Notes on Fourier Analysis (XVI).*)

By

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This paper consists of four independent parts. The first is devoted to the study of the Cesàro summability of Fourier series, the second to the divergence character of Fourier series, the third to a new definition of the Stieltjes integral, and the last to a certain series of functions.

Part I.

§ **1.** It is well known that, if

$$(1) \qquad \int_0^t |\varphi_x(u)|\, du = o(t),$$

then the Fourier series of $f(t)$ is summable (C, k) $(k>0)$ at x.

The condition (1) may be replaced by the more general condition

$$(2) \qquad \int_0^t \varphi_x(u)\, du = o(t),$$

$$(3) \qquad \int_0^t |\varphi_x(u)|\, du = O(t).$$

But these conditions does not depends on the order k of summability. On the other hand, the Hardy-Littlewood condition is that for $(C, k+\varepsilon)$ summability, but not for (C, k) summability. It will be interesting to find the (C, k) summability condition depending on k, which becomes as weaker as k increases. In this case, it must be remarked that (1) holds almost everywhere, so that the seeked condition must also be so for k.

§ **2. Theorem 1.** *If*

$$(2) \qquad \int_0^t \varphi_x(u)\, du = o(t)$$

and

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$$(4) \qquad \int_{\pi/n}^{\pi} \frac{|\varphi_x(t)-\varphi_x(t-\pi/n)|}{t^2} dt = o(n),$$

then the Fourier series of $f(t)$ is summable $(C, 1)$ at x.

Proof. We have

$$I \equiv f(x)-\sigma_n(x) = \frac{1}{2\pi(n+1)} \int_0^{\pi} \varphi_x(t) \frac{\sin^2 (n+1)t/2}{\sin^2 t/2} dt$$

$$= \frac{1}{2\pi n} \int_0^{\pi} \varphi_x(t) \frac{\sin^2 nt}{t^2} dt + o(1).$$

As usual we divide the last integral into two parts, that is,

$$I' \equiv \frac{1}{2\pi n} \int_0^{\pi} \varphi_x(t) \frac{\sin^2 nt}{t^2} dt = \frac{1}{2\pi n} \left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right) \varphi_x(t) \frac{\sin^2 nt}{t^2} dt \equiv J_1 + J_2,$$

say. By integration by parts we have

$$J_1 = \frac{1}{2\pi n} \int_0^{\pi/n} \varphi_x(t) \frac{\sin^2 nt}{t^2} dt$$

$$= \frac{1}{2\pi n} \left\{ \left[\Phi_x(t) \frac{\sin^2 nt}{t^2} \right]_0^{\pi/n} - n \int_0^{\pi/n} \Phi_x(t) \frac{\sin^2 nt}{t^2} dt + 2 \int_0^{\pi/n} \Phi_x(t) \frac{\sin^2 nt}{t^3} dt \right\}$$

$$= o(1) + o\left(\int_0^{\pi/n} n \, dt \right) + o\left(\frac{1}{n} \int_0^{\pi/n} n^2 \, dt \right) = o(1)$$

by (1), where $\Phi_x(t) = \int_0^t \varphi_x(u) \, du$.

$$J_2 = \frac{1}{2\pi n} \int_{\pi/n}^{\pi} \varphi_x(t) \frac{\sin^2 nt}{t^2} dt$$

$$= \frac{1}{4\pi n} \int_{\pi/n}^{\pi} \varphi_x(t) \frac{dt}{t^2} - \frac{1}{4\pi n} \int_{\pi/n}^{\pi} \varphi_x(t) \frac{\cos 2nt}{t^2} dt$$

$$\equiv \frac{1}{4\pi} (K_1 - K_2),$$

say. Now, by (1), we have

$$K_1 = \frac{1}{n}\int_{\pi/n}^{\pi} \varphi_x(t)\frac{dt}{t^2}$$

$$= \frac{1}{n}\left\{\left[\frac{\Phi_x(t)}{t^2}\right]_{\pi/n}^{\pi} + 2\int_{\pi/n}^{\pi}\frac{\Phi_x(t)}{t^3}\,dt\right\}$$

$$= \frac{1}{n}\left\{o(n) + o\left(\int_{\pi/n}^{\pi}\frac{dt}{t^2}\right)\right\} = o(1).$$

On the other hand

$$K_2 = \frac{1}{n}\int_{\pi/n}^{\pi}\varphi_x(t)\,\frac{\cos 2nt}{t^2}\,dt$$

$$= -\frac{1}{n}\int_{3\pi/2n}^{\pi+\pi/2n}\varphi_x\left(t-\frac{\pi}{2n}\right)\frac{\cos 2nt}{(t-\pi/2n)^2}\,dt,$$

$$2K_2 = \frac{1}{n}\int_{\pi/n}^{3\pi/2n}\varphi_x(t)\frac{\cos 2nt}{t^2}\,dt - \frac{1}{n}\int_{\pi}^{\pi+\pi/2n}\varphi_x\left(t-\frac{\pi}{2n}\right)\frac{\cos 2nt}{(t-\pi/2n)^2}\,dt$$

$$+\frac{1}{n}\int_{3\pi/2n}^{\pi}\left\{\frac{\varphi_x(t)}{t^2} - \frac{\varphi_x(t-\pi/2n)}{(t-\pi/2n)^2}\right\}\cos 2nt\,dt$$

$$\equiv L_1 - L_2 + L_3,$$

say. We have easily $L_1 = o(1)$ and $L_2 = o(1)$.

$$L_3 = \frac{1}{n}\int_{3\pi/2n}^{\pi}\left\{\frac{\varphi_x(t)}{t^2} - \frac{\varphi_x(t-\pi/2n)}{(t-\pi/2n)^2}\right\}\cos 2nt\,dt$$

$$= \frac{1}{n}\int_{3\pi/2n}^{\pi}\frac{\varphi_x(t)-\varphi_x(t-\pi/2n)}{t^2}\cos 2nt\,dt$$

$$+\frac{\pi}{2n^2}\int_{3\pi/2n}^{\pi}\varphi_x(t-\pi/2n)\frac{2t-\pi/2n}{t^2(t-\pi/2n)^2}\,dt$$

$$\equiv M_1 + M_2,$$

say. By integration by parts and (1), we have $M_2 = o(1)$, and

$$M_1 = o\left(\frac{1}{n}\int_{\pi/n}^{\pi}\frac{|\varphi_x(t)-\varphi_x(t-\pi/2n)|}{t^2}\,dt\right) = o(1)$$

by (4).

We remark that (4) is derived from (1). For

$$\int_{\pi/n}^{\pi} \frac{|\varphi_x(t)-\varphi_x(t-\pi/n)|}{t^2}\,dt \leqq 2\int_{\pi/n}^{\pi} \frac{|\varphi_x(t)|}{t^2}\,dt$$

$$=2\left\{\left[\frac{\Phi_x(t)}{t^2}\right]_{\pi/n}^{\pi}+\int_{\pi/n}^{\pi}\frac{\Phi_x^*(t)}{t^3}\,dt\right\}=o(n),$$

where $\Phi_x^*(t)=\int_0^t|\varphi_x(u)|\,du$. Hence (2) and (4) holds almost everywhere. But (4) and (3) are mutually exclusive.

§ **3. Theorem 2.** *If*

$$\text{(2)} \qquad \int_0^t \varphi_x(u)\,du=o(t)$$

and

$$\text{(5)} \qquad \int_{\pi/n}^{\pi}\frac{|\varphi_x(t)-\varphi_x(t-\pi/n)|}{t^{1+k}}\,dt=o(n^k),$$

then the Fourier series of $f(t)$ *is summable* (C,k) *at* x, k *being* >-1[1].

We prove the case $-1<k<0$, since the contrary case is similarly and more easily proved.

For the proof we need a lemma due to Szegö :

Lemma. *The n-th Casàro mean of order* $k\,(0>k>-1)$ *of the series*

$$1/2+\cos x+\cos 2x+\cdots\cdot+\cos nx+\cdots\cdot$$

becomes

$$\frac{\cos\left[\left(n+\frac{k+1}{2}\right)x-\frac{k+1}{2}\pi\right]}{A_n^k\,(2\sin x/2)^{k+1}}+\frac{k}{2(n+1)}\sum_{\mu=1}^{\infty}p_{\mu,n}\left(\frac{\sin\mu x/2}{\sin x/2}\right)^2$$

where $(p_{\mu,n})$ *is a positive sequence such that*

$$\sum_{\mu=1}^{\infty}p_{\mu,n}=1,\quad \sum_{\mu=1}^{\infty}\mu\,p_{\mu,n}=O(n).$$

We will now prove Theorem 2.

$$\Delta\equiv\sigma_n^k(x)-f(x)=\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)K_n^{(k)}(t)\,dt$$

$$=\frac{1}{\pi}\left(\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right)\varphi_x(t)K_n^{(k)}(t)\,dt\equiv I+J,$$

say. For positive k, $|K_n^{(k)}(t)|\leqq 2n$ and then $I=o(1)$. For $0>k>-1$ we will use the expression in Lemma of the kernel $K_n^{(k)}(t)$. Then

$$\Delta=\int_0^{\pi}\varphi_x(t)\frac{\cos\left[\left(n+\frac{k+1}{2}\right)t-\frac{k+1}{2}\pi\right]}{A_n^k(2\sin t/2)^{k+1}}dt$$

$$+\frac{k}{2(n+1)}\sum_{\mu=1}^{\infty}p_{\mu,n}\int_0^{\pi}\varphi_x(t)\left(\frac{\sin\mu t/2}{\sin t/2}\right)^2dt\equiv K+L,$$

say. Since

$$\frac{1}{n}\int_{\pi/n}^{\pi}\frac{|\varphi_x(t)-\varphi_x(t-\pi/n)|}{t^2}dt\leqq\frac{1}{n^k}\int_{\pi/n}^{\pi}\frac{|\varphi_x(t)-\varphi_x(t-\pi/n)|}{t^{1+k}}dt.$$

We get

$$\int_0^{\pi}\varphi_x(t)\left(\frac{\sin\mu t/2}{\sin t/2}\right)^2dt=o(\mu),$$

by Theorem 1, and then

$$Q=o\left(\frac{1}{n+1}\sum_{\mu=1}^{\infty}\mu p_{\mu,n}\right)=o(1).$$

Thus we have

$$I=\int_0^{\pi/n}\varphi_x(t)\frac{\cos\left[\left(n+\frac{k+1}{2}\right)t-\frac{k+1}{2}\pi\right]}{A_n^k(2\sin t/2)^{k+1}}dt+o(1)$$

$$=\left[\Phi_x(t)\frac{\cos\left[\left(n+\frac{k+1}{2}\right)t-\frac{k+1}{2}\pi\right]}{A_n^k(2\sin t/2)^{k+1}}\right]_0^{\pi/n}$$

$$+\left(n+\frac{k+1}{2}\right)\int_0^{\pi/n}\Phi_x(t)\frac{\sin\left[\left(n+\frac{k+1}{2}\right)t-\frac{k+1}{2}\pi\right]}{A_n^k(2\sin t/2)^{k+1}}dt$$

$$+(k+1)\int_0^{\pi/n}\Phi_x(t)\frac{\cos\left[\left(n+\frac{k+1}{2}\right)t-\frac{k+1}{2}\pi\right]}{A_n^k(2\sin t/2)^{k+2}}\cos\frac{t}{2}dt+o(1)$$

$$=o(1).$$

Thus we have $I=o(1)$.

Now

$$\Delta=K+o(1)$$

$$= \int_{\pi/\left(n+\frac{k+1}{2}\right)}^{\pi} \varphi_x(t) \frac{\cos\left[\left(n+\frac{k+1}{2}\right)t - \frac{k+1}{2}\pi\right]}{A_n^k (2\sin t/2)^{k+1}} dt + o(1).$$

In the estimation of the last integral, we can follow the line of calculation of K_2 in the proof of Theorem 1.

The case $k=0$ is the Lebesgue's convergence criterion of the Fourier series and the case $k=1$ is Theorem 1. (1) implies (5) for $k>0$, but not for $k \leqq 0$. In fact (4) is not the local property for $k<0$, which, combined with (6), is consistent with the fact that the Cesàro summability of Fourier series of negative order is not the local property.

Incidentally we have proved that

Theorem 3. *If* (2) *and* (4) *holds, then the necessary and sufficient condition that the Fourier series of $f(t)$ is* (C,k) *summable* $(0>k>-1)$, *is that*

$$\lim_{n\to\infty} \frac{1}{n^k} \int_0^{\pi} \varphi_x(t) \frac{\cos\left[\left(n+\frac{k+1}{2}\right)t - \frac{k+1}{2}\pi\right]}{(2\sin t/2)^{k+1}} dt = 0.$$

Part II.

§ 1. **Partial sum of Fourier series.** It is well known that the Fourier series of $f(t)$ is summable (C,δ) $(\delta>0)$ at a point x, provided that

$$(1) \qquad \int_0^t |\varphi_x(u)|\,du = o(t), \quad \varphi_x(t) = f(x+t)+f(x-t)-2f(x).$$

Then the condition (1) may be replaced by the following more general one :

$$(2) \qquad \int_0^t \varphi_x(u)\,du = o(t), \quad \int_0^t |\varphi_x(u)|\,du = O(t).$$

The generalization of this kind was done in analogous problems by many writers. Importance of such generalization lies in that, if the latter of the condition (2) is supposed, then the first becomes necessary in such theorems.

We will show that such generalizations are sometimes impossible. Let $s_n(x)$ be the n-th partial sum of Fourier series of $f(x)$. It is well known that

$$(3) \qquad s_n(x) = o(\log n)$$

under the condition (1). But (3) does not hold under the condition (2)[2].

For the proof[3], let (n_k) and (μ_k) be the increasing sequences of odd integers and they will be determined later. Let us put

$$N_k \equiv n_0 n_1 \cdots n_k, \quad M_k \equiv N_k \mu_k$$

and I_k be the interval $(\pi/N_k, \pi/N_{k-1})$. Finally, let $f(t)$ be an even function such that

$$f(2t)/2 \equiv \sin M_k t \quad (t \varepsilon I_k).$$

Let $\pi/N_k < t \leqq \pi/N_{k-1}$. Then

$$\int_{I_p} \sin M_p u\, du = \int_{\pi/N_p}^{\pi/N_{p-1}} \sin M_p u\, du = \frac{1}{N_p}\int_{\pi}^{n_p \pi} \sin \mu_p u\, du = 0,$$

$$\left| \int_{\pi/N_k}^{t} \sin M_k u\, du \right| \leqq \frac{1}{M_k} \left| \int_{\mu_k \pi}^{M_k t} \sin u\, du \right| \leqq \frac{\pi}{M_k} \leqq \frac{t}{\mu_k}.$$

Since $\mu_k \to \infty$, we have

$$\int_0^t f(u)\, du = O(t).$$

On the other hand, the boundedness of $f(x)$ implies

$$\int_0^t |f(u)| du = o(t).$$

We will now show that $s_n(0)$ is not $o(\log n)$. Putting $s_n(0) \equiv s_n$, we have

$$s_n = \frac{2}{\pi}\int_0^{\pi} f(u) \frac{\sin (n+1/2)u}{2 \sin u/2} du = \frac{1}{\pi}\int_0^{\pi/2} f(2u) \frac{\sin (2n+1) u}{\sin u} du$$

$$= \sum_{k=1}^{\infty} \frac{1}{\pi} \int_{I_k} \sin M_k u \frac{\sin (2n+1)u}{\sin u} du.$$

Let us put $n \equiv (M_k - 1)/2 \equiv m_k$, and

$$\pi s_{m_k} \equiv \sum_{p=1}^{k} + \sum_{p=k+1}^{\infty} \equiv I + J.$$

For $p > k$

$$i_p \equiv \int_{I_p} \sin M_p t \frac{\sin M_k t}{\sin t} dt$$

$$= \int_{I_p} \frac{\cos(M_p - M_k)t}{t} dt - \int_{I_p} \frac{\cos (M_p + M_k)t}{t} dt + o\left(\frac{1}{N_p}\right)$$

$$= \int_{\pi(M_p-M_k)/N_{p-1}}^{\pi(M_p-M_k)/N_{p-1}} \frac{\cos t}{t}\, dt - \int_{\pi(M_p+M_k)/N_{p-1}}^{\pi(M_p+M_k)/N_{p-1}} \frac{\cos t}{t}\, dt + o\left(\frac{1}{N_p}\right)$$

$$= \int_{\pi\left(\frac{M_p}{N_{p-1}}-\frac{M_k}{N_{p-1}}\right)}^{\pi\left(\frac{M_p}{N_{p-1}}+\frac{M_k}{N_{p-1}}\right)} \frac{\cos t}{t}\, dt - \int_{\pi\left(\frac{M_p}{N_p}-\frac{M_k}{N_p}\right)}^{\pi\left(\frac{M_p}{N_p}+\frac{M_k}{N_p}\right)} \frac{\cos t}{t}\, dt + o\left(\frac{1}{N_p}\right)$$

$$= o\left(\frac{N_{p-1}}{M_p}\frac{M_k}{N_{p-1}}\right) + o\left(\frac{N_p}{M_p}\frac{M_k}{N_p}\right) + o\left(\frac{1}{N_p}\right) = o\left(\frac{M_k}{M_p}\right) + o\left(\frac{1}{N_p}\right).$$

Thus we have

$$2J = \sum_{p=k+1}^{\infty} i_p = O\left(M_k \sum_{p=k+1}^{\infty} \frac{1}{M_p}\right) + O\left(\sum_{p=k+1}^{\infty} \frac{1}{N_p}\right)$$

$$= O\left(\frac{M_k}{M_{k+1}}\right) + O\left(\frac{1}{N_{k+1}}\right) = O\left(\frac{1}{\mu_{k+1}\, n_{k+1}}\right) + O\left(\frac{1}{N_{k+1}}\right) = o(1).$$

Let us estimate I.

$$I = \sum_{p=1}^{k} \int_{I_p} \sin M_k t \frac{\sin M_k t}{\sin t}\, dt = \sum_{p=1}^{k} i_p .$$

For $p < k$ we have

$$|i_p| \leqq \int_{I_p} \frac{dt}{\sin t} = \int_{I_p} \frac{dt}{t} + O\left(\frac{1}{N_{p-1}}\right) = \log n_p + O\left(\frac{1}{N_{p-1}}\right).$$

We have also

$$i_k = \int_{I_k} \frac{\sin^2 M_k t}{\sin t}\, dt = \frac{1}{2}\int_{I_k} \frac{dt}{t} - \frac{1}{2}\int_{I_k} \frac{\cos 2M_k t}{t}\, dt + o(1)$$

$$= \frac{1}{2} \log n_k - O(1).$$

Thus we have $$I \geqq \frac{1}{2} \log n_k - \sum_{i=1}^{k-1} \log n_p - O(1),$$

and then

$$\pi\, s_{m_k} \geqq \frac{1}{2} \log n_k - \sum_{i=1}^{k-1} \log n_p - O(1).$$

If $n_k > n^5_{k-1}$ $(k = 1, 2, \cdots)$, then $\pi\, s_{mk} \geqq (1/8) \log m_k$. This shows that $s_n \neq o(\log n)$, which is the required.

§ 2. **The Hardy-Lettlewood problem.** Hardy and Lettlewood proved that, if $f(x)$ satisfies the condition

$$\int_0^t |\varphi_x(u)|\,du = o\left(t\Big/\log\frac{1}{t}\right) \tag{4}$$

and the Fourier coefficients of $f(t)$ are $O(1/n^\delta)$ $(\delta > 0)$, then the Fourier series of $f(t)$ converges at $t = x$. They proposed whether (4) may be replaced by

$$\int_0^t \varphi_x(u)\,du = o\left(t\Big/\log\frac{1}{t}\right),\quad \int_0^t |\varphi_x(u)|\,du = O\left(t/\log\frac{1}{t}\right). \tag{5}$$

We can answer this problem negatively.

For the proof we take (n_k), (μ_k), (M_k) and (N_k) as in the begining of § 1, and put $c_k \equiv 1/\log N_k$. Let $f(t)$ be an even function such that

$$f(2t)/2 = c_k \sin M_k t \quad (t \in I_k).$$

Then $f(t)$ is continuous at $t = 0$. For t in $I_k \equiv (\pi/N_k, \pi/N_{k-1})$,

$$\int_0^t f(u)\,du = c_k \int_{\pi/N_k}^t \sin M_k u\,du = o\left(\frac{1}{M_k \log N_k}\right)$$

$$= O\left(\frac{t}{\mu_k}\Big/\log\frac{1}{t}\right) = o\left(t\Big/\log\frac{1}{t}\right),$$

and also

$$\int_0^t |f(u)|\,du = c_k \int_{\pi/N_k}^t |\sin M_k u|\,du + \sum_{p=k+1}^{\infty} c_p \int_{I_p} |\sin M_p u|\,du$$

$$\leqq c_k t + \pi \sum_{p=k+1}^{\infty} \frac{c_p}{N_{p-1}} = o\left(t\Big/\log\frac{1}{t}\right).$$

Thus the above defined function satisfies the condition (5).

Let us estimate s_{M_k} as in § 1. Dividing s_{M_k} into I and J, we can see easily $J = o(1)$. We will now estimate I. For $p < k$

$$i_p \equiv 2\int_{I_p} \sin M_p t \,\frac{\sin M_k t}{\sin t}\,dt$$

$$= \int_{I_p} \frac{\cos (M_k - M_p)\,t}{t}\,dt - \int_{I_p} \frac{\cos (M_k + M_p)\,t}{t}\,dt + o\left(\frac{1}{N_p}\right)$$

$$= \int_{\pi\left(\frac{M_k}{N_p}-\frac{M_p}{N_p}\right)}^{\pi\left(\frac{M_k}{N_{p-1}}-\frac{M_p}{N_{p-1}}\right)} \frac{\cos t}{t} dt - \int_{\pi\left(\frac{M_k}{N_p}+\frac{M_p}{N_p}\right)}^{\pi\left(\frac{M_k}{N_{p-1}}+\frac{M_p}{N_{p-1}}\right)} \frac{\cos t}{t} dt + o\left(\frac{1}{N_p}\right)$$

$$= O\left(\frac{N_{p-1}}{M_k}\frac{M_p}{N_{p-1}}\right) + O\left(\frac{N_p}{M_k}\frac{M_p}{N_p}\right) + O\left(\frac{1}{N_p}\right) = O\left(\frac{M_p}{M_k}\right) + O\left(\frac{1}{N_p}\right)$$

Thus

$$\sum_{p=3}^{k-1} i_p = O\left(\frac{1}{M_k}\sum_{p=3}^{k-1}\frac{1}{M_p}\right) + O\left(\sum_{p=3}^{k-1}\frac{1}{N_p}\right) + O\left(\frac{M_{k-1}}{M_k}\right) + \frac{1}{5} \leqq \frac{2}{5}.$$

Hence if we take n_k such as $n_k = n_{k-1}^3$, we have

$$\pi\, s_{M_k} \geqq \frac{c_k}{2}\log n_k - \frac{2}{5} + o(1) \geqq \frac{1}{2}\frac{\log n_k}{\log N_k} - \frac{2}{5} - o(1) \geqq \frac{1}{20} + o(1).$$

Thus s_n does not converge.

Concerning Fourier coefficients of $f(t)$ we have

$$\pi\, a_n = \int_0^\pi f(t) \cos nt\, dt = \sum_{p=3}^{\infty} c_p \int_{I_p} \sin M_p t \cos nt\, dt.$$

If $N_{k+1} > M_k > N_k$, $N_{k+1} > n \geqq N_k$, then we put

$$\pi\, a_n = c_k \int_{I_k} \sin M_k t \cos nt\, dt + r_k,$$

where the integral term is

$$c_k \int_{I_k} \sin M_k t \cos nt\, dt = O\left(\frac{1}{(M_k - n)\log N_k}\right) = O\left(\frac{1}{N_{k-1}}\right) = O(1/n^{1/9})$$

for $|M_k - n| > N_{k-1}$, and for $|M_k - n| \leqq N_{k-1}$ the left hand side is

$$O\left(\frac{1}{|M_k - n|}\right)\int_0^{\pi |M_k - n|/N_{k-1}} dt\Big) = O\left(\frac{1}{N_{k-1}}\right) = O\left(\frac{1}{n^{1/9}}\right).$$

Since r_k is of lower order, we have completed the proof.

§ 3. **Jump of functions.** Lukàcs proved that, if

$$(6) \qquad \int_0^t |\psi_x(u) - l(x)|\, du = o(t), \quad \psi_x(u) = f(x+u) - f(x-u),$$

then

$$\lim_{n\to\infty} \bar{s}_n(x)/\log n = -l(x)/\pi,$$

where $\bar{s}_n(x)$ denotes the n-th partial sum of conjugate Fourier series of $f(x)$. Mr. Matsuyama proposed whether (6) may be replaced by

$$\int_0^t \psi_x((u) - l(x))\, du = o(t), \int_0^t |\psi_x(u) - l(x)|\, du = O(t)$$

or not[4]. This can be answered negatively. For this proof it is sufficient to take $f(t)$ as odd function and $c_k \equiv (-1)^k$ in the example in § 2.

§ 4. The Riesz summability of the derived Fourier series. It is known that if[6]

$$\int_0^t \frac{\psi_x(t) - 2st}{t}\, dt = o(t),$$

$$(7) \qquad \int_0^t \frac{|\psi_x(t) - 2st|}{t}\, dt = O(t),$$

then the derived Fourier series of $f(t)$ is summable $(C, 1+\varepsilon)$ $(\varepsilon > 0)$, but not summable $(C, 1)$. Therefore it arises the problem whether it is summable by the Riesz logarithmic mean of order 1, or not under the condition (7)[7].

But this can also be answered negatively.

Part III.

§ 1. Let $f(x)$ and $g(x)$ be integrable functions with period 2π such that

$$(1) \qquad \int_0^{2\pi} f(x)\, dx = \int_0^{2\pi} g(x)\, dx = 0$$

and their Fourier series be

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$g(x) \sim \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx).$$

If $f(x)$ is continuous in $(0, 2\pi)$ and $g(x)$ is of bounded variation, then we have

$$(2) \qquad \int_0^{2\pi} f(x)\, dg(x) = \pi \sum_{n=1}^{\infty} n(a_n \beta_n - b_n \alpha_n) \quad (C, 1),$$

which means that the right-hand side series is summable $(C, 1)$ to the left hand side integral. This relation holds for Young-Stieltjes integral[8], that is, if $f \varepsilon V_p$, $g \varepsilon V_q$ $(1/p + 1/q > 1)$, then (2) holds. Therefore we can adopt (2) as the definition of the Stieltjes integral[9].

In general, let $f(x)$ be an integrable function defined in $(0, 2\pi)$ and $g(x)$ be continuous there, and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$g(x) \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx).$$

The Stieltjes integral of $f(x)$ with respect to $g(x)$ is defined by

$$(3) \quad \int_0^{2\pi} f(x)\, dg(x) = \frac{1}{2\pi} \{g(2\pi) - g(0)\} \int_0^{2\pi} f(x)\, dx + \pi \sum_{n=1}^{\infty} n\, (a_n \beta_n - b_n \alpha_n) \ (C, 1).$$

§ 2. In the following we will consider the case when (1) is satisfied. The general case can be treated quite similarly.

Let $K_n(x)$ be the Fejér Kernel, that is,

$$K_n(x) = \frac{1}{2(n+1)} \frac{\sin^2 (n+1)x/2}{\sin^2 x/2}.$$

Then (2) becomes

$$\int_0^{2\pi} f(x)\, dg(x) = \lim_{n \to \infty} \frac{1}{\pi} \int_0^{2\pi} f(x) dx \int_0^{2\pi} g(y) K_n'(x-y)\, dy.$$

Thus (4) may be used as definition. Since $K_n'(x)$ has singularity at $x = 0$, the integral is a sort of singular integral. Singular integrals are used as representation of functions, but it is rare to use them as the definition of the integral. Definition of the fractional integral are on this line.[10]

§ 3. Let us consider the existence condition of our integral. If $f(x)$ is bounded almost everywhere and $g(x)$ is differentiable almost everywhere, with integrable differential coefficient, then the Stieltjes integral exists. For, the right hand side of (2) may be written as

$$\lim_{n \to \infty} \int_0^{2\pi} \sigma_n(x, f)\, g'(x)\, dx,$$

where $\sigma_n(x, f)$ denotes the n-th arithmetic mean of the Fourier series of

$f(x)$. Since $\sigma_n(x, f)$ is essentially bounded, above limit exists.

Since the definition of the integral is symmetric with respect to $f(x)$ and $g(x)$, it is desirable to find the symmetric existence condition. For this purpose we prove the following theorem.

Theorem 1. *If*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(x)g(y)-f(y)g(x)|}{(x-y)^2}\,dx\,dy<\infty,$$

then the Stieltjes integral of $f(x)$ with respect to $g(x)$ exists and is equal to zero.

Proof. It is sufficient to prove that

$$I\equiv\lim\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx\int_{-\pi}^{\pi}g(y)\,K_n'(x-y)\,dy=0.$$

If we put $x-y\equiv 2u$, $x+y\equiv 2v$, then $x=u+v$, $y=v-u$, and

$$I=\lim_{n\to\infty}\frac{2}{\pi}\int_{-\pi\sqrt{2}}^{\pi\sqrt{2}}dv\int_{-\pi\sqrt{2}+v}^{\pi\sqrt{2}+v}f(u+v)\,g(v-u)\,K_n'(2u)\,du,$$

where

$$K_n'(2u)=\frac{1}{2(n+1)}\cdot\frac{\sin^2(n+1)u}{\sin^2 u}$$

$$=-\frac{\cos u}{(n+1)\sin^3 u}\sin^2(n+1)u+\frac{\sin 2(n+1)u}{2\sin^2 u}$$

$$=-L_n(u)+M_n(u),$$

say. Now

$$I_n'\equiv\int_{-\pi\sqrt{2}}^{\pi\sqrt{2}}dv\int_{-\pi\sqrt{2}+v}^{\pi\sqrt{2}-v}f(v+u)\,g(v-u)\,L_n(u)\,du$$

$$=\int_{\pi-\sqrt{2}}^{\pi\sqrt{2}}dv\int_{0}^{n\sqrt{2}-v}F(u,v)\,L_n(u)\,du,$$

where

$$F(u,v)\equiv f(v+u)\,g(v-u)-f(v-u)\,g(v+u).$$

Dividing the inner integral of I_n',

$$I_n'=\int_0^{\pi\sqrt{2}}dv\left(\int_0^{\pi/(n+1)}+\int_{\pi/(n+1)}^{\pi\sqrt{2}-v}\right)du\equiv P_n+Q_n,$$

say. Then we have

$$|P_n| \leqq \int_0^{\pi\sqrt{2}} dv \int_0^{\pi/(n+1)} |F(u,\ v)| \frac{n+1}{u} du = (n+1) \int_0^{\pi/(n+1)} \frac{G(u)}{u} du$$

$$= (n+1) \int_0^{\pi/(n+1)} \frac{G(u)}{u^2} u\, du \leqq \int_0^{\pi/(n+1)} \frac{G(u)}{u^2} du$$

where we put

$$G(u) \equiv \int_{-\pi/2}^{\pi/2} |F(u,\ v)|\, dv.$$

By the hypothesis

$$\int_0^{\pi\sqrt{2}} \frac{G(u)}{u^2} du \leqq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x)g(y) - f(y)g(x)|}{(x-y)^2} dx\, dy < \infty.$$

Thus $P_n = o(1)$.

We have

$$|Q_n| \leqq \int_{-\pi\sqrt{2}}^{\pi\sqrt{2}} dv \int_{\pi/(n+1)}^{\pi\sqrt{2}-v} |F(u,\ v)| \frac{du}{(n+1)u^3}$$

$$= \frac{\pi}{n+1} \int_{\pi/(n+1)}^{\pi\sqrt{2}} \frac{du}{u^3} \int_0^{\pi\sqrt{2}-u} |F(u,\ v)|\, dv = o(1).$$

Thus we have proved $I_n' = P_n + Q_n = o(1)$.

On the other hand

$$I_n'' \equiv \int_{-\pi\sqrt{2}}^{\pi\sqrt{2}} dv \int_{-\pi\sqrt{2}+v}^{\pi\sqrt{2}-v} f(u+v)\, g(v-u)\, M_n(u)\, du$$

$$= \int_{-\pi\sqrt{2}}^{\pi\sqrt{2}} dv \int_0^{\pi\sqrt{2}-v} F(u,\ v)\, M_n(u)\, du$$

$$= \int_0^{\pi\sqrt{2}} \frac{\sin 2(n+1)u}{\sin^2 u} du \int_{-\pi\sqrt{2}+v}^{\pi\sqrt{2}-v} F(u,\ v)\, dv.$$

We see $I_n'' = o(1)$ by the Riemann-Lebesgue theorem. Thus we have

$$I = \lim_{n\to\infty} (I_n' + I_n'') = 0,$$

which is the required.

Theorem 2. *If there is an s such that*

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x)g(y) - f(y)g(x) - 2s(x-y)|}{(x-y)^2} dx\, dy < \infty,$$

then the Stielies integral of $f(x)$ with respect to $g(x)$ exists.

Proof. If we consider

$$I_n - \frac{s}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(x-y)\,K_n\,(x-y)\,dx\,dy$$

instead of $I_n \equiv I_n' + I_n''$, then we can proceed as the proof of Theorem 1.

§ 4. If $f(x)$ and $g(x)$ satisfy the condition of Theorem 2, then (3) and the proof of Theorem 1 imply

$$\frac{1}{2h}\int_{x-h}^{x+h} f(x)\,dg(x) = \frac{g(x+h)-g(x-h)}{(2h)^2}\int_{x-h}^{x+h} f(t)\,dt + o(1)$$

as $h \to 0$. If $f(x)$ is continuous,

$$\frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dg(t) = f(x)\frac{g(x+h)-g(x-h)}{2h} + o(1).$$

This is the differential property of the Stieltjes integral gotten by Burkill for Young-Stieltjes integral[8].

§5. We will now extend the above method to the Hellinger integral. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

$$g(x) \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}(\alpha_n \cos nx + \beta_n \sin nx).$$

Since the Hellinger integral $\int \frac{df(x)\,dg(x)}{dx}$ is ordinarily defined as the limit of

$$\sum_{i=1}^{n}\frac{[f(x_i)-f(x_{i-1})][g(x_i)-g(x_{i-1})]}{x_i - x_{i-1}}$$

as the norm of division (x_i) tends to zero, we can suppose that $a_0 = \alpha_0 = 0$. If

$$(1) \qquad \pi\sum_{n=1}^{\infty} n^2(a_n\alpha_n + b_n\beta_n)$$

converges in the $(C, 1)$ sense, then we say that $f(x)$ and $g(x)$ are integrable in the Hellinger sense, and denote the integral by

$$(2) \qquad \int_{-\pi}^{\pi}\frac{df(x)\,dg(x)}{dx}.$$

We can easily see that, if $g(x)$ is absolutely continuous, then the above integral reduces to the Stieltjes integral, defined in § 1. The $(C, 1)$ mean of (1) is given by

$$(3) \qquad \pi^2 \lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\, dx \int_{-\pi}^{\pi} g(t)\, K_n''(x-t) dt.$$

Corresponding to Theorem 1, we have

Theorem 3. *If*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \frac{f(x)\,g(y) - f(y)\,g(x)}{(x-y)^3} \right| dx\,dy < \infty,$$

then the integral (2) *exist and is equal to zero.*

Proof. If is sufficient to prove that the limit (3) is equal to zero. Puttnig $x-t=2u$, $x+t=2v$, the integral in (3) becomes as $h \to 0$.

$$(4) \qquad \frac{2}{\pi} \int_{-\pi\sqrt{2}}^{\pi\sqrt{2}} dv \int_{-\pi\sqrt{2}+v}^{\pi\sqrt{2}-v} f(u+v)\, g(v-u)\, K_n''(2u)\, du.$$

Now,

$$K_n''(2u) = \frac{\sin^2(n+1)u}{(n+1)\sin^2 u} - \frac{\cos u \sin^2(n+1)u}{\sin^3 u} + \frac{3\cos^2 u \sin^2(n+1)u}{\sin^4 u}$$

$$+ 2(n+1) \frac{\cos^2(n+1)u}{2\sin^2 u} - \frac{\cos u \sin^2(n+1)u}{\sin^3 u}$$

$$\equiv L_1(u) + L_2(u) + L_3(u) + L_4(u) + L_5(u),$$

say. Let I_i be the integral (4), replaced $K_n'(2u)$ by $L_i(u)$. Then, putting $F(u,v) \equiv f(v+u)\,g(v-u) - f(v-u)\,g(v+u)$,

$$I_3 = \int_{-\pi\sqrt{2}}^{\pi\sqrt{2}} du \int_{0}^{\pi\sqrt{2}-v} F(u,v)\, L_3(u)\, du$$

$$= \int_{-\pi\sqrt{2}}^{\pi\sqrt{2}} du \left(\int_0^{\pi/(n+1)} + \int_{\pi/(n+1)}^{\pi\sqrt{2}-v} \right) du \equiv P + Q,$$

say. Concerning P, we have

$$|P| = \left| \frac{3}{n+1} \int_{-\pi\sqrt{2}}^{\pi\sqrt{2}} dv \int_0^{\pi/(n+1)} F(u,v)\, \frac{\cos^2 u \sin^2(n+1)u}{\sin^4 u}\, du \right|$$

$$\leqq 6\,(n+1) \int_0^{\pi\sqrt{2}} du \int_0^{\pi/(n+1)} |F(u,v)|\, \frac{du}{u^2}$$

$$\leqq 6\pi \int^{\pi\sqrt{2}} du \int_0^{\pi/(n+1)} \frac{|F(u,v)|}{u^3}\, du,$$

which is $o(1)$ as $n\to\infty$, by the hypothesis. We have also

$$|Q|\leqq\int_{-\pi\sqrt{2}}^{\pi\sqrt{2}}du\int_{-\pi\sqrt{2}+v}^{\pi\sqrt{2}-v}|F(u,v)|\frac{du}{(n+\ )u^4}$$

$$\leqq\frac{2}{n+1}\int_{\pi/(n+1)}^{\pi/v2}\frac{du}{u^4}\int_0^{\pi\sqrt{2}-u}|F(u,v)|du=o(1)$$

Thus $I_3=P+Q=o(1)$. Similarly we have $I_1+I_2+I_4+I_5=o(1)$. Hence the theorem ie proved.

Theorem 4. If

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\frac{f(x)g(y)-f(y)g(x)-s(x-y)^{2/2}}{(x-y)^3}\right|dxdy<\infty,$$

the Hellinger integral (3) *exists and equal to s.*

Part IV.

§1. M. Kac proved the following theorem(1).

Theorem. *Let $\varphi(x)$ be a periodic function with period 2π, belonging to the class* Lip α $(0<\alpha\leqq1)$, *such as*

$$(1)\qquad \int_0^{2\pi}\varphi(x)\,dx=0.$$

If n_k is the integral multiple of 2π and $n_{k+1}/n_k>q>1$ $(k=1,2,\cdots\cdots)$, and $\sum_{k=1}^{\infty}c_k^2<\infty$, then the series

$$(2)\qquad \sum_{k=1}^{\infty}c_k\,\varphi(n_k x)$$

converges almost everywhere to a function in (L^2).

In this part we consider the series (2) with $\varphi(x)$ belonging to Lip (α,r). The class Lip (α,r) $(0<\alpha\leqq1,\ r>1)$ consists of functions $\varphi(x)$ such as

$$\left(\int_0^{2\pi}|\varphi(x+t)-\varphi(x)|^r dx\right)^{1/r}=O(t^\alpha),\ (t>0).$$

More generally we consider the class Lip (α,β,r) which consists of function $\varphi(x)$ such as

$$\left(\int_0^{2\pi} |\varphi(x+t)-\varphi(x)|^r dx\right)^{1/r} = O\left(t^\alpha \log^3 \frac{1}{t}\right), \qquad (t>0).$$

Especially we concern the class Lip $(0, \beta, r)$.

§ 2. Theorem 1. *Let $\varphi(x)$ be a periodic function with period 2π, belonging to the class* Lip $(\alpha, 2)$ $(0<\alpha\leqq 1,\ r>1)$. *If*

$$m_{k+1}/m_k > p > 1,\ \ n_{k+1}/n_k > q > 1\ \ (k = 1, 2, \cdots,)$$

and

$$\sum_{k=1}^{\infty} c_k^2 < \infty,$$

then the sequence

$$\sum_{k=1}^{m_i} c_k \varphi(n_k x)$$

converges almost everywhere to a function in (L^2).

Lemma 1. *For $\varphi(x)$ satisfying the condition of Theorem* 1,

$$\left|\int_0^{2\pi} \varphi(n_i x)\varphi(n_j x)dx\right| \leqq A/q^{\alpha|i-j|}.$$

Proof. Let $i<j$. By the periodicity of $\varphi(x)$, we have

$$I \equiv \int_0^{2\pi} \varphi(n_j x)\,\varphi(n_i x)dx = \int_0^{2\pi} \varphi(n_j x)\,\varphi(n_i x + 2\pi k n_i/n_j)\,dx$$

for $k = 1, 2, \cdots$. Let N be the greatest integer such as $Nn_i/n_j \leqq 1$. Putting $\xi \equiv 2\pi N n_i/n_j$, we have

$$I = \int_0^{2\pi} \varphi(n_j x)\left[\frac{1}{N}\sum_{k=1}^{N} \varphi\left(n_i x + \frac{2\pi k n_i}{n_j}\right)\right] dx$$

$$= \int_0^{2\pi} \varphi(n_j x)\left[\frac{1}{N}\sum_{k=1}^{N} \varphi\left(n_i x + \frac{2\pi k n_i}{n_j}\right) - \int_0^{2\pi} \varphi(n_i x + t)\,dt\right] dx$$

$$= \int_0^{2\pi} \varphi(n_j x)\left[\frac{1}{N}\sum_{k=1}^{N}\int_{2\pi k n_i/n_j}^{2\pi(k+1)n_i/n_j} \left\{\varphi\left(n_i x + \frac{2\pi k n_i}{n_j}\right) - \varphi(n_i x + t)\right\}dt\right.$$

$$\left. + \int_\xi^{2\pi} \varphi(n_j x + t)\,dt\right] dx.$$

Now

$$\left|\int_{2\pi k n_i/n_j}^{2\pi(k+1)n_i/n_j} dt \int_0^{2\pi} \varphi(n_j x)\left\{\varphi\left(n_i x + \frac{2\pi k n_i}{n_j}\right) - \varphi(n_i x + t)\right\} dx\right|$$

$$\leqq \int_{2\pi k n_i/n_j}^{2\pi(k+1)n_i/n_j} dt \left(\int_0^{2\pi} \varphi^2(n_j x)\,dx\right)^{1/2} \left(\int_0^{2\pi} \left\{\varphi\left(n_i x + \frac{2\pi k n_i}{n_j}\right) - \varphi(n_i x + t)\right\}^2 dx\right)^{1/2}$$

$$\leqq \int_{2\pi k n_i/n_j}^{2\pi(k+1)n_i/n_j} \left(\frac{n_i}{n_j}\right)^{\alpha} dt.$$

Since $2\pi - \xi \leqq n_i/n_j$, we have

$$|I| \leqq A\,(n_i/n_j)^{\alpha} \leqq A/q^{\alpha(j-i)},$$

which was to be proved.

Lemma 2. *Under the assumption of Theorem 1 the series (2) converges in the (L^2)-mean.*

Proof. Let $1 \leqq m \leqq n$.

$$\int_0^{2\pi} \left(\sum_{k=m}^{n} c_k \varphi(n_k x)\right)^2 dx = \sum_{j,k=m}^{n} c_j c_k \int_0^{2\pi} \varphi(n_j x)\varphi(n_k x)\,dx$$

$$\leqq A \sum_{j,k=m}^{n} \frac{|c_j|\cdot|c_k|}{q^{\alpha|k-j|}} \leqq A \sum_{r=0}^{n-m} \sum_{s=m+r}^{n} \frac{|c_s|\cdot|c_{s-r}|}{q^{\alpha r}}$$

$$\leqq A \sum_{r=0}^{n-m} \frac{1}{q^{\alpha r}} \sum_{s=m}^{n} c_s^2 \leqq A \sum_{r=0}^{\infty} \frac{1}{q^{\alpha r}} \sum_{s=m}^{n} c_s^2 \to 0 \quad (m, n \to \infty).$$

Poof of Theorem 1. We can suppose that

$$\varphi(x) \sim \sum_{\nu=1}^{\infty} a_\nu e^{i\nu x}$$

and put

$$s_n(x) \equiv \sum_{\nu=1}^{n} a_\nu e^{i\nu x}.$$

Hence

$$\varphi(n_k x) \sim \sum_{\nu=1}^{\infty} a_\nu e^{i\nu n_k x}.$$

By $\varphi \in \text{Lip}\,(\alpha, 2)$

$$\left(\int_0^{2\pi} [\varphi(n_k x) - s_{\mu_k}(n_k x)]^2\,dx\right)^{1/2} \leqq \left(\int_0^{2\pi} [\varphi(x) - s_{\mu_k}(x)]^2\,dx\right)^{1/2}$$

$$\leqq A/\mu_k^{\alpha} \quad (k = 1, 2, \cdots).$$

If we take $\mu_k \equiv k^{(1+\epsilon)\alpha}$, then

$$\int_0^{2\pi} \left(\sum_{k=1}^{\infty} |c_k \psi(n_k x)|\right)^2 dx \leqq \sum_{k=1}^{\infty} c_k^2 \cdot \sum_{k=1}^{\infty} \int_0^{2\pi} \psi(n_k x)^2\,dx \leqq A \sum_{k=1}^{\infty} c_k^2,$$

where $\psi(n_k x) \equiv \varphi(n_k x) - s_{\mu_k}(n_k x)$. Thus the series $\sum c_k \psi(n_k x)$ converges almost everywhere. By Lemma 2, the series

$$\sum_{k=1}^{\infty} c_k s_{\mu_k}(n_k x)$$

converges in the (L^2)-mean. Let us take p such as

$$\mu_{m_{k+1}} > p\, \mu_{m_k} \ (k = 1, 2, \cdots\cdots),$$

and put

$$c_i' \equiv 0 \ \ (m_{2^{\nu-1}} < i \leqq m_{2^\nu}), \ c_i' \equiv c_i \ \ (m_{2^\nu} < i \leqq m_{2^{\nu+1}}),$$

$$c_i \equiv c_i' + c_i'' \ (i = 1, 2, \cdots\cdots),$$

then the series

$$(3) \qquad \sum_{k=1}^{\infty} c_k' s_{\mu_k}(n_k x), \quad \sum_{k=1}^{\infty} c_k'' s_{\mu_k}(n_k x)$$

converge in the (L^2)-mean. The m_{2i+1}-th partial sum of the first series of (3) is the $\mu_{m_{2i+1}}$-th partial sum of Fourier of the function represented by the series. Hence, by the Kolmogoroff theorem[13], the sequence

$$\sum_{k=1}^{m_{2i+1}} c_k' s_{\mu_k}(n_k x)$$

converges almost everywhere. Similarly, concerning the second series of (3), the sequence

$$\sum_{k=1}^{m_{2i}} c_k'' s_{\mu_k}(n_k x)$$

converges almost everywhere. Hence

$$\sum_{k=1}^{m_i} c_k \varphi(n_k x) = \sum_{k=1}^{m_i} c_k' \varphi(n_k x) + \sum_{k=1}^{m_i} c_k'' \varphi(n_k x)$$

converges almost everywhere.

Theorem 2. *In Theorem* 1, *the condition* $\varphi \in \mathrm{Lip}\,(\alpha, 2)$ *may be replaced by* $\varphi \in \mathrm{Lip}\,(0, \beta, 2)$ $(\beta > 1)$.

Theorem 3. *In Theorem* 1, *we can replace the condition of* (m_k) *and* (n_k) *by*

$$0 < p < n_k/k^\gamma < q \ (\gamma < 1, \ k = 1, 2, 3, \cdots\cdots),$$

$$m_{k+1} > m_k^{1+(1+\varepsilon)2\alpha\gamma} \quad (\varepsilon > 0, \ k = 1, 2, \cdots\cdots).$$

§ 3. Theorem 4. *If* $\varphi(x)$, (m_i) *and* (n_i) *satisfy the conditions in Theorem* 1 *or* 2, *then*

$$\left(\int_0^{2\pi} \left[\sup_i \sum_{k=1}^{m_i} c_k \varphi(n_k x)\right]^2 dx\right)^{1/2} \leqq C \sum_{k=1}^{\infty} c_k^2 .$$

Proof is contained in that of Theorem 1.

Theorem 5. *Under the conditions of Theorem* 1 *or* 2,

$$\sum_{i=1}^{\infty}\left|\sum_{k=m_i+1}^{m_{i+1}} c_k \varphi(n_k x)\right|^2 < \infty$$

almost everywhere.

Proof. Theorem 1 holds good even if we replace c_k by $\pm c_k$ $(k = 1, 2, \dots)$. If $\{r_k(u)\}$ denotes the Rademacher system, then, for almost all u,

$$\limsup_{n\to\infty} \sum_{k=1}^{n}\left(\sum_{j=m_k+1}^{m_{k+1}} c_j \varphi(n_j x)\right) r_k(u) < \infty$$

for almost all x. Hence, by Fubini's Theorem, we have, for almost all x,

$$\limsup_{n\to\infty} \sum_{k=1}^{n}\left(\sum_{j=m_k+1}^{m_{k+1}} c_j \varphi(n_j x)\right) r_k(u) < \infty$$

for almost all u, and then the series of the square of coefficients converges for almost all x, which is the required.

§4. We will now prove some category theorems.

Theorem 6. *If* $n_{k+1}/n_k > q > 1$ *and* $\sum_{k=1}^{\infty} c_k^2 < \infty$, *then there is a measurable set* L *in* $(0, 2\pi)$ *such that*

1) *For all* $\varphi \in \mathrm{Lip}\,(0, \beta, r)$ *the series* (2) *converges almost everywhere in* L, and

2) *For all* $\varphi \in \mathrm{Lip}\,(0, \beta, r)$, *except a set of the first category, we have*

$$\limsup_{n\to\infty}\left|\sum_{k=1}^{n} c_k \varphi(n_k x)\right| < \infty$$

almost everywhere in CL.

Proof. If we put

$$u_n(\varphi) \equiv \sum_{k=1}^{n} c_k \varphi(n_k x),$$

then $u_n(\varphi)$ is linear in Lip $(0, \beta, r)$, norm being that of (L^2). Since the set of trigonometrical polynomials is a dense set in Lip $(0, \beta, r)$, and the series

$$\sum_{k=1}^{\infty} c_k \cos n_k x$$

converges almost everywhere, $u_n(\varphi)$ converges in a dense set of Lip $(0, \beta, r)$. By a theorem due to Saks, we get the theorem.

Theorem 7. *If* $n_k | n_{k+1}$ $(k = 1, 2, \dots)$ *and* $\sum_{k=1}^{\infty} c_k^2 < \infty$, *then the following* *two cases are possible*:

1) *For all* $\varphi \in \mathrm{Lip}\,(0, \beta, r)$ (2) *converges almost everywhere.*

2) *For all* $\varphi \in \mathrm{Lip}\,(0, \beta, r)$, *except a set of the first category,*

$$\limsup_{m,n\to\infty} \left| \sum_{k=m}^{n} c_k \varphi(n_k t) \right| = +\infty \tag{4}$$

almost everywhere.

Proof. Without loss of generality we can suppose $r = 2$. Let us suppose that there exists $\varphi_0 \in \mathrm{Lip}\,(0, \beta, 2)$ such that $\sum_{k=1}^{\infty} c_k \varphi(n_k t)$ does not converge almost everywhere. Since $n_k | n_{k+1}$,

$$\limsup_{m,n\to\infty} \left| \sum_{k=m}^{n} c_k \varphi_0(n_k t) \right| = +\infty \tag{5}$$

almost everywhere. In order to prove the theorem, it is sufficient to prove that the set H of φ such that (5) does not hold, is of the first category. If we put

$$\Phi_{M,\eta} \equiv \left(\varphi;\ \mathrm{meas}\left(t; \left| \sum_{k=m}^{n} c_k \varphi(n_k t) \right| \leqq M\,(m < n;\ m, n = 1, 2, \cdots)\right) \geqq \eta\right),$$

then $H = \vee(\Phi_{r,1/s};\ r, s = 1, 2, \cdots)$. Now $\Phi_{M,\eta}$ are closed. For, $\int_0^{2\pi} |\varphi - \psi|^2 dx < \varepsilon^2$ implies

$$\int_0^{2\pi} \left(\sum_{k=m}^{n} c_k [\varphi(n_k t) - \psi(n_k t)] \right)^2 dt \leqq \varepsilon \cdot \mathrm{const.},$$

which may be seen from the proof of Theorem 1 and 2. If we put

$$E \equiv \left(t; \left| \sum_{k=m}^{n} c_k [\varphi(n_k t) - \psi(n_k t)] \right| \geqq M_1 \right),$$

then we have $|E| \cdot M_1^2 < \varepsilon \cdot$ const. Thus $C\,\Phi_{M,\eta}$ is open, and then $\Phi_{M,\eta}$ is closed.

It remains to prove that $\Phi_{M,n}$ is non-dense. Otherwise $\Phi_{M,\eta}$ contains a sphere S. Since trigonometrical polynomials form a set D dense in Lip $(0, \beta, 2)$, there are a $w \in D$ and $r > 0$ such that

$$\left(\varphi;\ \int_0^{2\pi} |\varphi - w|^2 dt \leqq r^2 \right) \supseteqq S.$$

Since $\{n_k\}$ has the Hadamard gap, $\sum c_k w(n_k t)$ converges almost everywhere. $\int^{2\pi} |\varphi_0|^2\, dt \leqq r^2$ and the hypotheses imply that $\psi \equiv \varphi_0 + w \in S$. On the otherhand, by (5) $\psi \overline{\in} \Phi_{M,\eta}$, which contradicts $S \subseteqq \Phi_{M,n}$.

Foot-Notes.

1) Mr. G. Sunouchi kindly remarked me that analogue of Theorem 2 is proved by J.J. Gergen in Qurterly Journal of Math., vol. 1(1930). Since we can easily see, from the proof, that the condition may be replaced by (C, r) mean of $f(t)$ is $o(1)$ and difference in (5) may be replaced by the difference of any order, Theorem 2 may be written in the Gergen form.

2) This problem was proposed by Mr. G. Sunouchi, whom the author expresses his hearty thanks.

3) Cf. Lebesgue, Séries trigonométriques, 1906, p. 85.

4) Matsuyama, Real Analysis Monthly, vol. 1, No. 6 (1946) (in Japanese). cf. O. Szàsz, Trans. Am. Math. Soc., 42 (1942) and S. Izumi, Journal of Math. Soc., vol. 1, No. 2 (1948).

5) S. Izumi, Tôhoku Math. Journ., 28 (1930).

6) Concenning this result the author owes much to Messrs N. Matsuyama, G. Sunouchi and S. Yano.

7) L. C. Young, Acta Math., 67 (1937); Burkill, Journ. London Math. Soc., 23 (1948).

8) Cf. S. Izumi and T. Kawata, Tôhoku Math. Journ., 44 (1938).

9) Zygmund, Trigonometrical series.

10) M. Kac, Annals of Math., 43 (1942).

11) We can prove this lemma by the method of M. Kac, but our Method allows us a generalization used in Theorem 4.

12) For the general case we use the Littlewood-Paley Theorem.

13) This theorem contains a theorem due to T. Kawata and the author, Tôhoku Math. Journ., 1940.

14) After written up this paper, A. Zygmund sent me the paper due to him, Kac and Salem, Trans. Am. Math. Soc., 48 (1948), where is found a theorem near Theorem 2 in this parts. Author expresses his hearty thanks to Prof. A. Zygmund who gave him valuable remarks.