ON A CERTAIN GROUP CONCERNING THE p-ADIC NUMBER FIELD.

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In the local class field theory, we consider the norm group of a finite extension field of a p-adic number field k. An abelian extension K of k is uniquely determined by this subgroup of k^*, where k^* is the multiplicative group of all non zero elements of k. We denote this norm group of K by $N^*_{K/k}$. Then the galois group of K/k is isomorphic to the factor group $k^*/N^*_{K/k}$.

We may consider, in some sense dually to the above fact, a subgroup $G(k/K)$ of K^* which consists of all the elements of K^* whose norms to k are unity. It is likely that $G(k/K)$ has close connections with the subfield K. When K/k is cyclic, the structure of $G(k/K)$ was determined by Hilbert. When K/k is abelian, a certain property of $G(k/K)$ was given by Prof. T. Tannaka,[1] who gave also another theorem which is analogous to the ordering theorem of local class field theory. The former property was extended to non-abelian cases, by Mr. T. Nakayama and Mr. Y. Matsushima.[2]

In this paper, restricting to the abelian case, I shall give a detailed structure of $G(k/K)$, and add a certain remark to a particular non-abelian case.

1. The structure of $G(k/\Omega)$.

Let k be a p-adic number field, and K be a finite extension of k. We denote the multiplicaitve groups of their non zero elements by k^*, K^*, respectively, and norm group of K/k, by $N^*_{K/k}$. The elements of K whose norm to k are unity, form a subgroup of K^* and we denote this by $G(k/K)$. When K is a normal extension of k with its galois group G, we mean by a factor set of K/k a system of elements $a_{\sigma,\tau}$ $(\sigma, \tau \in G)$ of K satisfying

(1) $$a^{\rho}_{\sigma,\tau}\, a_{\sigma\tau,\rho} = a_{\sigma,\tau\rho}\, a_{\tau,\rho}.$$

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1) T. Tannaka [8].

2) T. Nakayama and Y. Matsushima [4], T. Nakayama [7].

Further, we shall denote by $K_G^{1-\lambda}$ the group generated by $\theta^{1-\sigma}$, $\theta \in K$, $\sigma \in G$.

One of Tannaka's results runs as follows:

Theorem 1.[3] *Let Ω be a finite abelian extension field of k with its galois group A, and $(a_{\sigma,\tau})$ be a factor set of Ω/k whose exponent is equal to the degree of extension Ω/k. Then $G(k/\Omega)$ is generated by $\Omega_A^{1-\lambda}$ and $a_{\sigma,\tau}/a_{\tau,\sigma}$, where σ, τ run over A:*

$$(2) \qquad G(k/\Omega) = \left(\frac{a_{\sigma,\tau}}{a_{\tau,\sigma}},\ \Omega_A^{1-\lambda}\right).$$

Let

$$(3) \qquad (n_1, \cdots, n_r) \qquad n_{i+1} | n_i$$

be an invariant system of A, then A decomposes directly in cyclic groups Z_i of order n_i:

$$(4) \qquad A = Z_1 \times Z_2 \times \cdots \times Z_r.$$

This decomposition is up to isomorphism unique. Let σ_i be a generator of Z_i, and we shall fix it throughout this section.

In the Theorem 1, it is not necessary to take all the elements of A, but sufficient to do with σ_i of (4). We show this fact in next

Lemma 1.

$$(5) \qquad G(k/\Omega) = \left(\frac{a_{\sigma_i,\sigma_j}}{a_{\sigma_j,\sigma_i}},\ \Omega_A^{1-\lambda}\right).$$

We prove this by induction. Let $N = Z_1 \times \cdots\cdots \times Z_{r-1}$ and M be the corresponding intermediate field. We assume the lemma for the extension Ω/M. Then we take an element θ of $G(k/\Omega)$,

$$N_{\Omega/k}\, \theta = 1.$$

As N/k is cyclic, it follows from Hilbert's lemma that

$$(6) \qquad N_{\Omega/M}\, \theta = n^{1-\sigma_r},\ n \in M.$$

Furthermore, as Ω/M is abelian extension with its galois group N, there exists[4] an element σ of N such that

$$n \equiv a_{\sigma,N} \qquad \text{mod } N^*_{\Omega/M},\text{[5]}$$

where $\qquad \sigma = \Pi_i \sigma_i^{x_i}.$

Then

$$(7) \qquad n \equiv \Pi_i\, a^{x_i}_{\sigma_i,N} \qquad \text{mod } N^*_{\Omega/N}.\text{[4]}$$

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3) We refer this theorem to [8].

4) T. Nakayama [6] and Y. Akizuki [1].

5) We denote a product $\prod_{\tau\in N} a_{\sigma,\tau}$ by $a_{\sigma,N}$, and in a similar way $\prod_{\tau\in N} a^{\tau}_{\sigma,\rho}$ by $a^N_{\sigma,\rho}$.

Next, we calculate $a_{\sigma_i, N}^{1-\sigma_r}$, using the relation (1),

$$(8) \qquad a_{\sigma_i, N}^{1-\sigma_r} = \frac{a_{\sigma_i, N}}{a_{\sigma_i \cdot N}^{\sigma_r}} = \frac{a_{\sigma_i, N}}{a_{\sigma_i, N\sigma_r}} \frac{a_{\sigma_i N, \sigma_r}}{a_{N, \sigma_r}}$$

$$= \frac{a_{\sigma_i, N}}{a_{\sigma_i, \sigma_r N}} = \frac{a_{\sigma_r, \sigma_i}^{N} \, a_{\sigma_r \sigma_i, N}}{a_{\sigma_r, \sigma_i N}} \quad \frac{a_{\sigma_r, N}}{a_{\sigma_i, \sigma_r}^{N} \, a_{\sigma_i \sigma_r, N}} = N_{\Omega/M} \frac{a_{\sigma_r, \sigma_i}}{a_{\sigma_i, \sigma_r}}.$$

It follows from (6), (7) and (8) that

$$N_{\Omega/M}\theta = (\Pi_i \, a_{\sigma_i, N}^{x_i} \cdot N_{\Omega/M}\,\omega)^{1-\sigma_r}$$

$$= \Pi_i \, (a_{\sigma_i, N}^{1-\sigma_r})^{x_i} N_{\Omega/M}(\omega^{1-\sigma_r}) = N_{\Omega/M}\Big(\Pi_i \Big(\frac{a_{\sigma_r, \sigma_i}}{a_{\sigma_i, \sigma_r}}\Big)^{x_i} \cdot \omega^{1-\sigma_r}\Big).$$

therefore

$$N_{\Omega/M}\Big[\; \theta / \Pi_i \Big(\frac{a_{\sigma_r, \sigma_i}}{a_{\sigma_i, \sigma_r}}\Big)^{x_i} \omega^{1-\sigma_r}\Big] = 1.$$

From the assumption of the induction, we obtain

$$G(k/\Omega) = \Big(\frac{a_{\sigma_i, \sigma_j}}{a_{\sigma_j, \sigma_i}},\; \Omega_A^{1-\lambda}\Big) \qquad \text{q.e.d.}$$

Let K be an abelian extension field of k, whose galois group H has invariant system

$$(n_1, n_2) \qquad n_2 | n_1.$$

Then

$$(9) \qquad H = H_1 \times H_2, \qquad H_i = \{\tau_i\}$$

where K_i are the cyclic groups of order n_i, and τ_i their fixed generaters. Let (b) be a factor set of K/k whose exponent is equal to the degree of K/k. From the lemma 1

$$G(k/K) = \Big(\frac{b_{\tau_1, \tau_2}}{b_{\tau_2, \tau_1}},\; K_H^{1-\lambda}\Big).$$

Concerning the order of $b_{\tau_1, \tau_2}/b_{\tau_2, \tau_1}$ mod $K_H^{1-\lambda}$, we obtain next

Lemma 2. *If* $(b_{\tau_2, \tau_1}/b_{\tau_2, \tau_1})^x$ *belongs to* $K_H^{1-\lambda}$, *then*

$$n_2 | x.$$

proof. Let K_i be the intermediate field which corresponds to H_i. From the assumption of the lemma and (9), we have

$$(10) \qquad \Big(\frac{b_{\tau_1, \tau_2}}{b_{\tau_2, \tau_1}}\Big)^x = \theta_1^{1-\tau_1} \theta_2^{1-\tau_2}, \quad \theta_i \in K.$$

Taking the norm with respect to K_2, the left-hand side of the equation (10) becomes

$$N_{K/K_2}\Big(\frac{b_{\tau_1, \tau_2}}{b_{\tau_2 \; \tau_1}}\Big)^x = (b_{\tau_2, H_2}^{1-\tau_1})^x = (b_{\tau_2, H_2}^{x})^{1-\tau_1} = (b_{\tau_2, H_2}^{x} N_{K/K_2}\theta'')^{1-\tau_1},$$

and the right-hand side

$$N_{K/K_2}\,\theta_1^{1-\tau_1},$$

therefore,

$$b^{1-\tau_1}_{\tau_2^x, H_2} = (N_{K/K_2}\theta)^{1-\tau_1} \qquad \theta \in K.$$

In this relation, $b_{\tau_2^x, H_2}$ and $N_{K/K_2}\theta$ belong to the field K_2,[5] and as the galois group H_1 of K_2/k is generated by τ_1, it follows that

(11) $$b_{\tau_2^x, H_2} = \alpha \cdot N_{K/K_2}\theta$$

where α belongs to the field k.

On the other hand we have

(12) $$\alpha \in N^*_{K/K_2}{}^{7)}$$

where if we regard α as an element of K_2, for

$$N_{K_2/k}\alpha = \alpha^{n_1} = \left(\alpha^{\frac{n_1}{n_2}}\right)^{n_2} \in N^*_{K_1/k}$$

implies (12), owing to the "*verschiebungssatz*" of the local class field theory. From (11) and (12) follows

$$b_{\tau_2^x, H_2} \in N^*_{K/K_2}$$

and from this using the Nakayama's theorem[5] we get

$$\tau_2^x = 1,$$

hence $$n_2 | x.$$ q. e. d.

Again we return to the extension Ω/k, and use the same notations as in the Theorem 1 and the Lemma 1.

Lemma 3.

(13) $$\left(\frac{a_{\sigma_i, \sigma_j}}{a_{\sigma_j, \sigma_i}}\right)^{n_j} \in \Omega_A^{1-\lambda}.$$

Proof. Let L_j be an intermediate field which corresponds to the subgroup Z_j of A, then Ω/L_j is a cyclic extension with its galois group Z_j. From this and (8) we get

$$N_{\Omega/L_j}\left(\frac{a_{\sigma_i, \sigma_j}}{a_{\sigma_j, \sigma_i}}\right)^{n_j} = (a^{n_j}_{\sigma_j, Z_j})^{1-\sigma_i} = (a_{\sigma_j, Z_j}^{n_j} N_{\Omega/L_j}\omega')^{1-\sigma_i} = N_{\Omega/L_j}\omega'^{1-\sigma_i}.$$

Hence,

$$\left(\frac{a_{\sigma_i, \sigma_j}}{a_{\sigma_j, \sigma_i}}\right)^{n_j} = \omega'^{1-\sigma_i}\,\omega''^{1-\sigma_j} \in \Omega_A^{1-\lambda}.$$ q. e. d.

Now, we point out a relation between the galois gloup A of Ω/k and the group $G(k/\Omega)$.

Theorem 2.

$$G(k/\Omega)/\Omega_A^{1-\lambda} = A_2 \times A_3 \times \cdots \times A_r$$

7) This proof is given by prof. T. Tannaka. Our original proof was much longer and considerably complicated.

where $A_i \cong Z_i \times Z_{i+1} \times \cdots \times Z_r$
and Z_i *are the cyclic groups of* (4).

Proof. We assume a relation between $\frac{a_{\sigma_i, \sigma_j}}{a_{\sigma_j, \sigma_i}}$ and $\Omega_A^{1-\lambda}$, i.e.

$$\Pi\left(\frac{a_{\sigma_i, \sigma_j}}{a_{\sigma_j, \sigma_i}}\right)^{x_{i,j}} \in \Omega_A^{1-\gamma}. \tag{14}$$

We choose Z_i, Z_j, $i < j$ arbitrary, and let N' be a direct factor excluding $Z_i \times Z_j$, and Z be the corresponding intermediate field, then Z/k is a normal extension with its galois group $Z_i \times Z_j$. We take norm of (14) with respect to Z, then a simple calculation will show that

$$N_{\Omega/Z} \frac{a_{\sigma_s, \sigma_t}}{a_{\sigma_t, \sigma_s}} = 1 \qquad \begin{pmatrix} t \neq i,j \\ s \neq i,j \end{pmatrix},$$

$$N_{\Omega/Z} \frac{a_{\sigma_i, \sigma_t}}{a_{\sigma_t, \sigma_i}} = a_{\sigma_t, N}^{1-\sigma_i} \qquad (t \neq i,j),$$

hence (14) changes to the form

$$\left(\frac{a^N_{\sigma_i, \sigma_j}}{a^N_{\sigma_j, \sigma_i}}\right)^{x_{i,j}} \in Z^{1-\lambda}_{Z_i \times Z_j}. \tag{15}$$

From Chevalley's lemma,[8] $(a^N_{\sigma_i, \sigma_j})$ is also a factor set of Z/k whose exponent is equal to $(Z:k)$. Thus we can regard $(a^N_{\sigma_i, \sigma_j})$ and Z as $(b_{\sigma, \tau})$, and K respectively in the lemma 2, hence

$$n_j | x_{i,j}. \tag{16}$$

This shows that in the relation (14) no $\frac{a_{\sigma_i, \sigma_j}}{a_{\sigma_j, \sigma_i}}$ can really appear, and there exists essentially only relations of the form (15) with (16). It follows that $a_{\sigma_i, \sigma_j}/a_{\sigma_j, \sigma_i} \bmod \Omega_A^{1-\lambda}$ forms a cyclic subgroup $Z_{j,i}$ of degree n_j, and

$$G(k/\Omega)/\Omega_A^{1-\lambda} = Z_{2,1} \times Z_{3,1} \times Z_{3,2} \times \cdots \times Z_{r,1} \times \cdots \times Z_{r,r-1}.$$

Then putting

$$A_i = Z_{i+1,i} \times Z_{i+2,i} \times \cdots \times Z_{r,i}$$

we obtain the desired theorem. q.e.d.

From this, as an immediate consequence, we obtain the Matsushima's result, namely:

Theorem 3. *Let k be a p-adic number field and Ω be a finite abelian extension field. If*

$$G(k/\Omega) = \Omega_A^{1-\lambda},$$

then Ω/k is a cyclic extension.

This theorem is not true for a nonabelian extension K/k. For example, let K/k be a nonabelian extension with galois group G. And we assume

8) C. Chevalley [3] or E. Witt [9].

that G/G' and G' are both cyclic groups, G' being the commutator subgroup of G. Then after a slight calculation we get

$$G(k/K) = K_G^{1-\lambda}.$$

2. Connections with the class field theory.

Let Ω and k denote the local fields as in the section 1. There exists the maximum abelian extension field $\overline{\Omega}$ of k, and obiously $\overline{\Omega} > \Omega$. Let $\overline{A}$ be an infinite abelian extension field of k and we put

(16) $$H(\overline{A}/k) = \bigwedge_A N^*_{A/k}$$

where A is any intermediate field of $\overline{A}/k$ of finite degree over k. For the infinite abelian extension $\overline{A}$ of k, we are able to constitute similar theory with finite abelian extension fields by using $H(A/k)$ instead of N^*.

Now, we shall show that $G(k/\Omega)$ is closely connected with the maximum abelian extension field $\overline{\Omega}$ of k.

Lemma 4.

(17) $$H(\overline{\Omega}/k) = 1.$$

Proof. Let $\alpha \in H(\overline{\Omega}/k)$, and we put α in the from

$$\alpha = P^{\varepsilon} \cdot e$$

where P is a fixed prime element of k, and e an unit element. If $\varepsilon \neq 0$, we denote the group of all the units by E, and construct a subgroup H of k^* generated by E and P^{β}, $|\beta| \equiv 0$ $(2\ |\varepsilon|)$. Then H_1 has finite index in k^*

$$(k^* : H_1) < \infty,$$

hence from the existence theorem of the local class field theory, there exists a finite abelian extension A_1 of k such that

$$H_1 = N^*_{A_1/k}.$$

Furthermore from (16)

(18) $$\alpha \in H(\overline{\Omega}/k) < N^*_{A_1/k} = H_1.$$

On the other hand, from the construction of H_1, it is obvious that

$$\alpha \overline{\in} H_1,$$

and this contradicts with (18). Therefore $\varepsilon = 0$, α is a unit.

If $\alpha \neq 1$, there exists a natural number n such that

(19) $$\alpha \neq 1 \qquad \mod p^n,$$

and we denote by E_n the group of all the element e_n of k^* congruent with unity modulus p^n:

$$e_n \equiv 1 \qquad \mod p^n.$$

From E_n and P, we construct a subgroup of k^* which has a finite group index in k^*.

(20) $$H_n = (P, E_n).$$

Analogously to the above discussion, we get an abelian extension A_n of k such that

$$H_n = N^*_{A_n/k}.$$

And similarly

(21) $$\alpha \in H(\overline{\Omega}/k) < N^*_{A_n/k} = H.$$

From (19) and (20) obviously

$$\alpha \overline{\in} H.$$

Thus we lead to a contradiction, and the lemma is proved.

Theorem 4.

Let K/k be any finite extension, then

(22) $$G(k/K) = H(K\overline{\Omega}/K).$$

Proof. Obviously

$$G(k/K) < H(K\overline{\Omega}/K).$$

Conversely, we take an element Θ from $H(K\overline{\Omega}/K)$ and put

$$\theta = N_{K/k}\Theta.$$

We assume $\theta \neq 1$ and lead to a contradiction. If $\theta \neq 1$ there exists an abelian extension A of k such that

(23) $$\theta \overline{\in} N^*_{A/k}.$$

From (16) follows

$$\Theta \in N^*_{AK/K}.$$

Therefore, using the *Verschiebungssatz* we get

$$N_{K/k}\Theta = \theta \in N^*_{A/k}.$$

This contradicts with (23), hence we have

$$N_{K/k}\Theta = \theta = 1, \qquad \Theta \in G(k/K).$$ q. e. d.

As an immediate consequence of this theorem, using the ordering theorem of the local class field theory, we get one of Chevalley's results [2] :

Corollary. *Let k be a p-adic number field and K be its finite extension field. When we take a finite abelian extension A of K, then A/k is abelian, if and only if*

$$G(k/K) < N^*_{A/K}.$$

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