NOTES ON FOURIER ANALYSIS (XV) ON THE ABSOLUTE CONVERGENCE OF TRIGONOMETRICAL SERIES.*)[1]

By

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I. Let us conder the trigonometrical series

(1) $$\sum_{n=1}^{\infty} \rho_n \cos(nx-\alpha_n)$$

where $\rho_n \geqq 0$ $(n=1,2,\ldots\ldots)$ and

(2) $$\sum_{n=1}^{\infty} \rho_n = \infty.$$

We have proved that if

(3) $$\rho_n = O(1/n),$$

then the set of points where the series (1) converges absolutely is of α-caqacity zero $(0<\alpha<1)$[2].

We can now prove more precise result;

Theorem[3] If $\rho_n=O(1/n)$ and $\sum_{n=1}^{\infty}\rho_n=\infty$, then we have

(4) $$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \rho_k |\cos(kx-\alpha_k)|}{\sum_{k=1}^{n}\rho_k} = 2/\pi$$

except a set of α-capacity zero $(0<\alpha<1)$.

II. We shall firstly prove the following lemma.

Lemma. If (γ_n) is a sequence of complex quantities such that

$$\sum_{n=1}^{\infty} |\gamma_n| = \infty, \text{ and } |\gamma_n| = O(1/n),$$

then we have

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*) Received May 5th, 1946.

1) Read before the annual meeting of the Mathematical Society at May, 1946.

2) T. Tsuchikura and S. Yano, Notes on Fourier Analysis (V): Absolute convergence of trigonometrical series, under the press.

3) cf. R. Salem, The absolute convergence of trigonometrical series, Duke Math. Journ., 8 (1941).

(5) $$R_n(x)=\frac{\sum_{k=1}^{n}\gamma_k e^{ikx}}{\sum_{k=1}^{n}|\gamma_k|}\to 0$$

except a set of α-capacity zero.

Proof. Let E be the set of x such as $R_n(x)$ does not tend to zero. If we suppose that E is of α-capacity positive, then there is a positive distribution μ which concentrates on E and belongs to Lip α[4]. If we denote the Fourier-Stieltjes series of $\mu(x)$ by

$$d\mu(x)\sim 1/2\pi+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

then we have

(6) $$\sum_{n=1}^{\infty}(|a_n|+|b_n|)/n<\infty.$$

Now we have

$$\int_0^{2\pi}|R_n(x)|^2\,d\mu=2\pi\frac{\sum_{k=1}^{n}|\gamma_n|^2}{(\sum_{k=1}^{n}|\gamma_k|)^2}+2\frac{\int_0^{2\pi}\sum_{j\neq l}^{n}\gamma_j\bar{\gamma_j}e^{i(j-l)x}d\mu}{(\sum_{k=1}^{n}|\gamma_k|)^2}.$$

Since

$$\left|\int_0^{2\pi}\gamma^j\bar{\gamma}^l e^{i(j-l)x}\,d\mu\right|\leqq|\gamma_j|\,|\gamma_l|\,(|a_{j-l}|+|b_{j-l}|),$$

we have

$$\left|\sum_{j\neq l}^{n}\int_0^{2\pi}\gamma_j\gamma_l e^{j(l-l)x}\,d\mu\right|$$

$$\leqq\sum_{j\neq l}^{n}|\gamma_j|\,|\gamma_l|(|a_{j-l}|+|b_{j-l}|)=O(\sum_{k=1}^{n}|\gamma_\gamma|),$$

by (6) and $\gamma_n=O(1/n)$. Hence

$$\int_0^{2\pi}|R_n(x)|^2d\mu\leqq M/\sum_{k=1}^{n}|\gamma_k|,$$

M being a constant.

If we take a suquence of integers (n_ν) such that

$$\nu^2\leqq\sum_{k=1}^{n\nu}|\gamma_k|<(\nu+1)^2,$$

then the series $\sum_{k=1}^{\infty}\int_0^{2\pi}|R(x)|^2d\mu$ converges and then $\sum_{\nu=1}^{\infty}|R_{n\nu}(x)|^2$ converges

4) R. Salem and A. Zygmund, Capacity of sets and Fourier series, Trans. Amer. Math. Soc., 59 (1946), p. 23–41. Especially see the corollary of Theorem 1.

expect a set N of μ-measure zero.

In particular

$$R_{n_\nu}(x)\to 0$$

for $x \varepsilon$ N. For n such as $n_\nu \leqq n < n_{\nu+1}$, we have

$$\left| R_n(x)\sum_{k=1}^{n}|\gamma_k| - R_{n_\nu}(x)\sum_{k=1}^{n_\nu}|\gamma_k| \right| < \sum_{k=n_\nu}^{n_{\nu+1}}|\gamma_k|,$$

hence

$$\left| R_n(x) - \frac{\sum_{k=1}^{n_\nu}|\gamma_k|}{\sum_{k=1}^{n}|\gamma_k|} R_{n_\nu}(x) \right| \leqq \frac{\sum_{k=n_\nu+1}^{n_{\nu+1}}|\gamma_k|}{\sum_{k=1}^{n_\nu}|\gamma_k|}$$

$$= \frac{\sum_{k=1}^{n_{\nu+1}}|\gamma_k|}{\sum_{k=1}^{n_\nu}|\gamma_k|} - 1 < \frac{(\nu+1)^2}{\nu^2} - 1 = o(1),$$

which proves that $R_n(x)\to 0$ in E$-$N. This contradicts the definition of E. Thus the lemma is proved.

III. We will now prove the theorem.

Since

$$|\cos x| = \sum_{\nu=-\infty}^{+\infty} c_\nu e^{i\nu x} \quad (c_0 = 2/\pi)$$

with $|c_\nu| = O(\nu^{-2})$, we have

$$|\cos(nx-\alpha_n)| = \sum_{k=-\infty}^{+\infty} c_n\, e^{ik\alpha_n} e^{iknx}$$

and

$$\frac{\sum_{k=1}^{k} \rho_k |\cos(kx-\alpha_k)|}{\sum_{k=1}^{n}\rho_k} = \sum_{k=-\infty}^{+\infty} c_k\, Q_{k,n}(x),$$

where

$$Q_{k,n}(x) = \frac{\sum_{\nu=1}^{n} \rho_\nu\, e^{-ik\alpha_\nu} e^{ik\nu x}}{\sum_{\nu=1}^{n}\rho_\nu}.$$

By the lemma we have, for each $k \neq 0$,

(7) $$\lim_{n\to\infty} Q_{k,n}(x) = 0$$

except a set E_k of μ-measure zero. Hence there is a set E of μ-measure zero such that (7) holds good for any integer $k \neq 0$ expect x in E. Since $|Q_{k,n}(x)|$ are uniformly bounded and $|c_k| = O(1/k^2)$, we can easily see that

$$\frac{\sum_{k=1}^{n} \rho_k |\cos(kx - \alpha_k)|}{\sum_{k=1}^{n} \rho_k} \to c_0 = 2/\pi$$

except x in E. This proves the theorem.

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