NOTES ON FOURIER ANALYSIS (XXXV)*)

By

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Part I. Order of the partial sum of Fourier series.

§ 1. Mr. F. T. Wang [1] proved that, if

(1.1) $$\Phi_\alpha(t) \equiv \frac{1}{\Gamma(\alpha)}\int_0^t \varphi_x(u)(t-u)^{\alpha-1}du = o(t^\alpha), \quad t \to 0,$$

then

(1.2) $$s_n(x) = o(n^{\alpha/(1+\alpha)}), \quad n \to \infty,$$

where $\varphi_x(t) = f(x-t) + f(x-t) - 2f(x)$ and $s_n(x)$ is the n-th partial sum of Fourier series of $f(t)$ at $t=x$.

Further he proved [2] that, in the case $\alpha=1$, (1.2) cannot be replaced by

(1.3) $$s_n(x) = o(n^\delta), \quad n \to \infty,$$

where $0 < \delta < 1/2$. We prove more precisely that we cannot replace (1.2) by

(1.4) $$|s_n(x)| \leqq \varepsilon_n n^{1/2}, \quad n \to \infty,$$

(ε_n) being a given null sequence, even if the condition (1.1) replaced by that the integral

(1.5) $$\int_0^\pi \frac{\varphi(u)}{u} du$$

exists in the Cauchy sense.

Mr. N. Matsuyama [3] proved that there is an integrable function $f(t)$ such that

(1.6) $$\int_0^\pi \frac{\varphi(u)}{u^{1+\alpha}} du$$

exists in the Cauchy sense and

(1.7) $$s_n(x) \neq o(n^\delta)$$

for $\delta < 1/(2+\alpha)$. We prove that (1.7) can be replaced by

(1.8) $$s_n(x) \geqq \varepsilon_n n^{1/(2+\alpha)}$$

for any assigned null sequence (ε_n), and for infinitely many n.

Further we prove that, for any α, there is an integrable function $f(t)$ satisfying (1.1) and

$$s_n(x) \geqq \varepsilon_n n^{\alpha/(1+\alpha)}$$

for any assigned null sequence (ε_n) and infinitely many n.

The method of construction of examples is that used by the author in the previous paper [4].

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§ **2.** THEOREM 1. *For any sequence* (ε_n) *tending to zero, there is an integrable function* $f(t)$ *such that*

(2.1) $$\Phi_\alpha(t) = o(t^\alpha)$$

and that there is a sequence (M_k) *such that*

(2.2) $$s_{M_k}(x) \geqq \varepsilon_{M_k} M_k^{\alpha/(1+\alpha)} \quad (k = 1, 2, \ldots).$$

PROOF. Let (Δ_k) be a sequence of disjoint intervals such as

$$\Delta_k \equiv \left(\frac{\pi}{n_k}, \ \frac{\pi}{n_k} + \frac{\pi}{m_k}\right) \quad (k = 1, 2, \ldots),$$

where (m_k) and (n_k) are increasing sequences of integers determined later, and we put

(2.3) $$f(t) \equiv c_k t \sin M_k t \quad (t \varepsilon \Delta_k)$$

for $k=1, 2, \ldots$ and $f(t) \equiv 0$ in $(0, \pi) - \vee \Delta_k$, $f(t) \equiv f(-t)$, where (c_k) is a sequence of positive numbers and (M_k) an increasing sequence of integers, both determined later. Let us suppose

(2.4) $$m_k/n_k \to 0 \quad (k \to \infty).$$

Since

$$\sum c_k \int_{\Delta_k} t|\sin M_k t|\,dt \leqq 3\sum c_k\left(\frac{\pi}{m_k}\right)^2,$$

$f(t)$ is integrable when

(2.5) $$\sum c_k/m_k^2 < \infty.$$

We have

$$s_{M_k}(0) = \frac{2}{\pi}\int_0^\pi f(t)\frac{\sin M_k t}{t}dt + o(1)$$
$$= c_k \int_{\Delta_k} \sin^2 M_k t dt + \sum_{i \neq k} c_i \int_{\Delta_i} \sin M_i t \sin M_k t\, dt + o(1),$$

where the first term is $\geqq c_k/4m_k$ for sufficiently large M_k comparing with m_k, and

$$c_i \int_{\Delta i} \sin M_i t \sin M_k t\, dt = O\left(\frac{c_i}{|M_k - M_i|}\right).$$

If we suppose that

(2.6) $$\sum_{\substack{i=1 \\ i \neq k}}^{\infty} \frac{c_i}{|M_k - M_i|} = O(1)$$

and

(2.7) $$\frac{c_k}{m_k} \geqq \eta_k M_k^{\alpha/(1+\alpha)} \quad (k = 1, 2, \ldots),$$

$\eta_k \equiv \varepsilon_{M_k}$, then we get (2.2).

Let us first confine ourselves to the case $0 < \alpha < 1$. For $t \varepsilon \Delta_k$, we have

$$\Phi_\alpha(t) = \int_0^t f(u)(t-u)^{\alpha-1}du$$
$$= c_k \int_{\pi/n_k}^t u \sin M_k u(t-u)^{\alpha-1}du + \sum_{i=k+1}^{\infty} c_i \int_{\pi/n_i}^{\pi/n_i+\pi/m_i} u \sin M_i u (t-u)^{\alpha-1}du,$$

where

$$c_k \int_{\pi/n_k}^{t} u \sin M_k u (t-u)^{\alpha-1} du = O\left(\frac{c_k t}{M_k^\alpha}\right)$$

and

$$c_i \int_{\pi/n_i}^{\pi/n_i+\pi/m_i} u \sin M_i u (t-u)^{\alpha-1} du = O\left(\frac{c_i}{m_i M_i}\left(t-\frac{\pi}{m_i}\right)^{\alpha-1}\right)$$

which is $O(c_i t^{\alpha-1}/m_i M_i)$, if $i \geqq k+1$ and

(2.8) $$\frac{\pi}{n_{i+1}} + \frac{\pi}{m_{i+1}} \leqq \frac{\pi}{2n_i} \quad (i=1, 2, \ldots).$$

Thus, if

(2.9) $$\frac{c_k}{M_k^\alpha} \leqq \delta_k t^{\alpha-1} \quad (\pi/n_k \leqq t \leqq \pi/n_{k-1})$$

and

(2.10) $$\sum_{i=k+1}^{\infty} \frac{c_i}{m_i M_i} \leqq \delta_k t \quad (\pi/n_k \leqq t \leqq \pi/n_{k-1})$$

for a null sequence (δ_k), then we have (2.1).

We shall now define (M_k), (m_k), (n_k) and (c_k) such that the conditions (2.4)—(2.10) are satisfied. Let

$$c_k = 2^{2k^2}, \ m_k = k_2^{k^2},$$

then (2.5) is satisfied. If we take any null sequence (η_k) and put

$$M_k = \left(\frac{1}{k\eta_k}\right)^{(1+\alpha)/\alpha} 2^{(1+\alpha)k^2/\alpha},$$

then we get (2.6) and (2.7). Further taking (δ_k) such as

$$(k\eta_k)^{1+\alpha} = \delta_k$$

and (n_k) such as

$$n_k = 2^{(k+1)^2},$$

we get (2.9) and (2.10). The conditions (2.4) and (2.8) are evident by the construction. Thus we get the required.

§ 3. We will next consider the case $1<\alpha<2$. We have

$$\int_{\pi/n_k}^{t} u \sin M_k u (t-u)^{\alpha-1} du = O\left(\frac{t}{M_k^\alpha} + \frac{t^{\alpha-1}}{n_k M_k} + \frac{1}{M_k^{\alpha+1}}\right).$$

Hence, in this case, (2.9) must be replaced by

(3.1) $$\frac{c_k}{M_k^\alpha} \leqq \delta_k t^{\alpha-1}, \ \frac{c_k}{n_k M_k} \leqq \delta_k t, \ \frac{c_k}{M_k^{\alpha+1}} \leqq \delta_k t^\alpha$$

for $\pi/n_k \leqq t \leqq \pi/n_{k-1}$. Condition (2.8) is not needed.

We take

$$c_k = 2^{2.2^{k^2}}, \quad m_k = k 2^{2^{k^2}},$$

then (2.5) is satisfied. If we take a null sequence (η_k) and put

$$M_k = (k\eta_k)^{\frac{1+\alpha}{\alpha}} 2^{\frac{1+\alpha}{\alpha} \cdot 2^{k^2}}$$

then we get (2.6) and (2.7). Supposing

(3.2) $$\eta_k \leqq 1/k^2,$$

we put

$$n_k \equiv \left(\frac{\delta_k}{(k\eta_k)^{1+\alpha}}\right)^{\frac{1}{\alpha-1}} \cdot 2^{2^{k^2}}$$

where $\delta_k \equiv 1/k^{3-\alpha}$. Then the other conditions are all satisfied.

Now, (3.2) is equivalent to

$$\varepsilon_{K_k} \leqq c/\log\log M_k.$$

For more slowly decreasing (ε_n), it is sufficient to take higher power numbers as m_k, c_k, etc. For example $m_k \equiv k2^{2^{k^2}}$, and so on.

We can treat the case $n<\alpha<n+1 (n=2, 3, \ldots)$ similarly. The integral case is also similarly proved.

§ 4. In the case $\alpha=1$, Theorem 1 may be generalized in the following form.

THEOREM 2. *For any sequence (ε_n) tending to zero, there is an integrable function $f(t)$ such that the integral*

$$\int_0^\pi \frac{\varphi(u)}{u} du \tag{4.1}$$

converges in the Cauchy sense and that there is a sequence (M_k) such as

$$s_{M_k}(x) \geqq \varepsilon_{M_k} M_k^{1/2}.$$

PROOF. Let us take a function defined by (2.3), where we suppose that m_k and n_k divide M_k, and M_k/m_k and M_k/n_k are even. Then the integral (4.1) exists and equals to zero. Then it is required the conditions (2.4)—(2.7). Hence, modifying the example in § 2, we get the required.

THEOREM 3. *For any sequence (ε_n) tending to zero, there is an integrable function $f(t)$ such that the integral*

$$\int_0^\pi \frac{\varphi(u)}{u^{1+\alpha}} du \tag{4.2}$$

converges in the Cauchy sense and that there is a sequence (M_k) such as

$$s_{M_k}(x) \geqq \varepsilon_{M_k} M_k^{1/(2+\alpha)}. \tag{4.3}$$

PROOF. Let us take a function defined by (2.3). Then it is sufficient to take (M_k), (m_k), (n_k) and (c_k) such that (2.4), (2.5). (2.6) are satisfied and

$$\frac{c_k}{m_k} \geqq \eta_k M_k^{1/(2+\alpha)}, \tag{4.4}$$

$$\sum_{k=1}^\infty \frac{c_k n_k^\alpha}{M_k} < \infty. \tag{4.5}$$

If we take

$$c_k \equiv 2^{2k^2}, \quad m_k \equiv k2^{k^2}, \quad n_k \equiv k^2 2^{k^2}$$

and

$$M_k \equiv \frac{1}{(k\eta_k)^{2+\alpha}} 2^{(2+\alpha)k^2},$$

then the conditions except (4.5) are all satisfisfied. If

(4.6) $$\eta_k \leqq 1/k^3,$$

then (4.5) is also satisfied. (4.6) is equivalent to

(4.7) $$\varepsilon_{M_k} \leqq c/(\log M_k)^{3/2}.$$

If we take $c_k = 2^{2\cdot 2^{k^2}}$, etc., then the condition (4.7) is replaced by

$$\varepsilon_{M_k} \leqq c/(\log\log M_k)^{3/2}.$$

Thus proceeding we get the theorem for any slowly tending to zero sequence.

Part II. On Zygmund's Method of summation

§ **1.** A. Zygmund [5] has introduced the following method of summation. Let $\sum_{n=1}^{\infty} a_n$ be a given series. If

(1) $$\lim_{\alpha\to 0}\frac{2}{\pi}\sum_{n=1}^{\infty} a_n \int_{\alpha}^{\pi} \frac{\sin nt}{2\operatorname{tg} t/2}dt = s,$$

the series being supposed to be convergent, then the series $\sum_{n=1}^{\infty} a_n$ is said to be $(K, 1)$ summable to s. And, if

(2) $$\lim_{\alpha\to 0}\frac{2}{\pi}\sum_{n=1}^{\infty} a_n \int_{\alpha}^{\pi} \frac{\sin^2 nt/2}{n\sin^2 t/2}dt = s,$$

the series being supposed to be convergent, the series $\sum_{n=1}^{\infty} a_n$ is said to be $(K, 2)$ summable. In this part we find necessary and sufficient conditions for $(K, 1)$- and $(K, 2)$-summabilities of Fourier series and get the relations between these and the Riemann summabilities.

§ **2.** Let us put

$$s_0 = 0, \quad s_n = \sum_{k=1}^{n} a_k, \quad s_n^* = s_n - \frac{a_n}{2}.$$

Then

$$\sum_{n=1}^{N} a_n \int_{\alpha}^{\pi} \frac{\sin nt}{2\operatorname{tg} t/2}dt = \sum_{n=1}^{N}(s_n - s_{n-1})\int_{\alpha}^{\pi} \frac{\sin nt}{2\operatorname{tg} t/2}dt$$
$$= -\sum_{n=1}^{N-1} s_n \int_{\alpha}^{\pi} \frac{\sin nt - \sin(n+1)t}{2\operatorname{tg} t/2}dt + s_N\int_{\alpha}^{\pi}\frac{\sin Nt}{2\operatorname{tg} t/2}dt$$
$$= \frac{1}{2}\sum_{n=1}^{N-1} s_n\left[\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{n+1}\right] + s_N\int_{\alpha}^{\pi}\frac{\sin Nt}{2\operatorname{tg} t/2}dt$$
$$= \sum_{n-1}^{N-1} s_n^* \frac{\sin n\alpha}{n} + s_N\int_{\alpha}^{\pi}\frac{\sin(N-1/2)t}{2\sin t/2}dt.$$

If we suppose that $s_N = o(N)$, then the sum in (1) becomes

$$\sum_{n=1}^{\infty} s_n^* \frac{\sin n\alpha}{n}$$

Moreover, when

$$(3)\qquad \sum_{n=1}^{\infty} a_n \frac{\sin n\alpha}{n}$$

converges and tends to zero as $\alpha \to 0$, (1) equals to

$$\lim_{\alpha\to 0}\sum_{n=1}^{\infty} s_n \frac{\sin n\alpha}{n}.$$

Thus we get

THEOREM 1. *If $s_n = o(n)$ and (3) converges and tends to zero as $\alpha \to 0$, then the series summable $(K, 1)$ is (R_1) summable and conversely.*

It is known that there is a series summable (R_1) and (3) diverges for a null sequence (α_i), and conversely, whence there is a sequence (R_1) summable but not $(K, 1)$ summable, and conversely.

Let us consider an integrable function $f(t)$ and its Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt).$$

We can suppose $a_0 = 0$ and $b_n = 0$ $(n = 1, 2, \ldots)$ without loss of generality. Then the partial sum $s_n = o(n)$ and the series

$$\sum_{n=1}^{\infty} \frac{a_n}{u} \sin n\alpha$$

converges and tends to zero as $\alpha \to 0$. Thus, by Theorem 1, we get

THEOREM 2. *For Fourier series, the $(K, 1)$ summability is equivalent to the (R_1) summability.*

For Fourier series (R_1)-and $(R, 1)$-summabilities are mutually exclusive [7], whence $(K, 1)$-and $(R, 1)$-summabilities are also. After Hardy and Rogosinski [7] we get the following theorem.

THEOREM 3. *The Fourier serie of $f(t)$ is $(K. 1)$ summable to $f(x)$ at $t = x$ if and only if*

$$\int_{\to 0}^{\pi} \frac{dt}{t} \int_{|h-t|}^{h+t} \frac{\phi(x, u)}{\operatorname{tg} u/2} du \to 0 \quad (h \to 0),$$

where $\phi(x, u) = f(x+u) + f(x-u) - 2f(x)$.

THEOREM 4. *Let $f \varepsilon L^p (p > 1)$. The necessary and sufficient condition that the Fourier series of $f(t)$ is $(K, 1)$ summable at $t = x$, is the existence of the integral*

$$(4)\qquad \int_{\to 0}^{\pi} \frac{\bar{f}(x+v) - \bar{f}(x-v)}{v} dv,$$

where $\bar{f}(t)$ is the conjugate function of $f(t)$.

We will now give a direct proof of the last theorem. Let the Fourier series of $f(t)$ be

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(x).$$

We can suppose that $a_0 \equiv A_0 \equiv 0$. We have

$$A_n(x) = \frac{1}{\pi}\int_0^{2\pi} f(x+u)\cos nu\, du.$$

Hence

$$\begin{aligned} S &\equiv \frac{2}{\pi}\sum_{n=1}^{\infty} A_n(x)\int_\alpha^\pi \frac{\sin nt}{2\operatorname{tg} t/2}dt \\ &= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\int_0^{2\pi} f(x+u)\cos nu\, du \int_\alpha^\pi \frac{\sin nt}{2\operatorname{tg} t/2}dt \\ &= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\int_\alpha^{2\pi} f(x+u)du\int_\alpha^n \frac{\cos nu \sin nt}{2\operatorname{tg} t/2}dt. \end{aligned}$$

Let s_n^* be the modified partial fum of S [6]. Then

$$\begin{aligned} S_n^* &= \frac{2}{\pi^2}\sum_{k=1}^{n}{}^* \int_0^{2n} f(x+u)du\int_\alpha^\pi \frac{\sin kt \cos ku}{2\operatorname{tg} t/2}dt \\ &= \frac{2}{\pi^2}\int_0^{2\pi} f(x+u)du\int_\alpha^\pi \left\{\sum_{k=1}^{n}{}^*\sin kt\cos ku\right\}\frac{dt}{2\operatorname{tg} t/2} \\ &= \frac{2}{\pi^2}\int_0^{2\pi} f(x+u)du\int_\alpha^\pi \frac{1-\cos n(t-u)}{2\operatorname{tg}(t-u)/2}\,\frac{dt}{2\operatorname{tg} t/2} \\ &\quad + \frac{2}{\pi^2}\int_0^{2\pi} f(x+u)du\int_\alpha^\pi \frac{1-\cos n(t+u)}{2\operatorname{tg}(t+u)/2}\,\frac{dt}{2\operatorname{tg} t/2}. \end{aligned}$$

If we put

$$\begin{aligned} g(t) &\equiv \cot t/2 \quad \text{in} \quad (\alpha, \pi), \\ &\equiv 0 \quad\quad\;\; \text{in} \quad (-\pi, \alpha) \end{aligned}$$

and $g(t) \equiv g(t+2\pi)$ for all t. Then the inner integrals of the last double integrals converge to the conjugate function at $t=u$ and $t=-u$, respectively. By $f \varepsilon L^p$,

$$S = \lim_{n\to\infty} S_n^* = \frac{1}{\pi}\int_0^{2\pi} f(x+u)du\int_0^\pi [\psi(u,t) + \psi(-u,t)]\frac{dt}{2\operatorname{tg} t/2}$$

where the inner integral is taken in the Lebesgue sense and $\psi(u,t) = g(u-t) - g(u+t)$[6].

Now, let D_α be the domain in $(0,\pi\,;\,0,2\pi)$ such that

$$|u-v| > 2\alpha,\ |2\pi - (u+v)| > 2\alpha.$$

Then we have, by an easy calcutation,

$$\begin{aligned} S &= \frac{1}{\pi}\int\!\!\int_{D_\alpha} f(x+u)\frac{\cos t/2}{\sin\dfrac{u-t}{2}\sin\dfrac{u+t}{2}}dt\,du + o(1) \\ &= \frac{1}{\pi}\int\!\!\int_{D_\alpha} f(x+v+w)\frac{\cos(v+w)/4}{\sin\dfrac{v}{2}\sin\dfrac{w}{2}}dv\,dw + o(1). \end{aligned}$$

By $D_{\alpha,\beta}$ we denote the domain in $(0,\pi\,;\,0,2\pi)$ such that

$$|u-v| > 2\alpha,\quad |2\pi - (u+v)| > 2\beta.$$

For the existence of the limit $\lim_{\alpha\to 0} S$, it is necessary and sufficient that the limit

$$(5) \qquad \lim_{\alpha,\beta\to 0}\int_\alpha^\pi \frac{dv}{\sin\frac{v}{4}}\int_\beta^\pi \frac{\psi(x+v,w)}{\sin\frac{w}{4}}dw$$

exists. Since by the hypothesis we can invert the integral of v and $\lim_{\beta\to 0}$, (5) is equivalent to

$$\lim_{\alpha\to 0}\int_\alpha^\pi \frac{f(x+v)-f(x-v)}{\sin\frac{v}{4}}dv.$$

Thus the theorem is proved.

§ **3.** THEOREM 5. *If the integral*

$$(6) \qquad \int_{-\pi}^\pi f(x+t)\log^+\frac{1}{t}dt$$

converges, then the necessary and sufficient condition that the Fourier series of $f(t)$ is summable $(K, 2)$ at $t=x$, is that the integral

$$(7) \qquad \int_{\to 0}^\pi \frac{dt}{t^2}\int_0^\pi \varphi(x+u,\ t)\log\left(\frac{1}{2\sin\frac{u}{2}}\right)du$$

converges, where

$$\varphi(x,t) = f(x+t) + f(x-t) - 2f(x).$$

PROOF. Using the notations in the proof of Theorem 1 and putting

$$S \equiv \frac{2}{\pi}\sum_{n=1}^\infty A_n(x)\int_\alpha^\pi \frac{\sin^2 nt/2}{n\sin^2 t/2}dt,$$

we have

$$S = \frac{2}{\pi}\sum_{n=1}^\infty\int_0^{2\pi} f(x+u)\cos nu\,du\int_\alpha^\pi \frac{\sin^2 nt/2}{n\sin^2 t/2}dt$$
$$= \frac{2}{\pi}\sum_{n=1}^\infty\int_0^{2\pi} f(x+u)du\int_\alpha^\pi \frac{\cos nu\,\sin^2 nt/2}{n\sin^2 t/2}dt.$$

Let S_n be the n-th partial sum of S. Then

$$S_n = \frac{2}{\pi}\int^{2\pi} f(x+u)du\int_\alpha^\pi\left\{\sum_{k=1}^n \frac{\cos ku\,\sin^2 nt/2}{k\sin^2 t/2}\right\}dt.$$

Now

$$\cos ku\sin^2 kt/2 = \cos ku(1-\cos kt)/2$$
$$= \{2\cos ku - \cos k(u-t) - \cos k(u+t)\}/4.$$

and, for $0<t<2\pi$,

$$\sum_{k=1}^\infty \frac{\cos kt}{k} = -\log\left(2\sin\frac{t}{2}\right),$$

$$\left|\sum_{k-1}^n \frac{\cos kt}{k}\right| < A\left(1+\log^+\frac{1}{t}+\log^+\frac{t}{2\pi-t}\right),$$

where A is an absolute constant[3]. Thus (6) and (7) give us

$$S=\lim_{n\to\infty} S_n = \frac{2}{\pi}\int_\alpha^\pi \frac{dt}{\sin^2 t/2}\int_\alpha^{2\pi} f(x+u)\cdot\Big[-\log\Big(2\sin\frac{u}{2}\Big) + \log\Big(2\Big|\sin\frac{u-t}{2}\Big|\Big)+\log\Big(2\Big|\sin\frac{u-t}{2}\Big|\Big)\Big]du$$

$$= \frac{2}{\pi}\int_\alpha^\pi \frac{dt}{\sin^2 t/2}\int_0^{2\pi}[f(x+u+t)+f(x+u-t) - 2f(x+u)]\log\Big(2\sin\frac{u}{2}\Big)du.$$

Thus the theorem is proved.

The necessary and sufficient condition for (R_2) summability is

$$\lim_{\alpha\to 0}\int_{\to 0}^{\pi}\frac{\varphi_1(x,t)}{t^2}\log\left|\frac{t+\alpha}{t-\alpha}\right|dt = 0, \tag{8}$$

where $\varphi_1(x,t)=\int_0^t \varphi(x,u)du$[3]. Since (7) and (8) are exclusive, $(K,2)$ and (R_2)-summabilities are exclusive.

Part III. Cesàro summability theorems.

§ 1. Let $\phi(t)$ be an even periodic function with Fourier series

$$\phi(t)\sim\sum_{n=0}^{\infty} a_n \cos nt,\quad a_0=0. \tag{1.1}$$

The α-th integral of $\phi(t)$ is defined by

$$\Phi_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \phi(u)(t-u)^{\alpha-1}du \quad (\alpha>0)$$

and the β-th Cesàro sum of (1.1) is defined by $s_n^\beta(\beta>-1)$. Especially we put $s_n^0 \equiv s_n$.

L. S. Bosanquet [8] has proved that

$$\Phi_\beta(t)=o(t^\beta)\quad (t\to 0)$$

implies

$$s_n^\alpha = o(n^\alpha)\quad (n\to\infty) \tag{1.2}$$

for $\alpha>\beta$, and conversely

$$s_n^\beta = o(n^\beta)\quad (n\to\infty)$$

implies

$$\Phi_\alpha(t)=o(t^\alpha)\quad (t\to 0) \tag{1.3}$$

for $\alpha>\beta+1$. This paper concerns the converse part of the Bosanquet theorem. Recently Hyslop [9] and the author [10] proved, that, if

$$s_n^\beta = o(n^\gamma)\quad (n\to\infty) \tag{1.4}$$

for $\beta>\gamma>0$, then

$$\Phi_\alpha(t)=o(t^{\alpha+\beta-\gamma})\quad (t\to 0) \tag{1.5}$$

for $\alpha>1+\gamma$.

Naturally it is required to find such α that (1.3) holds under the hypothesis (1.4). The solution is given by

$$\alpha \geqq (\beta + 1)/(\beta - \gamma + 1).$$

This is proved in Theorem 1.

Secondly we treat the case (1. 4) with some Tauberian condition. The theorem of this type was considered by Loo [11]. Tauberian condition used by him is

(1. 6) $$a_n = O(1/n^{1-\delta}) \quad (n \to \infty)$$

for $0<\delta<1$. Besides this we use a weaker one such as

(1. 7) $$s_n^{-\delta} = O(1) \quad (n \to \infty).$$

We prove in Theorem 2 that (1. 4) and (1. 7) imply (1. 3) for

$$\alpha \geqq 1 + \gamma\delta/(\beta - \gamma + \delta);$$

and in Theorem 3 that (1. 4) and (1. 6) imply (1. 3) for

$$\alpha \geqq \gamma(\beta + 1)/(\beta - \gamma + \delta).$$

In the latter case we need some restriction concerning β, γ and δ. These theorems imply Loo's theorems as special case.

In the proof we do not use Young functions, which were always used to prove theorems in this direction, but we use a method in the former paper. This method makes also easy the proof of the converse part of the Bosanquet theorem stated above.

§ 2. Theorem 1. *If*

(2. 1) $$s_n^\beta = o(n^\gamma) \quad (n \to \infty).$$

for $\beta>\gamma>0$, *then*

(2. 2) $$\Phi_\alpha(t) = o(t^\alpha) \quad (t \to 0)$$

for $\alpha \geqq (\beta+1)/(\beta-\gamma+1)$.

Proof. Let $\alpha \equiv (\beta+1)/(\beta-\gamma+1)$ and we will prove (2. 2) for such α. Then $1<\alpha<1+\gamma$. We distinguish sevearl cases, and begin by the case $1<\alpha<2$.

$$\begin{aligned}\Gamma(\alpha)\Phi_\alpha(t) &= \int_0^t \varphi(u)(t-u)^{\alpha-1}du \\ &= \sum_{n=0}^\infty a_n \int_0^t \cos nu(t-u)^{\alpha-1}du = \sum_{n=0}^\infty s_n \int_0^t \Delta \cos nu(t-u)^{\alpha-1}du \\ &= \sum_{n=0}^M + \sum_{n=M+1}^\infty \equiv I + J,\end{aligned}$$

say, where $\Delta \cos nu = \cos nu - \cos(n+1)u$ and M will be determined later. By the well known formula

$$s_n = \sum_{\nu=0}^n (-1)^{n-\nu}\binom{\beta}{n-\nu} s_\nu^\beta,$$

we have

$$\begin{aligned}I &= \sum_{n=0}^M s_n \int_0^t \Delta \cos nu(t-u)^{\alpha-1}du \\ &= \sum_{n=0}^M \int_0^t \Delta \cos nu(t-u)^{\alpha-1}du \sum_{\nu=0}^n (-1)^{n-\nu}\binom{\beta}{n-\nu} s_\nu^\beta \qquad (2.3) \\ &= \sum_{n=0}^M s_\nu^\beta \int_0^t \left\{\sum_{n=\nu}^M (-1)^{n-\nu}\binom{\beta}{n-\nu}\Delta \cos nu\right\}(t-u)^{\alpha-1}du.\end{aligned}$$

The inner sum is

$$\sum_{n=\nu}^{M}(-1)^{n-\nu}\binom{\beta}{n-\nu}\Delta\cos nu = 2^{\beta+1}\left(\sin\frac{u}{2}\right)^{\beta+1}\cos\left(\left(\nu+\frac{\beta+1}{2}\right)u+\frac{\beta+1}{2}\pi\right)$$
$$-\sum_{m=M-\nu+1}^{\infty}(-1)^{m}\binom{\beta}{m}\Delta\cos(m+\nu)u \equiv K_1(u)-K_2(u),$$

say. Let us decompose I in (2.3) such that

$$I=\sum_{\nu=0}^{M}=\sum_{\nu=0}^{N}+\sum_{\nu=N+1}^{M}\equiv I_1+I_2,$$

and I_1 and I_2 such that

$$I_1=\sum_{\nu=0}^{N}s_\nu^\beta\int_0^t K_1(u)(t-u)^{\alpha-1}du+\sum_{\nu=0}^{N}s_\nu^\beta\int_0^t K_2(u)(t-u)^{\alpha-1}du$$
$$\equiv I_1'+I_1'',$$
$$I_2=\sum_{\nu=N+1}^{M}s_\nu^\beta\int_0^t K_1(u)(t-u)^{\alpha-1}du+\sum_{\nu=N+1}^{M}s_\nu^\beta\int_0^t K_2(u)(t-u)^{\alpha-1}du$$
$$\equiv I_2'+I_2''.$$

Since

$$\int_0^t\left(\sin\frac{u}{2}\right)^{\beta+1}\cos((\nu+\lambda)u+\lambda\pi)(t-u)^{\alpha-1}du=O(t^{\alpha+\beta+1}),$$

we have

$$I_1'=o\left(\sum_{\nu=0}^{N}\nu^\gamma\cdot t^{\alpha+\beta+1}\right)=o\left(N^{\gamma+1}\cdot t^{\alpha+\beta+1}\right)$$

which is $o(t^\alpha)$, when

(2.4) $$N\equiv[1/t^{(\beta+1)/(\gamma+1)}].$$

By

$$\int_0^t\Delta\cos(m+\nu)u(t-u)^{\alpha-1}du=O\left(\frac{t^\alpha}{m+\nu}\right),$$

we have

$$I_1''=\sum_{\nu=0}^{N}s_\nu^\beta\sum_{m=M-\nu+1}^{\infty}(-1)^m\binom{\beta}{m}\int_0^t\Delta\cos(m+\nu)u(t-u)^{\alpha-1}du$$
$$=o\left(\sum_{\nu=0}^{N}\nu^\gamma\sum_{m=M-\nu+1}^{\infty}\frac{t^\alpha}{m^{\beta+1}(m+\nu)}\right)$$
$$=o\left(\frac{t^\alpha}{M}\sum_{\nu=0}^{N}\frac{\nu^\gamma}{(M-\nu+1)^\beta}\right)=o\left(t^\alpha\frac{N^{\gamma+1}}{M^{\beta+1}}\right)=o(t^\alpha),$$

if $M\geqq 2N$. Thus we have $I_1''=o(t^\alpha)$, whence $I_1=I_1'-I_1''=o(t^\alpha)$.

In order to estimate I_2, putting $\lambda\equiv(\beta+1)/2$ and using the Lebesgue's device, we obtain

(2.5) $$\int_0^t\left(\sin\frac{u}{2}\right)^{\beta+1}\cos((\nu+\lambda)u+\lambda\pi)(t-u)^{\alpha-1}du$$
$$=\frac{1}{2}\Bigg\{\int_0^t\left(\sin\frac{u}{2}\right)^{\beta+1}\cos((\nu+\lambda)u+\lambda\pi)(t-u)^{\alpha-1}du$$
$$-\int_{\pi/(\nu+\lambda)}^{t+\pi/(\nu+\lambda)}\left(\sin\frac{1}{2}\left(u+\frac{\pi}{v+\lambda}\right)\right)^{\beta+1}\cos((v+\lambda)u$$

$$+ \lambda\pi)\Big(t - u - \frac{\pi}{\nu + \lambda}\Big)^{\alpha-1}du\Big\}$$

$$= \frac{1}{2}\Big\{\int_0^{\pi/(\nu+\lambda)} + \int_{\pi/(\nu+\lambda)}^{t} + \int_t^{t+\pi/(\nu+\lambda)}\Big\}$$

$$= O\Big(\frac{t^{\beta+1}}{\nu^\alpha} + \frac{t^{\alpha+\beta-1}}{\nu^2} + \frac{t^{\alpha-1}}{\nu^{\beta+1}}\Big) = O\Big(\frac{t^{\beta+1}}{\nu^\alpha}\Big)$$

for $\nu t \geqq 1$. Hence

$$I_2' = \sum_{\nu=N+1}^{M} s_\nu^\beta \int_0^t \Big(\sin\frac{u}{2}\Big)^{\beta+1} \cos\,((\nu+\lambda)u + \lambda\pi)(t-u)^{\alpha-1}du$$

$$= o\Big(\sum_{\nu=N+1}^{M} \nu^\delta \frac{t^{\beta+1}}{\nu^\alpha}\Big) = o(t^{\beta+1}M^{\gamma+1-\alpha})$$

which is $o(t^\alpha)$, if

(2.6) $$M \leqq 1/t^{(\beta+1-\alpha)/(\gamma+1-\alpha)}.$$

On the other hand, we have

$$\int_0^t \Delta \cos nu(t-u)^{\alpha-1}du = 2\int_0^t \frac{u}{2}\cos\Big(n + \frac{1}{2}\Big)u(t-u)^{\alpha-1}du$$

$$= O(t/n^\alpha)$$

for $nt \geqq 1$, and then

(2.7) $$I_2'' = \sum_{\nu=N+1}^{M} s_\nu^\beta \int_0^t \Big\{\sum_{m=M-\nu+1}^{\infty} (-1)^m \binom{\beta}{m} \Delta \cos\,(m+\nu)\,u\Big\}(t-u)^{\alpha-1}du$$

$$= o\Big(\sum_{\nu=N+1}^{M} \nu^\gamma \sum_{m=M-\nu+1}^{\infty} \frac{t}{m^{\beta+1}(m+\nu)^\alpha}\Big)$$

$$= o\Big(\sum_{\nu=N+1}^{M} \nu^\gamma \cdot \frac{t}{M^\alpha(M-\nu+1)^\beta}\Big) = o\Big(\frac{t}{M^{\alpha+\beta-\gamma-1}}\Big)$$

for $0<\beta<1$, which is $o(t^\alpha)$, if

(2.8) $$M \geqq 1/t^{(\alpha-1)/(\alpha-1+(\beta-\gamma))}.$$

Thus $I_2 = I_2' - I_2'' = o(t^\alpha)$ when the conditions (2.6) and (2.7) are satisfied.

Finally

$$J = \sum_{n=M+1}^{\infty} s_n \int_0^t \Delta \cos nu(t-u)^{\alpha-1}du$$

$$= o\Big(\sum_{n=M+1}^{\infty} n^{\gamma/(\beta+1)} \cdot \frac{t}{n^\alpha}\Big) = o(t/M^{\alpha-1-\gamma/(\beta+1)})$$

which is $o(t^\alpha)$, if

(2.9) $$M \geqq 1/t^{(\alpha-1)/(\alpha-1-\gamma/(\beta+1))}.$$

By $\alpha = (\beta+1)/(\beta-\gamma+1)$,

$$\frac{\beta+1-\alpha}{\gamma+1-\alpha} = \frac{\alpha-1}{\alpha-1-\gamma/(\beta+1)} \geqq \frac{\alpha-1}{\alpha-1+(\beta-\gamma)}.$$

Hence the conditions (2.6), (2.8) and (2.9) are consistent, and it is sufficient to take M such as

$$M = [1/t^{(\beta+1-\alpha)/(\gamma+1-\alpha)}] = [1/t^{(\beta+1)/\gamma}],$$

which is sufficiently larger than N. Thus the theorem is proved for the case $1<\alpha<2$ and $0<\beta<1$.

§ **3.** Let us now consider the case $1<\alpha<2$ and $1<\beta<2$. It is enough to estimate I_2 only. Using the Abel lemma in the inner sum in (2. 7)

$$(5.1) \quad I_2'' = \sum_{\nu=N+1}^{M} s_\nu^\beta \int_0^t \left\{ \sum_{m=M-\nu+1}^{\infty} (-1)^m \binom{\beta-1}{m} \Delta^2 \cos(m+\nu)u \right\} (t-u)^{\alpha-1} du$$

$$- \sum_{\nu=N+1}^{M} (-1)^{M-\nu} s_\nu^\beta \binom{\beta-1}{M-\nu} \int_0^t \Delta \cos (M+1)u(t-u)^{\alpha-1} du$$

$$\equiv i_1 - i_2,$$

say, where $\Delta^2 \cos nu = \Delta(\Delta \cos nu) = 4\left(\sin\frac{u}{2}\right)^2 \cos (n+1)u$.

Now

$$i_1 = o\left(\sum_{\nu=N+1}^{M} \nu^\gamma \sum_{m=M-\nu+1}^{\infty} \frac{t^2}{m^\beta(m+\nu)^\alpha} \right)$$

$$= o\left(\sum_{\nu=N+1}^{M} \nu^\gamma \frac{t^2}{(M-\nu+1)^{\beta-1} M^\alpha} \right) = o\left(\frac{t^2}{M^{\alpha+\beta-\delta-2}} \right),$$

which is evidently $o(t^\alpha)$ if $\alpha+\beta-\gamma-2 \geqq 0$. If $\alpha+\beta+\gamma-2 \leqq 0$, it is sufficient that

$$(3.2) \qquad M \leqq 1/t^{(2-\alpha)/(2-\alpha-(\beta-\gamma))}.$$

Since M is taken such that

$$M = [1/t^{(\beta+1)/\gamma}] = [1/t^{\alpha/(\alpha+1)}].$$

(3. 2) is satisfied when $2/(1+\beta) \leqq 1$, which is the case for $1<\beta<2$.

Secondly, if we use the Abel lemma in i_2 again, then

$$i_2 = \left\{ - \sum_{\nu=N+1}^{M} s_\nu^{\beta-1} (-1)^{M-\nu} \binom{\beta-2}{M-\nu} \right.$$

$$\left. + s_{N+1}^\beta (-1)^{M-N+1} \binom{\beta-2}{M-N+1} \right\} \cdot \int_0^t \Delta \cos (M+1)u(t-u)^{\alpha-1} du$$

$$= o\left(\left\{ \sum_{\nu=N+1}^{M} \frac{\nu^{\gamma\beta/(\beta+1)}}{(M-\nu)^{\beta-1}} + \frac{N^\gamma}{M^{\beta-1}} \right\} \frac{t}{M^\alpha} \right)$$

$$= o\left(\frac{t}{M^{\alpha+\beta-\beta\gamma/(\beta+2)-2}} + \frac{t^{1-\gamma(\beta+1)/(\gamma+1)}}{M^{\alpha+\beta-1}} \right)$$

where $\alpha+\beta-\gamma\beta/(\beta+1)-2>0$ by $\alpha<2$. Thus $i_2=o(t^\alpha)$ when

$$(3.3) \qquad M \geqq 1/t^{((\alpha-1)/(\alpha+\beta-\xi\beta/(\beta+1)-2)}$$

and

$$M \geqq 1/t^{(\alpha-1)+\gamma(\beta+1)/(\gamma+1))/(\alpha+\beta-1)}.$$

These are easily verified. Thus the theorem is proved in the considering case.

Let us proceed to the case $1<\alpha<2$ and $2<\beta<3$. We use the Abel lemma in (3. 1) again. Using it in i_1,

$$i_1 = \sum_{\nu=N+1}^{M} s_\nu^\beta \int_0^t \left\{ \sum_{m=M-\nu+1}^{\infty} (-1)^m \binom{\beta-2}{m} \Delta^3 \cos (m+\nu)u \right\} (t-u)^{\alpha-1} du$$
$$- \sum_{\nu=N+1}^{M} s_\nu^\beta (-1)^{M-\nu} \binom{\beta-1}{M-\nu} \int_0^t \Delta^3 \cos (m+\nu) u (t-u)^{\alpha-1} du$$
$$\equiv j_1 - j_2,$$

say, where $\Delta^3 \cos nu = \Delta(\Delta^2 \cos nu) = 2^3 \left(\frac{u}{2}\right)^3 \cos\left(n+\frac{3}{2}\right)u$. Similarly as i_1 in the former case

$$j_1 = o\left(\sum_{\nu=N+1}^{M} \nu^\gamma \sum_{m=M-\nu+1}^{\infty} \frac{t^3}{m^{\beta-1}(m+\nu)^\alpha} \right)$$
$$= o\left(\sum_{\nu=N+1}^{M} \nu^\gamma \frac{t^3}{(M-\nu+1)^{\beta-2} M^\alpha} \right) = o\left(\frac{t^3}{M^{\alpha+\beta-\gamma-3}} \right),$$

which is evidently $o(t^\alpha)$ if $\alpha+\beta-\gamma-3 \geqq 0$. If $\alpha+\beta-\gamma-3 \leqq 0$, it is sufficient that

$$M \leqq 1/t^{(3-\alpha)/(3-\alpha-(\beta-\gamma))}. \tag{3.5}$$

This is satisfied when $3/(\beta+1) \leqq 1$, which is the case. In j_2 we use the Abel lemma once. Then we get the required estimation. Concerning i_2, it is sufficient to use the Abel lemma twice. Thus the theorem is proved for the case $1<\alpha<2$ and $2<\beta<3$.

Thus proceeding we can complete the proof of the theorem for the case $1<\alpha<2$, since the integral case of α is trivial.

§ **4.** Let us consider the case $2<\alpha<3$. In this case $\beta>1$. We have

$$\Gamma(\alpha)\Phi_\alpha(t) = \int_0^t \phi(u)(t-u)^{\alpha-1} du$$
$$= \sum_{n=0}^{\infty} s_n \int_0^t \Delta \cos nu (t-u)^{\alpha-1} du$$
$$= \sum_{n=0}^{\infty} s_n^1 \int_0^t \Delta^2 \cos nu (t-u)^{\alpha-1} du$$
$$= \sum_{n=0}^{M} + \sum_{n=M+1}^{\infty} \equiv I + J,$$

say. By the formula

$$s_n^1 = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{\beta-1}{n-\nu} s_\nu^\beta,$$

we have

$$I = \sum_{n=0}^{M} s_\nu^\beta \int_0^t \left\{ \sum_{n=\nu}^{M} (-1)^{n-\nu} \binom{\beta-1}{n-\nu} \Delta^2 \cos nu \right\} (t-u)^{\alpha-1} du,$$

where the inner sum is

$$\sum_{n=\nu}^{M} (-1)^{n-\nu} \binom{\beta-1}{n-\nu} \Delta^2 \cos nu = 2^{\beta+1} \left(\sin\frac{u}{2}\right)^{\beta+1} \cos\left(\left(v+\frac{\beta+1}{2}\right)u + \frac{\beta+1}{2}\pi\right)$$
$$- 2 \sum_{m=M-\nu+1}^{\infty} (-1)^m \binom{\beta-1}{m} \Delta^2 \cos (m+\nu) u,$$

say. Hence, similarly as in § 2, we decompose I such that

$$I = I_1 + I_2 = (I_1' + I_1'') + (I_2' + I_2'').$$

Defining N by (1.4), $I_1'=o(t^\alpha)$ and

$$I_1'' = o\left(\sum_{\nu=0}^{N}\nu^\gamma \sum_{m=M-\nu+1}^{\infty} \frac{t^{\alpha+1}}{m^\beta(m+\nu)}\right)$$
$$= o\left(\frac{t^{\alpha+1}}{M}\sum_{\nu=0}^{N}\frac{\nu^\gamma}{(M-\nu+1)^{\beta-1}}\right)$$
$$= o(t^{\alpha+1}N^{\gamma+1}/M^\beta) = o(t^\alpha).$$

Now, by the twice application of Lebesgue's device, we get

$$\int_0^t \left(\sin\frac{u}{2}\right)^{\beta+1} \cos\left((\nu+\lambda)u + \lambda\pi\right)(t-u)^{\alpha-1}du$$
$$= O\left(\frac{t^{\beta+1}}{\nu^\alpha} + \frac{t^{\alpha+\beta-2}}{\nu^3} + \frac{t^{\alpha-1}}{\nu^{\beta+2}}\right) = O\left(\frac{t^{\beta-1}}{\nu^\alpha}\right)$$

for $\nu t \geqq 1$. Thus, if the condition (2.6) is satisfied, then we obtain $I_2'=o(t^\alpha)$.

$$I_2'' = \sum_{\nu=N+1}^{M} s_\nu^\beta \int_0^t \left\{\sum_{m=M-\nu+1}^{\infty}(-1)^m\binom{\beta-1}{m}\Delta^2\cos(m+\nu)u\right\}(t-u)^{\alpha-1}du$$

may be esitimated similarly as the case $1<\alpha<2$, dividing the cases $n<\beta<n+1 (n=1, 2, \dots)$.

Finally

$$J = \sum_{n=M+1}^{\infty} s_n^1 \int_0^t \Delta^2 \cos nu(t-u)^{\alpha-1}du$$
$$= o\left(\sum_{n=M+1}^{\infty} n^{2\gamma/(\beta+1)}\frac{t^2}{n^\alpha}\right) = o\left(\frac{t^2}{M^{\alpha-1-2\gamma/(\beta+1)}}\right)$$

where $\alpha-1-2\gamma/(\beta+1)>0$ by $\alpha>2$. Hence $J=o(t^\alpha)$, if

(4.1) $$M \geqq 1/t^{(\alpha-2)/(\alpha-1-2\gamma/(\beta+1))}.$$

Since

$$\frac{\beta+1-\alpha}{\gamma+1-\alpha} = \frac{\alpha-2}{\alpha-1-2\gamma/(\beta+1)} = \frac{\alpha}{\alpha-1},$$

(2.6) and (4.1) are consistent.

Thus the theorem is proved for the case $2<\alpha<3$. The proof of the case $n<\alpha<n+1 (n=3, 4, \dots)$ is now in hand. Since the proof for integral α is easy, we have completed the proof of the theorem.

§ **5.** THEOREM 2. *If*

(5.1) $$s_n^{-\delta} = O(1) \quad (n\to\infty)$$

for $0<\delta<1$ *and*

(5.2) $$s_n^\beta = o(n^\gamma) \quad (n\to\infty)$$

for $\beta>\gamma>0$, *then*

(5.3) $$\Phi_\alpha(t) = o(t^\alpha) \quad (n\to\infty)$$

for $\alpha\geqq 1+\gamma\delta/(\beta-\gamma+\delta)$.

Proof of this theorem follows the similar lines of that of Theorem 1. In our cace, (5.1) and (5.2) imply

$$s_n = o(n^{\delta\gamma/(\beta+\delta)}), \quad s_n^1 = o(n^{(1+\delta)\gamma/(\beta+\delta)}), \dots$$

This attributes to the estimation of J. In the case $1<\alpha<2$, $J=o(t^\alpha)$ if

(5.4) $$M \geqq 1/t^{(\alpha-1)/(\alpha-1-\delta\gamma/(\beta+\delta))}.$$

(2.6) and (5.4) are consistent when

$$\frac{\beta+1-\alpha}{\gamma+1-\alpha} \geqq \frac{\alpha-1}{\alpha-1-\delta\gamma/(\beta+\delta)}$$

which gives $\alpha \geqq 1+\gamma\delta/(\beta-\gamma+\delta)$. In this case $M=[1/t^{(\beta+\delta)/\gamma}]$. For such M, it is easy to verify the conditions (3.2), (3.3) and so on.

In the case $2<\alpha<3$, we obtain, $J=o(t^\alpha)$ if

(5.5) $$M \geqq 1/t^{(\alpha-2)/(\alpha-1-(1+\delta)\gamma/(\beta+\delta))}.$$

(2.6) and (5.5) are consistent when

$$\frac{\beta+1-\alpha}{\gamma+1-\alpha} \geqq \frac{\alpha-2}{\alpha-1-(1+\delta)\gamma/(\beta+\delta)}$$

which gives also $\alpha \geqq 1+\gamma\delta/(\beta-\gamma+\delta)$. Thus the remaining estimation is the same as that of the former case.

We are now easy to prove the cases $n<\alpha<n+1$ $(n=3, 4, \dots)$.

§ **6.** THEOREM 3. *If*

(6.1) $$a_n = O(1/n^{1-\delta}) \quad (n \to \infty)$$

for $0<\delta<1$, *and*

(6.2) $$s_n^\beta = o(n^\gamma) \quad (n \to \infty)$$

for $\beta>\gamma\geqq 0$ *and further if*

(6.3) $$\delta(\beta-1) \leqq 2(\beta-\gamma), \quad 1-\delta \leqq \beta$$

(*that is,* $\beta \leqq 1$ *or* $\delta \leqq 2(\beta-\gamma)/(\beta-1)$), *then*

(6.4) $$\Phi_\alpha(t) = o(t^\alpha) \quad (t \to 0)$$

for $\alpha \geqq \max(1, \delta(\beta+1)/(\beta-\gamma+\delta))$.

We have

$$\Gamma(\alpha)\Phi_\alpha(t) = \sum_{n=0}^{\infty} a_n \int_0^t \cos nu(t-u)^{\alpha-1}du$$
$$= \sum_{n=0}^{M} + \sum_{n=M+1}^{\infty} \equiv I + J,$$

say. Estimation of I is similar as that in § 2. By the formula

$$a_n = \sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{\beta+1}{n-\nu} s_\nu^\beta,$$

we have

$$I = \sum_{n=0}^{M} a_n \int_0^t \cos nu(t-u)^{\alpha-1}du$$
$$= \sum_{\nu=0}^{M} s_\nu^\beta \int_0^t \left\{\sum_{n=\nu}^{M}(-1)^{n-\nu}\binom{\beta+1}{n-\nu}\cos nu\right\}(t-u)^{\alpha-1}du,$$

where the inner sum is

$$2^{\beta+1}\left(\sin\frac{u}{2}\right)^{\beta+1}\cos\left((\nu+\lambda)u+\lambda\pi\right)-\sum_{m=M-\nu+1}^{\infty}(-1)^m\binom{\beta+1}{m}\cos(m+\nu)u.$$

Hence, similarly as in § 2, we put

$$I=\sum_{\nu=0}^{M}=\sum_{\nu=0}^{N}+\sum_{\nu=N+1}^{M}=I_1+I_2$$
$$=(I_1'+I_1'')+(I_2'+I_2'').$$

Defining N by (2. 4), we get $I_1'=o(t^\alpha)$ and

$$I_1''=\sum_{\nu=0}^{N}s_\nu^\beta\sum_{m=M-\nu+1}^{\infty}(-1)^m\binom{\beta+1}{m}\int_0^t\cos(m+\nu)(t-u)^{\alpha-1}du$$
$$=o\left(\sum_{\nu=0}^{M}\nu^\gamma\sum_{m=M-\nu+1}^{\infty}\frac{t^{\alpha-1}}{m^{\beta+2}(m+\nu)}\right)$$
$$=o\left(\frac{t^{\alpha-1}}{M}\sum_{\nu=0}^{N}\frac{\nu^\gamma}{(M-v+1)^{\beta+1}}\right)$$
$$=o(t^{\alpha-1}N^{\gamma+1}/M^{\beta+2})=o(t^\alpha/(tM)^{\beta+2})=o(t^\alpha)$$

for $tM\geqq 1$. We have $I_2'=o(t^\alpha)$ by (1. 6). Since we can suppose $\alpha\leqq 2$, by (6. 3),

$$I_2''=\sum_{\nu=N+1}^{M}s_\nu^\beta\int_0^t\left\{\sum_{m=M-\nu+1}^{\infty}(-1)^m\binom{\beta+1}{m}\cos(m+\nu)u\right\}(t-u)^{\alpha-1}du$$
$$=\sum_{\nu=N+1}^{M}s_\nu^\beta\int^t\left\{\sum_{n=M-\nu+1}^{\infty}(-1)^m\binom{\beta}{m}\cos(m+\nu)u\right\}(t-u)^{\alpha-1}du$$
$$-\sum_{\nu=N+1}^{M}s_\nu^\beta(-1)^{M-\nu}\binom{\beta}{M-\nu}\int_0^t\cos(M+1)u(t-u)^{\alpha-1}du$$
$$\equiv i_1-i_2,$$

say. If we suppose $0<\beta<1$, then

$$i_1=o\left(\sum_{\nu=N+1}^{M}\nu^\gamma\sum_{m=M-\nu+1}^{\infty}\frac{t}{m^{\beta+1}(m+\nu)^\alpha}\right)$$
$$=o\left(\frac{t}{M^\alpha}\sum_{\nu=N+1}^{M}\frac{\nu^\gamma}{(M-\nu+1)^\beta}\right)=o\left(\frac{t}{M^{\alpha+\beta--1}}\right)$$

which is $o(t^\alpha)$, if

(6. 5) $$M\geqq 1/t^{(\alpha-1)/(\beta+\beta-\gamma-1)}.$$

By the Abel lemma

$$i_2=\sum_{\nu=N}^{M}s_\nu^{\beta-1}(-1)^{M-\nu}\binom{\beta-1}{M-\nu}+s_{N+1}^\beta(-1)^{M-N-1}\binom{\beta-1}{M-N-1}$$
$$=o\left(\sum_{\nu=N}^{M}\nu^{\gamma(\beta-1+\delta)/(\beta+\delta)}/(M-\nu)^{\alpha+\beta}\right)+o(N^\gamma/M^{\gamma+\beta})$$
$$=o(1/M^{\alpha+\beta-1-\gamma+\gamma/(\beta+\delta)})+o(N^\gamma/M^{\alpha+\beta})$$

which is $o(t^\alpha)$ if

(6. 6) $$M\geqq 1/t^{\alpha/(\alpha+\beta-\gamma-1+\gamma(\beta+\delta))}.$$

(6.7) $$M \geqq 1/t^{(\alpha-\gamma(\beta+1)/(\gamma+1))/(\alpha+\beta)}.$$

Finally

$$J = \sum_{n=M+1}^{\infty} a_n \int_0^t \cos nu(t-u)^{\alpha-1} du$$
$$= O\Big(\sum_{n=M+1}^{\infty} \frac{1}{n^{\alpha+1-\delta}}\Big) = O\Big(\frac{1}{M^{\alpha-\delta}}\Big)$$

which is $O(t^\alpha)$ if

(6.8) $$M \geqq 1/t^{\alpha/(\alpha-\delta)}.$$

Now the condition (6.8) implies (6.5), (6.6) and (6.7), (6.8) and (2.4) are consistent when

$$\alpha \geqq \delta(\beta+1)/(\beta-\gamma+\delta).$$

Thus we have proved (6.4) with O instead of o, for the case $0<\beta<1$. We can replace O by o by the ordinary method. The general case $n<\beta<n+1 (n=1, 2, \ldots)$ may be proved similarly as in § 3.

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