$$\frac{m[n\lambda\phi(m)]^n}{n!} = \alpha \log m.$$

Therefore

$$Pr[l_m > \phi(m)] = \exp\{ -\alpha \log m[1 + o(1)]\}$$

$$= \frac{1}{m^{\alpha[1+o(1)]}}.$$

So if $\alpha > 1$

$$\sum_{m=1}^{\infty} Pr[l_m > \phi(m)] \text{ converges.}$$

So

$$Pr\left[\operatorname*{Lim\,Sup}_{m\to\infty} \frac{l_m}{\phi(m)} \leqq 1\right] = 1.$$

Here

$$\phi(m) = \frac{1}{n\lambda}\left[\frac{\alpha \log m}{m} \cdot n!\right]^{1/n}$$

$$= \left[\frac{n!}{(n\lambda)^n} \cdot \frac{\alpha \log m}{m}\right]^{1/n}.$$

Therefore

$$Pr\left[\operatorname*{Lim\,Sup}_{m\to\infty} \frac{l_m}{\left[\frac{n!}{(n\lambda)^n} \frac{\alpha \log m}{m}\right]^{1/n}} \leqq 1\right] = 1.$$

Now $\beta = \alpha^{1/n}$, since $\alpha > 1$, $\beta > 1$. Hence we finally get:

If $\beta > 1$

$$Pr\left[\operatorname*{Lim\,Sup}_{m\to\infty} \frac{l_m}{\left[\frac{n!}{(n\lambda)^n} \frac{\alpha \log m}{m}\right]^{1/n}} \leqq 1\right] = 1.$$

By putting $n = 1$, in Theorems (1) and (2) we get the results obtained by Takeyuki Hida.

11. Takeyuki Hida [3] has proved the following results.

He defines

$$M_n = \operatorname{Max}[L_0(\omega),\ L_1(\omega), \ldots,\ L_{n-1}(\omega)]$$

and

$$Z_n = \frac{L_0(\omega) + L_1(\omega) + \ldots\ldots + L_{n-1}(\omega)}{M_n(\omega)}.$$

He has proved

$$E(M_n) = O(\log n),$$

$$E(Z_n) = O\left(\frac{n}{\log n}\right).$$

We can derive the following theorems from the above results. The fact that M_n and Z_n are non-negative almost everywhere may be noted in the proofs of the following theorems.

THEOREM 1. *For any $p > 0$*

$$Pr\left[\operatorname*{Lim}_{n\to\infty} \frac{M_n}{n(\log n)^{2+p}} = 0\right] = 1.$$

PROOF. We know that

$$E(M_n) = O(\log n).$$

So

$$E\left[\frac{M_n}{n(\log n)^{2+p}}\right] = O\left(\frac{1}{n(\log n)^{1+p}}\right).$$

Hence
$$E\left[\frac{M_n}{n(\log n)^{2+p}}\right] < \frac{K}{n(\log n)^{1+p}}$$

where K is a constant not depending upon n.
By Tchebycheff's inequality

$$Pr\left[\frac{M_n}{n(\log n)^{2+p}} > \varepsilon\right] < \frac{K}{\varepsilon\, n(\log n)^{1+p}}$$

for any $\varepsilon > 0$ and for all large n.
Hence

$$\sum_{n=2}^{\infty} Pr\left[\frac{M_n}{n(\log n)^{2+p}} > \varepsilon\right] < \frac{K}{\varepsilon}\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{1+p}}.$$

For any $p > 0$, the series on the right side is convergent.
Therefore

$$\sum_{n=2}^{\infty} Pr\left[\frac{M_n}{n(\log n)^{2+p}} > \varepsilon\right] < \infty.$$

By applying Borel-Cantelli Lemma

$$Pr\left[\lim_{n\to\infty}\frac{M_n}{n(\log n)^{2+p}} = 0\right] = 1.$$

THEOREM 2. *For any $p > 0$,*

$$Pr\left[\lim_{n\to\infty}\frac{Z_n}{n^2(\log n)^p} = 0\right] = 1.$$

PROOF. We know that

$$E(Z_n) = O\left(\frac{n}{\log n}\right).$$

So
$$E\left[\frac{Z_n}{n^2(\log n)^p}\right] < \frac{K}{n(\log n)^{1+p}},$$

where K does not depend upon n.
By Tchebycheff's inequality

$$Pr\left[\frac{Z_n}{n^2(\log n)^p} > \varepsilon\right] < \frac{K}{\varepsilon n(\log n)^{1+p}}$$

for any $\varepsilon > 0$ and for all large n.
Therefore for any $p > 0$

$$\sum_{n=2}^{\infty} Pr\left[\frac{Z_n}{n^2(\log n)^p} > \varepsilon\right] < \infty.$$

Hence by Borel-Cantelli Lemma

$$Pr\left[\lim_{n\to\infty} \frac{Z_n}{n^2(\log n)^p} = 0\right] = 1.$$

12. Asymptotic Behaviour of $\frac{t_1(\omega) + t_2(\omega) + \ldots + t_m(\omega)}{m^2}$.

We define

$$t_m(\omega) = \text{Min}\,(T, X(T, \omega) = m).$$

Hence from the definition

$$t_0(\omega) = 0.$$

We define

$$L_m(\omega) = t_{m+1}(\omega) - t_m(\omega).$$

Hence $\quad t_m(\omega) = L_0 + L_1 + \ldots\ldots + L_{m-1}.$

Therefore

$$t_1 + t_2 + \ldots + t_m = mL_0 + (m-1)\, L_1 + \ldots + L_{m-1}.$$

Hence

$$\frac{t_1 + t_2 + \ldots\ldots + t_m}{m^2} = \frac{m\, L_0 + (m-1)\, L_1 + \ldots\ldots + L_{m-1}}{m^2}.$$

THEOREM. $\frac{t_1 + t_2 + \ldots + t_m}{m^2}$ *converge in probability to* $\frac{1}{2\lambda}$.

PROOF. Let $\phi_m(t)$ denote the characteristic function of $\frac{t_1 + t_2 + \ldots + t_m}{m^2}$.

Hence

$$\begin{aligned}\phi_m(t) &= \phi_{(t_1+\ldots+t_m)/m^2}(t) \\ &= \phi_{\frac{mL_0+(m-1)L_1+\ldots+L_{m-1}}{m^2}}(t) \\ &= \frac{1}{\left(1-\frac{imt}{m^2\lambda}\right)\left(1-\frac{i(m-1)t}{m^2\lambda}\right)\ldots\left(1-\frac{it}{m^2\lambda}\right)} \\ &= \frac{1}{\prod_{r=1}^{m}\left[1-\frac{irt}{m^2\lambda}\right]}.\end{aligned}$$

We now find the limit of $\phi_m(t)$ as $m \to \infty$. For this we first consider the limit of the denominator

$$\prod_{r=1}^{m}\left(1-\frac{irt}{m^2\lambda}\right) = \left[\prod_{r=1}^{m}\left\{\left(1-\frac{irt}{m^2\lambda}\right)\exp\left(\frac{irt}{m^2\lambda}\right)\right\}\right] \times \exp\left(-\frac{it}{\lambda m^2}\sum_{r=1}^{m} r\right). \tag{1}$$

Consider now the limit as $m \to \infty$ of

$$\prod_{r=1}^{m}\left(1-\frac{irt}{m^2\lambda}\right)\exp\left(\frac{irt}{m^2\lambda}\right).$$

Put $1+W_r(m)=\left(1-\frac{ur}{m^2}\right)\exp\left(\frac{ur}{m^2}\right)$ where $u=\frac{it}{\lambda}$.

Therefore

$$W_r(m)=\left(1-\frac{ur}{m^2}\right)\exp\left(\frac{ur}{m^2}\right)-1.$$

Let $|u|\leqq k$, we can choose an $m>m_0$ such that

$$\left|\frac{ur}{m^2}\right|\leqq\frac{|k|}{m}\leqq 1 \quad \text{in } |u|\leqq k.$$

Now

$$\begin{aligned} W_r(m)&=\left(1-\frac{ur}{m^2}\right)\exp\left(\frac{ur}{m^2}\right)-1\\ &=\left(1-\frac{ur}{m^2}\right)\left(1+\frac{ur}{m^2}+\frac{(ur)^2}{2!m^4}+\dots\right)-1\\ &=-\frac{1}{2}\left(\frac{ur}{m^2}\right)^2\left[1+\frac{4}{3!}\left(\frac{ur}{m^2}\right)^3+\frac{6}{4!}\left(\frac{ur}{m^2}\right)^4+\dots\right].\end{aligned}$$

Therefore

$$|W_r(m)|<\frac{Ar^2}{m^4}<\frac{Ar^2}{r^4}<\frac{A}{r^2}, \text{ where } A \text{ is a constant.}$$

Since $\sum_{r=1}^{\infty}\frac{1}{r^2}$ is convergent, conditions of Tannery's theorem are fulfilled.

Hence

$$\lim_{m\to\infty}\prod_{r=1}^{m}\left(1-\frac{irt}{m^2\lambda}\right)\exp\left(\frac{irt}{m^2\lambda}\right)=1.$$

Also

$$\lim_{m\to\infty}\exp\left(-\frac{it}{m^2\lambda}\sum_{r=1}^{m}r\right)=\lim_{m\to\infty}\exp\left\{-\frac{it}{m^2\lambda}\frac{m(m+1)}{2}\right\}=\exp\left(-\frac{it}{2\lambda}\right).$$

Hence the denominator of $\phi_m(t)\to\exp(-it/2\lambda)$, as $m\to\infty$.

Therefore $\lim_{m\to\infty}\phi_m(t)=\exp(it/2\lambda)$, for all t, since k is arbitrary.

Hence

$$\lim_{m\to\infty}Pr\left[\frac{t_1+\dots+t_m}{m^2}\leqq x\right]=\begin{cases}1 & \text{if } x\geqq 1/2\lambda\\ 0 & \text{otherwise.}\end{cases}$$

In other words $\frac{t_1+t_2+\dots+t_m}{m^2}$ converges in probability to $\frac{1}{2\lambda}$.

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