Ulam-Hyers stability of undecic functional equation in quasi-β-normed spaces: Fixed point method

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Abstract

In this paper, we acquire the general solution of the undecic functional equation

$$\begin{aligned} f(x+6y) &- 11f(x+5y) + 55f(x+4y) - 165f(x+3y) + 330f(x+2y) \\ &- 462f(x+y) + 462f(x) - 330f(x-y) + 165f(x-2y) - 55f(x-3y) \\ &+ 11f(x-4y) - f(x-5y) = 39916800f(y). \end{aligned}$$

We also obtain the generalized Ulam-Hyers stability of the above functional equation in quasi-β-normed spaces using fixed point method. Moreover, we investigate the pertinent stabilities of the above functional equation using sum of powers of norms, product of powers of norms and mixed product-sum of powers of norms as upper bounds. We also present a counter-example for non-stability of the above functional equation in singular case.

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1 Introduction

The stability problem of functional equations originated from a famous "stability" question of Ulam [25] in 1940 concerning the stability of group homomorphisms. In fact, let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with a metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \varepsilon$ for all $x, y \in G_1$, then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$? Ulam's problem was affirmatively answered by D.H. Hyers [11] for Banach spaces. It was further generalized by a great number of mathematicians ([1], [24], [16], [17], [18], [10], [23]). For more detailed information about such problems on quadratic, cubic, quartic, quintic, sextic, septic, octic, nonic functional equations and other types of functional equations, one can see ([3], [4], [5], [6], [7], [8], [10], [12], [19], [20], [21], [22], [27], [30], [31], [32]).

In this paper, we find the general solution of undecic functional equation

$$\begin{aligned} f(x+6y) &- 11f(x+5y) + 55f(x+4y) - 165f(x+3y) + 330f(x+2y) \\ &- 462f(x+y) + 462f(x) - 330f(x-y) + 165f(x-2y) - 55f(x-3y) \\ &+ 11f(x-4y) - f(x-5y) = 39916800f(y). \end{aligned} \tag{1.1}$$

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Moreover, we obtain the generalized Ulam-Hyers stability of the undecic functional equation in quasi-β-normed spaces using fixed point method. Since $f(x) = x^{11}$ is a solution of (1.1), we say that it is an undecic functional equation. Every solution of the undecic functional equation is said to be an undecic mapping.

2 Preliminaries

In this section, we recall some basic concepts concerning quasi-β-normed spaces and m-additive symmetric mappings.

Let β be a fixed real number with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. Let X be a linear space over K. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following conditions:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda|^{\beta} \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that $\|x + y\| \leq K (\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

Definition 2.2. A quasi-β-Banach space is a complete quasi-β-normed space.

Definition 2.3. A quasi-β-norm $\|\cdot\|$ is called a (β, p)-norm $(0 < p < 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

Definition 2.4. A function $B : \mathbb{R} \to \mathbb{R}$ is said to be additive if $B(x + y) = B(x) + B(y)$, for all $x, y \in \mathbb{R}$.

Definition 2.5. A function $B_m : \mathbb{R}^m \to \mathbb{R}$ is called $m-$additive (for $m \in \mathbb{N}$) if it is additive in each of its variable.

Definition 2.6. A function B_m is called symmetric if

$$B_m(x_1, x_2, \ldots, x_m) = B_m\left(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(m)}\right),$$

for every permutation $\{\pi(1), \pi(2), \ldots, \pi(m)\}$ of $\{1, 2, \ldots, m\}$.

We denote the diagonal $B_m(x, x, \ldots, x)$ as $B^m(x)$, if $B_m(x_1, x_2, \ldots, x_m)$ is an $m-$additive symmetric map. If we substitute $x_1 = x_2 = \cdots = x_l = x$ and $x_{l+1} = x_{l+2} = \cdots = x_m = y$ in $B_m(x_1, x_2, \ldots, x_m)$, then we denote the resulting expression as $B^{l,m-l}(x, y)$.

The difference operator Δ_h for the function $f : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$\Delta_h f(x) = f(x+h) - f(x) \qquad \text{for } h \in \mathbb{R}$$

and

$$\Delta_h^0 f(x) = f(x), \Delta_h^1 f(x) = \Delta_h f(x) \text{ and } \Delta_h \circ \Delta_h^m f(x) = \Delta_h^{m+1} f(x)$$

for all $m \in \mathbb{N}$ and $h \in \mathbb{R}$, $\Delta_h \circ \Delta_h^m$ denotes the composition of the operators Δ_h and Δ_h^m. Using the above definition, consider the functional equation

$$\Delta_h^{m+1} f(x) = 0$$

which can be written in the explicit form as

$$\sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m+1}{k} f(x+kh) = 0$$

is equivalent to the Fr$\acute{e}$chet functional equation

$$\Delta_{h_1, h_2, \ldots, h_{m+1}} f(x) = 0,$$

where $x, h_1, h_2, \ldots, h_{m+1} \in \mathbb{R}$ and $\Delta_{h_1,\ldots,h_k} = \Delta_{h_k} \circ \cdots \circ \Delta_{h_1}$ for $k = 2, 3, \ldots, m+1$.

3 General solution of functional equation (1.1)

In this section, let X and Y be vector spaces. In the following theorem, we find the general solution of the undecic functional equation (1.1).

Theorem 3.1. A function $f : X \to Y$ is a solution of the functional equation (1.1) if and only if f is of the form $f(x) = A^{11}(x)$ for all $x \in X$, where $A^{11}(x)$ is the diagonal of the 11-additive symmetric map $A_{11} : X^{11} \to Y$.

Proof. Assume that f satisfies the functional equation (1.1). Replacing (x, y) by $(0, 0)$, one finds that $f(0) = 0$. Now, plugging (x, y) into $(0, x)$ and $(x, -x)$ in (1.1) and adding the two resulting equations, we get

$$f(-x) = -f(x), \tag{3.1}$$

for all $x \in X$. Substituting (x, y) by $(0, 2x)$ in (1.1) and using (3.1) , one obtains that

$$f(12x) - 10f(10x) + 44f(86x) - 110f(6x) + 165f(4x) - 39916932f(2x) = 0 \tag{3.2}$$

for all $x \in X$. Considering (x, y) as $(6x, x)$ in (1.1), one gets

$$f(12x) - 11f(11x) + 55f(10x) - 165f(9x) + 330f(8x) - 462f(7x) + 462f(6x) - 330f(5x) + 165f(4x) - 55f(3x) + 11f(2x) - 39916801f(x) = 0 \quad (3.3)$$

for all $x \in X$. Subtracting equation (3.3) from (3.2), we find

$$11(11x) - 65f(10x) + 165f(9x) - 286f(8x) + 462f(7x) - 572f(6x) + 330f(5x) + -55f(3x) - 39916943f(2x) + 39916801f(x) = 0 \quad (3.4)$$

for all $x \in X$. Letting (x, y) as $(5x, x)$ in (1.1), one gets

$$11(11x) - 121f(10x) + 605f(9x) - 1815f(8x) + 3630f(7x) - 5082f(6x) + 5082f(5x) - 3630f(4x) + 1815f(3x) - 605f(2x) - 439084679f(x) = 0 \quad (3.5)$$

for all $x \in X$. Now, subtracting equation (3.5) from (3.4), we obtain

$$56f(10x) - 440(9x) + 1529f(8x) - 3168f(7x) + 4510f(6x) - 4752f(5x) + 3630f(4x) - 1760f(3x) - 39916338f(2x) + 479001480f(x) = 0. \quad (3.6)$$

for all $x \in X$. Plugging (x, y) into $(4x, x)$ in (1.1), and multiplying the resulting equation by 56, one obtains

$$56f(10x) - 616f(9x) + 3080f(8x) - 9240f(7x) + 18480f(6x) - 25872f(5x) - 18480f(3x) + 9240f(2x) - 2235343824f(x) = 0 \quad (3.7)$$

for all $x \in X$. Subtracting equation (3.7) from (3.6), we get

$$176f(9x) - 1551f(8x) + 6072f(7x) - 13970f(6x) + 21120f(5x) - 22242f(4x) + 16720f(3x) - 39925578f(2x) + 2714345304f(x) = 0 \quad (3.8)$$

for all $x, y \in X$. Replacing (x, y) with $(3x, x)$ in (1.1), and multiplying the resulting equation by 176, one finds

$$176f(9x) - 1936f(8x) + 9680f(7x) - 29040f(6x) + 58080f(5x) - 81312f(4x) + 81312f(3x) - 57904f(2x) - 7025329696f(x) = 0 \quad (3.9)$$

for all $x \in X$. Subtracting equations (3.8) and (3.9), we arrive at

$$385f(8x) - 3608f(7x) + 15070f(5x) - 36960f(5x) + 59070f(4x) \\ - 64592f(3x) - 39867674f(2x) + 9739675000f(x) = 0 \tag{3.10}$$

for all $x \in X$. Taking (x, y) as $(2x, x)$ in (1.1), and multiplying the resulting equation by 385, one gets

$$385f(8x) - 4235f(7x) + 21175f(6x) - 63525f(5x) + 127050f(4x) \\ - 177485f(3x) + 173636f(2x) - 15368073875f(x) = 0 \tag{3.11}$$

for all $x \in X$. Subtracting equation (3.11) from (3.10), we arrive at

$$627f(7x) - 6105f(6x) + 26564f(5x)067980f(4x) \\ + 112893f(3x) - 40041309f(2x) + 25107748875f(x) = 0 \tag{3.12}$$

for all $x \in X$. Substituting (x, y) as (x, x) in (1.1), and multiplying the resulting equation by 627, one finds

$$627f(7x) - 6897f(6x) + 34485f(5x) - 102828f(4x) \\ + 200013f(3x) - 255189f(2x) - 25027647381f(x) = 0 \tag{3.13}$$

for all $x \in X$. Subtracting equation (3.13) from (3.12), we arrive at

$$792f(6x) - 7920f(5x) + 34848f(4x) - 87120f(3x) \\ - 39786120f(2x) + 50135396256f(x) = 0 \tag{3.14}$$

for all $x \in X$. Replacing (x, y) with $(0, x)$ in (1.1), and multiplying the resulting equation by 792, one obtains

$$792f(6x) - 7920(5x) + 34848f(4x) - 87120f(3x) \\ + 130680f(2x) - 31614210144f(x) = 0 \tag{3.15}$$

for all $x \in X$. Subtracting equation (3.15) from (3.14), we arrive at

$$f(2x) = 2^{11}f(x) \tag{3.16}$$

for all $x \in X$.

On the other hand, one can rewrite the functional equation (1.1) in the form

$$\begin{aligned} f(x) &+ \frac{1}{462}f(x+6y) - \frac{11}{462}f(x+5y) + \frac{55}{462}f(x+4y) - \frac{165}{462}f(x+3y) \\ &+ \frac{165}{462}f(x+2y) - f(x+y) - \frac{165}{231}f(x-y) + \frac{165}{462}f(x-2y) \\ &- \frac{55}{462}f(x-3y) + \frac{11}{462}f(x-4y) - \frac{1}{86400}f(y) = 0, \end{aligned} \tag{3.17}$$

for all $x \in X$. By Theorems 3.5 and 3.6 in [29], f is a generalized polynomial function of degree at most 11, that is, f is of the form

$$\begin{aligned} f(x) &= A^{11}(x) + A^{10}(x) + A^9(x) + A^8(x) + A^7(x) + A^6(x) \\ &+ A^5(x) + A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x), \quad \forall x \in X, \end{aligned} \tag{3.18}$$

where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 1, 2, 3, \ldots, 11$. By $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$ and the function f is odd. Thus we have $A^{10}(x) = A^8(x) = A^6(x) = A^4(x) = A^2(x) = 0$. It follows that $f(x) = A^{11}(x) + A^9(x) + A^7(x) + A^5(x) + A^3(x) + A^1(x)$. By (3.16) and $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$, we obtain

$$\begin{aligned} &2^{11}\left(A^{11}(x) + A^9(x) + A^7(x) + A^5(x) + A^3(x) + A^1(x)\right) \\ &\quad = 2^{11}A^{11}(x) + 2^9A^9(x) + 2^7A^7(x) + 2^5A^5(x) + 2^3A^3(x) + 2A^1(x). \end{aligned}$$

Moreover, $2^{11}A^9(x) + 2^{11}A^7(x) = 2^9A^9(x) + 2^7A^7(x)$ yields $A^7(x) = -4A^9(x)/5$. Also, $2^9A^7(x) + 2^9A^5(x) = 2^9A^7(x) + 2^7A^5(x)$ implies $A^5(x) = -4A^7(x)/5 = 16A^9(x)/25$. Similarly, $2^7A^5(x) + 2^7A^3(x) = 2^5A^5(x) + 2^3A^3(x)$ gives $A^3(x) = -4A^5(x)/5 = -64A^9(x)/125$. Further, $2^5A^3(x) + 2^5A^1(x) = 2^3A^3(x) + 2A^1(x)$ arrives at $A^1(x) = -4A^3(x)/5 = 256A^9(x)/625$. It follows that $A^9(x) = A^7(x) = A^5(x) = A^3(x) = A^1(x) = 0$, $x \in X$. Therefore, $f(x) = A^{11}(x)$.

Conversely, assume that $f(x) = A^{11}(x)$ for all $x \in X$, where $A^{11}(x)$ is the diagonal of the 11-additive symmetric map $A_{11} : X^{11} \to Y$. From $A^{11}(x+y) = A^{11}(x) + A^{11}(y) + 11A^{10,1}(x,y) + 55A^{9,2}(x,y) + 165A^{8,3}(x,y) + 330A^{7,4}(x,y) + 462A^{6,5}(x,y) + 462A^{5,6}(x,y) + 330A^{4,7}(x,y) + 165A^{3,8}(x,y) + 55A^{2,9}(x,y) + 11A^{1,10}(x,y)$, $A^{11}(rx) = r^{11}A^{11}(x)$, $A^{10,1}(x,ry) = rA^{10,1}(x,y)$, $A^{9,2}(x,ry) = r^2A^{9,2}(x,y)$, $A^{8,3}(x,ry) = r^3A^{8,3}(x,y)$, $A^{7,4}(x,ry) = r^4A^{7,4}(x,y)$, $A^{6,5}(x,ry) = r^5A^{6,5}(x,y)$, $A^{5,6}(x,ry) = r^6A^{5,6}(x,y)$, $A^{4,7}(x,ry) = r^7A^{4,7}(x,y)$, $A^{3,8}(x,ry) = r^8A^{3,8}(x,y)$, $A^{2,9}(x,ry) = r^9A^{2,9}(x,y)$, $A^{1,10}(x,ry) = r^{10}A^{1,10}(x,y)$ $(x, y \in X, r \in \mathbb{Q})$, we see that f satisfies (1.1), which completes the proof of Theorem 3.1. Q.E.D.

4 Generalized Ulam-Hyers stability of equation (1.1)

Throughout this section, we assume that X is a linear space and Y is a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_Y$. Let K be the modulus of concavity of $\|\cdot\|_Y$. For notational convenience, we define the difference operator for a given mapping $f : X \to Y$ as

$$\begin{aligned} D_u f(x,y) = {} & f(x+6y) - 11f(x+5y) + 55f(x+4y) - 165f(x+3y) + 330f(x+2y) \\ & - 462f(x+y) + 462f(x) - 330f(x-y) + 165f(x-2y) \\ & - 55f(x-3y) + 11f(x-4y) - f(x-5y) - 39916800f(y) \end{aligned}$$

for all $x, y \in X$.

Lemma 4.1. (see [27]). Let $i \in \{-1, 1\}$ be fixed, $s, a \in \mathbb{N}$ with $a \geq 2$ and $\Psi : X \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\Psi\left(a^i x\right) \leq a^{is\beta} L \Psi(x)$ for all $x \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\|f(ax) - a^s f(x)\|_Y \leq \Psi(x) \tag{4.1}$$

for all $x \in X$, then there exists a uniquely determined mapping $F : X \to Y$ such that $F(ax) = a^s F(x)$ and

$$\|f(x) - F(x)\|_Y \leq \frac{1}{a^{s\beta}\left|1 - L^i\right|} \Psi(x) \tag{4.2}$$

for all $x \in X$.

Theorem 4.2. Let $i \in \{-1, 1\}$ be fixed. Let $\varphi : X \times X \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\varphi\left(2^i x, 2^i y\right) \leq 2048^{i\beta} L \varphi(x, y)$ for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\|D_u f(x,y)\|_Y \leq \varphi(x,y) \tag{4.3}$$

for all $x, y \in X$. Then there exists a unique undecic mapping $U : X \to Y$ such that

$$\|f(x) - U(x)\|_Y \leq \frac{1}{2048^{\beta}\left|1 - L^i\right|} \Psi(x) \tag{4.4}$$

for all $x \in X$, where

$$
\begin{aligned}
&\Psi(x)\\
&= \frac{1}{39916800^\beta}\Big[K^7\varphi(6x,x) + 11^\beta K^6\varphi(5x,x) + 56^\beta K^6\varphi(4x,x) + 176^\beta K^5\varphi(3x,x)\\
&+ 385^\beta K^4\varphi(2x,x) + 627^\beta K^3\varphi(x,x) + K^8\varphi(0,2x) + 792^\beta K^2\varphi(0,x)\\
&+ \left(\frac{K^9}{86400^\beta} + \frac{11^\beta K^7}{3628800^\beta} + \frac{56^\beta K^6}{3628800^\beta} + \frac{176^\beta K^5}{725760^\beta} + \frac{385^\beta K^4}{241920^\beta} + \frac{627^\beta K^3}{120960^\beta}\right)\varphi(0,0)\\
&+ \Big(\frac{56^\beta K^7}{39916800^\beta} + \frac{176^\beta K^6}{3628800^\beta} + \frac{385^\beta K^5}{725760^\beta} + \frac{627^\beta K^4}{241920^\beta}\\
&\qquad + \frac{792^\beta K^3}{120960^\beta}\Big)[\varphi(0,x) + \varphi(x,-x)]\\
&+ \Big(\frac{K^{10}}{120960^\beta} + \frac{176^\beta K^7}{39916800^\beta} + \frac{385^\beta K^6}{3628800^\beta} + \frac{627^\beta K^5}{725760^\beta}\\
&\qquad + \frac{792^\beta K^4}{241920^\beta}\Big)[\varphi(0,2x) + \varphi(2x,-2x)]\\
&+ \left(\frac{385^\beta K^7}{39916800^\beta} + \frac{627^\beta K^6}{3628800^\beta} + \frac{792^\beta K^5}{725760^\beta}\right)[\varphi(0,3x) + \varphi(3x,-3x)]\\
&+ \left(\frac{K^{11}}{241920^\beta} + \frac{627^\beta K^7}{39916800^\beta} + \frac{792^\beta K^6}{3628800^\beta}\right)[\varphi(0,4x) + \varphi(4x,-4x)]\\
&+ \frac{792^\beta K^7}{39916800^\beta}[\varphi(0,5x) + \varphi(5x,-5x)] + \frac{K^{12}}{725760^\beta}[\varphi(0,6x) + \varphi(6x,-6x)]\\
&+ \frac{K^{13}}{3628800^\beta}[\varphi(0,8x) + \varphi(8x,-8x)] + \frac{K^{14}}{39916800^\beta}[\varphi(0,10x) + \varphi(10x,-10x)]\Big].
\end{aligned}
$$

Proof. Substituting $x = y = 0$ in (4.3), we get

$$\|f(0)\|_Y \le \frac{1}{39916800^\beta}\varphi(0,0). \tag{4.5}$$

Now, replacing (x,y) by $(0,x)$ and $(x,-x)$in (4.3) and then adding the resulting equations, we obtain

$$\|f(x) + f(-x)\|_Y \le \frac{K}{39916800^\beta}[\varphi(0,x) + \varphi(x,-x)] \tag{4.6}$$

for all $x \in X$. Now, plugging (x,y) into $(0,2x)$ in (4.3), we obtain

$$
\begin{aligned}
\Big\| f(12x) - 11f(10x) + 55f(8x) - 165f(6x) + 330f(4x) - 39917162f(2x)\\
+ 462f(0) - 330f(-2x) + 165f(-4x) - 55f(-6x)\\
+ 11f(-8x) - f(-10x)\Big\|_Y \le \varphi(0,2x)
\end{aligned} \tag{4.7}
$$

for all $x \in X$. Using (4.5), (4.6) and (4.7), one finds

$$\begin{aligned}
&\|f(12x) - 10f(10x) + 44f(8x) - 110f(6x) + 165f(4x) - 39916932f(2x)\|_Y \\
&\leq K\varphi(0,2x) + \frac{K^2}{86400^\beta}\varphi(0,0) + \frac{K^3}{120960^\beta}[\varphi(0,2x) + \varphi(2x,-2x)] \\
&+ \frac{K^4}{241920^\beta}[\varphi(0,4x) + \varphi(4x,-4x)] + \frac{K^5}{725760^\beta}[\varphi(0,5x) + \varphi(6x,-6x)] \\
&+ \frac{K^6}{3628800^\beta}[\varphi(0,,8x) + \varphi(8x,-8x)] + \frac{K^7}{39916800^\beta}[\varphi(0,10x) + \varphi(10x,-10x)] \qquad (4.8)
\end{aligned}$$

for all $x \in X$. Letting (x, y) as $(6x, x)$ in (4.3) and subtracting the resulting equation from (4.8), we find

$$\begin{aligned}
&\Big\|11f(11x) - 65f(10x) + 165f(9x) - 286f(8x) + 462f(7x) - 572f(6x) \\
&\quad + 330f(5x) + 55f(3x) - 39916943f(2x) + 39916801f(x)\Big\|_Y \\
&\leq K\varphi(6x,x) + K^2\varphi(0,2x) + \frac{K^3}{86400^\beta}\varphi(0,x) + \frac{K^4}{120960^\beta}[\varphi(0,2x) + \varphi(2x,-2x)] \\
&+ \frac{K^5}{241920^\beta}[\varphi(0,4x) + \varphi(4x,-4x)] + \frac{K^6}{725760^\beta}[\varphi(0,6x) + \varphi(6x,-6x)] \\
&+ \frac{K^7}{3628800^\beta}[\varphi(0,8x) + \varphi(8x,-8x)] + \frac{K^8}{39916800^\beta}[\varphi(0,10x) + \varphi(10x,-10x)] \qquad (4.9)
\end{aligned}$$

for all $x \in X$. Considering (x, y) as $(5x, x)$ in (4.3) and multiplying the resulting equation by 11^β, we have

$$\begin{aligned}
&\Big\|11f(11x) - 121f(10x) + 605f(9x) - 1815f(8x) + 3630f(7x) - 5082f(6x) \\
&\quad + 5082f(5x) - 3630f(4x) + 1815f(3x) - 605f(2x) - 439084679f(x)\Big\|_Y \\
&\qquad \leq 11^\beta\varphi(5x,x) + \frac{11^\beta}{3628800^\beta}\varphi(0,0) \qquad (4.10)
\end{aligned}$$

for all $x \in X$. Subtracting (4.10) from (4.9), we get

$$\begin{aligned}
&\Big\| 56f(10x) - 440f(9x) + 1529f(8x) - 3168f(7x) + 4510f(6x) - 4752f(5x) \\
&\quad + 3630f(4x) - 1760f(3x) - 39916338f(2x) + 479001480f(x) \Big\|_Y \\
&\leq K^2\varphi(6x,x) + 11^\beta K\varphi(5x,x) + K^3\varphi(0,2x) + \left(\frac{K^4}{86400^\beta} + \frac{11^\beta K}{3628800^\beta}\right)\varphi(0,0) \\
&\quad + \frac{K^5}{120960^\beta}[\varphi(0,2x)+\varphi(2x,-2x)] + \frac{K^6}{241920^\beta}[\varphi(0,4x)+\varphi(4x,-4x)] \\
&\quad + \frac{K^7}{725760^\beta}[\varphi(0,6x)+\varphi(6x,-6x)] + \frac{K^8}{3628800^\beta}[\varphi(0,8x)+\varphi(8x,-8x)] \\
&\quad + \frac{K^9}{39916800^\beta}[\varphi(0,10x)+\varphi(10x,-10x)] \qquad (4.11)
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(4x, x)$ in (4.3), multiplying by 56^β and then using (4.5), (4.6) in the resulting equation, we obtain

$$\begin{aligned}
&\Big\| 56f(10x) - 616f(9x) + 3080f(8x) - 9240f(7x) + 18480f(6x) - 25872f(5x) \\
&\quad + 25872f(4x) - 18480f(3x) + 9240f(2x) - 2235343824f(x) \Big\|_Y \\
&\quad \leq 56^\beta K\varphi(4x,x) + \frac{56^\beta K}{3628800^\beta}\varphi(0,0) + \frac{56^\beta K^2}{39916800^\beta}[\varphi(0,x)+\varphi(x,-x)] \qquad (4.12)
\end{aligned}$$

for all $x \in X$. Subtracting (4.11) from (4.11), we obtain

$$\begin{aligned}
&\Big\| 176f(9x) - 1551f(8x) + 6072f(7x) - 13970f(6x) + 21120f(5x) - 22242f(4x) \\
&\qquad + 16720f(3x) - 39925578f(2x) + 2714345304f(x) \Big\|_Y \\
&\leq K^3\varphi(6x,x) + 11^\beta K^2\varphi(5x,x) + 56^\beta K^2\varphi(4x,x) + K^4\varphi(0,2x) \\
&+ \left(\frac{K^5}{86400^\beta} + \frac{11^\beta K^3}{3628800^\beta} + \frac{56^\beta K^2}{3628800^\beta}\right)\varphi(0,0) + \frac{56^\beta K^3}{39916800^\beta}[\varphi(0,x0+\varphi(x,-x)] \\
&\quad + \frac{K^6}{120960^\beta}[\varphi(0,2x)+\varphi(2x,-2x)] + \frac{K^7}{241920^\beta}[\varphi(0,4x)+\varphi(4x,-4x)] \\
&\quad + \frac{K^8}{725760^\beta}[\varphi(0,6x)+\varphi(6x,-6x)] + \frac{K^9}{3628800^\beta}[\varphi(0,8x)+\varphi(8x,-8x)] \\
&\quad + \frac{K^{10}}{39916800^\beta}[\varphi(0,10x)+\varphi(10x,-10x)] \qquad (4.13)
\end{aligned}$$

for all $x \in X$. Taking (x, y) as $(3x, x)$ in (4.3), multiplying by 176^β and then using (4.5), (4.6) in the resulting equation, we find

$$\begin{aligned}
&\Big\| 176f(9x) - 1936f(8x) + 9680f(7x) - 29040f(6x) + 58080f(5x) - 81312f(4x) \\
&\quad + 81312f(3x) - 57904f(2x) - 7025329696f(x)\Big\|_Y \\
&\leq 176^\beta K\varphi(3x,x) + \frac{176^\beta K}{725760^\beta}\varphi(0,0) + \frac{176^\beta K^2}{3628800^\beta}[\varphi(0,x) + \varphi(x,-x)] \\
&\qquad + \frac{176^\beta K^3}{39916800^\beta}[\varphi(0,2x) + \varphi(2x,-2x)] \qquad (4.14)
\end{aligned}$$

for all $x \in X$. Subtracting (4.14) from (4.13), we obtain

$$\begin{aligned}
&\Big\| 385f(8x) - 3608f(7x) + 15070f(6x) - 36960f(5x) + 59070f(4x) - 64592f(3x) \\
&\quad - 39867674f(2x) + 9739675000f(x)\Big\|_Y \\
&\leq K^4\varphi(6x,x) + 11^\beta K^3\varphi(5x,x) + 56^\beta K^3\varphi(4x,x) + 176^\beta K^2\varphi(3x,x) + K^5\varphi(0,2x) \\
&\quad + \left(\frac{K^6}{86400^\beta} + \frac{11^\beta K^4}{3628800^\beta} + \frac{56^\beta K^3}{3628800^\beta} + \frac{176^\beta K^2}{725760^\beta}\right)\varphi(0,0) \\
&\quad + \left(\frac{56^\beta K^4}{39916800^\beta} + \frac{176^\beta K^3}{3628800^\beta}\right)[\varphi(0,x) + \varphi(x,-x)] \\
&\quad + \left(\frac{K^7}{120960^\beta} + \frac{176^\beta K^4}{39916800^\beta}\right)[\varphi(0,2x) + \varphi(2x,-2x)] \\
&\quad + \frac{K^8}{241920^\beta}[\varphi(0,4x) + \varphi(4x,-4x)] + \frac{K^9}{725760^\beta}[\varphi(0,6x) + \varphi(6x,-6x)] \\
&\quad + \frac{K^{10}}{3628800^\beta}[\varphi(0,8x) + \varphi(8x,-8x)] \\
&\quad + \frac{K^{11}}{39916800^\beta}[\varphi(0,10x) + \varphi(10x,-10x)] \qquad (4.15)
\end{aligned}$$

for all $x \in X$. Replacing (x, y) by $(2x, x)$ in (4.3), multiplying by 385^β and using (4.5), (4.6) in the resulting equation, we find

$$\begin{aligned}
&\Big\| 385f(8x) - 4235f(7x) + 21175f(6x) - 63525f(5x) + 127050f(4x) \\
&\quad - 177485f(3x) + 173635f(2x) - 15368073875f(x)\Big\|_Y \\
&\leq 385^\beta K\varphi(2x,x) + \frac{385^\beta K}{241920^\beta}\varphi(0,0) + \frac{385^\beta K^2}{725760^\beta}[\varphi(0,x) + \varphi(x,-x)] \\
&\quad + \frac{385^\beta K^3}{3628800^\beta}[\varphi(0,2x) + \varphi(2x,-2x)] + \frac{385^\beta K^4}{39916800^\beta}[\varphi(0,3x) + \varphi(3x,-3x)] \qquad (4.16)
\end{aligned}$$

for all $x \in X$. Subtracting (4.16) from (4.15), we obtain

$$\begin{aligned}
&\Big\| 627f(7x) - 6105f(6x) + 26565f(5x) - 67980f(4x) + 112893f(3x) \\
&\qquad - 40041309f(2x) + 25107748875f(x) \Big\|_Y \\
&\leq K^5\varphi(6x,x) + 11^\beta K^4\varphi(5x,x) + 56^\beta K^4\varphi(4x,x) + 176^\beta K^3\varphi(3x,x) \\
&\quad + 385^\beta K^2\varphi(2x,x) + K^6\varphi(0,2x) \\
&\quad + \left(\frac{K^7}{86400^\beta} + \frac{11^\beta K^5}{3628800^\beta} + \frac{56^\beta K^4}{3628800^\beta} + \frac{176^\beta K^3}{725760^\beta} + \frac{385^\beta K^2}{241920^\beta}\right)\varphi(0,0) \\
&\quad + \left(\frac{56^\beta K^5}{39916800^\beta} + \frac{176^\beta K^4}{3628800^\beta} + \frac{385^\beta K^3}{725760^\beta}\right)[\varphi(0,x) + \varphi(x,-x)] \\
&\quad + \left(\frac{K^8}{120960^\beta} + \frac{176^\beta K^5}{39916800^\beta} + \frac{385^\beta K^4}{3628800^\beta}\right)[\varphi(0,2x) + \varphi(2x,-2x)] \\
&\quad + \frac{385^\beta K^5}{39916800^\beta}[\varphi(0,3x) + \varphi(3x,-3x)] + \frac{K^9}{241920^\beta}[\varphi(0,4x) + \varphi(4x,-4x)] \\
&\quad + \frac{K^{10}}{725760^\beta}[\varphi(0,6x) + \varphi(6x,-6x)] + \frac{K^{11}}{3628800^\beta}[\varphi(0,8x) + \varphi(8x,-8x)] \\
&\quad + \frac{K^{12}}{39916800^\beta}[\varphi(0,10x) + \varphi(10x,-10x)] \qquad (4.17)
\end{aligned}$$

for all $x \in X$. Now, changing (x, y) into (x, x) in (4.3), multiplying by 627^β and then using (4.5), (4.6) in the resulting equation, one finds

$$\begin{aligned}
&\Big\| 627f(7x) - 6897f(6x) + 34485f(5x) - 102828f(4x) + 20013f(3x) \\
&\qquad - 255189f(2x) - 25027647381f(x) \Big\|_Y \\
&\leq 627^\beta K\varphi(x,x) + \frac{627^\beta K}{120960^\beta}\varphi(0,0) + \frac{627^\beta K^2}{241920^\beta}[\varphi(0,x) + \varphi(x,-x)] \\
&\quad + \frac{627^\beta K^3}{725760^\beta}[\varphi(0,2x) + \varphi(2x,-2x)] + \frac{627^\beta K^4}{3628800^\beta}[\varphi(0,3x) + \varphi(3x,-3x)] \\
&\qquad\qquad + \frac{627^\beta K^5}{39916800^\beta}[\varphi(0,4x) + \varphi(4x,-4x)] \qquad (4.18)
\end{aligned}$$

for all $x \in X$. Subtracting (4.18) from (4.17), we get

$$\begin{aligned}
&\Big\| 792f(6x) - 7920f(5x) + 34848f(4x) - 87120f(3x) - 39786120f(2x) \\
&\qquad\qquad\qquad + 50135396256f(x)\Big\|_Y \\
&\le K^6\varphi(6x,x) + 11^\beta K^5\varphi(5x,x) + 56^\beta K^5\varphi(4x,x) + 176^\beta K^4\varphi(3x,x) \\
&+ 385^\beta K^3\varphi(2x,x) + 627^\beta K^2\varphi(x,x) + K^7\varphi(0,2x) \\
&+ \left(\frac{K^8}{84600^\beta} + \frac{11^\beta K^6}{3628800^\beta} + \frac{56^\beta K^5}{3628800^\beta} + \frac{176^\beta K^4}{725760^\beta} + \frac{385^\beta K^3}{241920^\beta} + \frac{627^\beta K^2}{120960^\beta}\right)\varphi(0,0) \\
&+ \left(\frac{56^\beta K^6}{39916800^\beta} + \frac{176^\beta K^5}{3628800^\beta} + \frac{385^\beta K^4}{725760^\beta} + \frac{627^\beta K^3}{241920^\beta}\right)[\varphi(0,x) + \varphi(x,-x)] \\
&+ \left(\frac{K^9}{120960^\beta} + \frac{176^\beta K^6}{39916800^\beta} + \frac{385^\beta K^5}{3628800^\beta} + + \frac{627^\beta K^4}{725760^\beta}\right)[\varphi(0,2x) + \varphi(2x,-2x)] \\
&+ \left(\frac{385^\beta K^6}{39916800^\beta} + \frac{627^\beta K^5}{3628800^\beta}\right)[\varphi(0,3x) + \varphi(3x,-3x)] \\
&+ \left(\frac{K^{10}}{241920^\beta} + \frac{627^\beta K^6}{39916800^\beta}\right)[\varphi(0,4x) + \varphi(4x,-4x)] \\
&+ \frac{K^{11}}{725760^\beta}[\varphi(0,6x) + \varphi(6x,-6x)] + \frac{K^{12}}{3628800^\beta}[\varphi(0,8x) + \varphi(8x - 8x)] \\
&\qquad + \frac{K^{13}}{39916800^\beta}[\varphi(0,10x) + \varphi(10x,-10x)]
\end{aligned} \tag{4.19}$$

for all $x \in X$. Multiplying (4.6) by 792^β and using (4.5), we have

$$\begin{aligned}
&\Big\| 792f(6x) - 7920f(5x) + 34848f(4x) - 87120f(3x) \\
&\qquad\qquad\qquad + 130680f(2x) - 31614210144f(x)\Big\|_Y \\
&\le 792^\beta K\varphi(0,x) + \frac{792^\beta K}{86400^\beta}\varphi(0,0) + \frac{792^\beta K^2}{120960^\beta}[\varphi(0,x) + \varphi(x,-x)] \\
&\quad + \frac{792^\beta K^3}{241920^\beta}[\varphi(0,2x) + \varphi(2x,-2x)] + \frac{792^\beta K^4}{725760^\beta}[\varphi(0,3x) + \varphi(3x,-3x)] \\
&\quad + \frac{792^\beta K^5}{3628800^\beta}[\varphi(0,4x) + \varphi(4x,-4x)] + \frac{792^\beta K^6}{39916800^\beta}[\varphi(0,5x) + \varphi(5x,-5x)]
\end{aligned} \tag{4.20}$$

for all $x \in X$. Subtracting (4.19) from (4.20), we obtain

$$\begin{aligned}
&\|39916800f(2x) - 81749606400f(x)\|_Y \\
&\leq K^7\varphi(6x,x) + 11^\beta K^6\varphi(5x,x) + 56^\beta K^6\varphi(4x,x) + 176^\beta K^5\varphi(3x,x) \\
&+ 385^\beta K^4\varphi(2x,x) + 627^\beta K^3\varphi(x,x) + K^8\varphi(0,2x) + 792^\beta K^2\varphi(0,x) \\
&+ \left(\frac{K^9}{86400^\beta} + \frac{11^\beta K^7}{3628800^\beta} + \frac{56^\beta K^6}{3628800^\beta} + \frac{176^\beta K^5}{725760^\beta} + \frac{385^\beta K^4}{241920^\beta} + \frac{627^\beta K^3}{120960^\beta}\right)\varphi(0,0) \\
&+ \Big(\frac{56^\beta K^7}{39916800^\beta} + \frac{176^\beta K^6}{3628800^\beta} + \frac{385^\beta K^5}{725760^\beta} + \frac{627^\beta K^4}{241920^\beta} \\
&\qquad + \frac{792^\beta K^3}{120960^\beta}\Big)[\varphi(0,x) + \varphi(x,-x)] \\
&+ \Big(\frac{K^{10}}{120960^\beta} + \frac{176^\beta K^7}{39916800^\beta} + \frac{385^\beta K^6}{3628800^\beta} + \frac{627^\beta K^5}{725760^\beta} \\
&\qquad + \frac{792^\beta K^4}{241920^\beta}\Big)[\varphi(0,2x) + \varphi(2x,-2x)] \\
&+ \left(\frac{385^\beta K^7}{39916800^\beta} + \frac{627^\beta K^6}{3628800^\beta} + \frac{792^\beta K^5}{725760^\beta}\right)[\varphi(0,3x) + \varphi(3x,-3x)] \\
&+ \left(\frac{K^{11}}{241920^\beta} + \frac{627^\beta K^7}{39916800^\beta} + \frac{792^\beta K^6}{3628800^\beta}\right)[\varphi(0,4x) + \varphi(4x,-4x)] \\
&\quad + \frac{792^\beta K^7}{39916800^\beta}[\varphi(0,5x) + \varphi(5x,-5x)] + \frac{K^{12}}{725760^\beta}[\varphi(0,6x) + \varphi(6x,-6x)] \\
&\qquad + \frac{K^{13}}{3628800^\beta}[\varphi(0,8x) + \varphi(8x,-8x)] \\
&\qquad\quad + \frac{K^{14}}{39916800^\beta}[\varphi(0,10x) + \varphi(10x,-10x)] \qquad (4.21)
\end{aligned}$$

for all $x \in X$. From (4.21), we arrive at

$$\left\|f(2x) - 2^{11}f(x)\right\|_Y \leq \Psi(x)$$

for all $x \in X$. By Lemma 4.1, there exists a unique mapping $U : X \to Y$ such that $U(2x) = 2^{11}(x)$ and

$$\|f(x) - U(x)\|_Y \leq \frac{1}{2048^\beta\,|1-L^i|}\Psi(x)$$

for all $x \in X$. It remains to show that U is an undecic map. By (4.3), we have

$$\begin{aligned}
\left\|\frac{1}{2048^{in}}D_uf\left(2^{in}x, 2^{in}y\right)\right\|_Y &\leq 2048^{-in\beta}\varphi\left(2^{in}x, 2^{in}y\right) \\
&\leq 2048^{-in\beta}\left(2048^{i\beta}L\right)^n\varphi(x,y) \\
&= L^n\varphi(x,y)
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. So $\|D_u U(x,y)\|_Y = 0$ for all $x, y \in X$. Thus the mapping $U : X \to Y$ is undecic. Q.E.D.

Corollary 4.3. Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_Y$. Let c_1, a be positive numbers with $a \neq \frac{11\beta}{\alpha}$ and $f : X \to Y$ be a mapping satisfying

$$\|D_u f(x,y)\|_Y \leq c_1 \left(\|x\|_X^a + \|y\|_X^a\right)$$

for all $x, y \in X$. Then there exists a unique undecic mapping $U : X \to Y$such that

$$\|f(x) - U(x)\|_Y \leq \begin{cases} \frac{c_1 \delta_a}{2048^\beta - 2^{a\alpha}} \|x\|_X^a, & a \in \left(0, \frac{11\beta}{\alpha}\right) \\ \frac{2^{a\alpha} c_1 \delta_a}{2048^\beta (2^{a\alpha} - 2048^\beta)} \|x\|_X^a, & a \in \left(\frac{11\beta}{\alpha}, \infty\right) \end{cases}$$

for all $x \in X$, where

$$\begin{aligned}
\delta_a = \frac{1}{39916800^\beta} \Big[& K^7 \left(6^{a\alpha} + 1\right) + 11^\beta K^7 \left(5^{a\alpha} + 1\right) + 56^\beta K^6 \left(4^{a\alpha} + 1\right) \\
& + 176^\beta K^5 \left(3^{a\alpha} + 1\right) + 385^\beta K^4 \left(2^{a\alpha} + 1\right) + 2 \cdot 627^\beta K^3 + K^8 2^{a\alpha} + 792^\beta K^2 \\
& + 3 \left(\frac{56^\beta K^7}{39916800^\beta} + \frac{176^\beta K^6}{3628800^\beta} + \frac{385^\beta K^5}{725760^\beta} + \frac{627^\beta K^4}{241920^\beta} + \frac{792^\beta K^3}{120960^\beta} \right) \\
& + 3 \cdot 2^{a\alpha} \left(\frac{K^{10}}{120960^\beta} + \frac{176^\beta K^7}{39916800^\beta} + \frac{385^\beta K^6}{3628800^\beta} + \frac{627^\beta K^5}{725760^\beta} + \frac{792^\beta K^4}{241920^\beta} \right) \\
& + 3 \cdot 3^{a\alpha} \left(\frac{385^\beta K^7}{39916800^\beta} + \frac{627^\beta K^6}{3628800^\beta} + \frac{792^\beta K^5}{725760^\beta} \right) \\
& + 3 \cdot 4^{a\alpha} + \left(\frac{K^{11}}{241920^\beta} + \frac{627^\beta K^7}{39916800^\beta} + \frac{792^\beta K^6}{3628800^\beta} \right) \\
& + \frac{3 \cdot 5^{a\alpha} \cdot 792^\beta K^7}{39916800^\beta} + \frac{3 \cdot 6^{a\alpha} K^{12}}{725760^\beta} + \frac{3 \cdot 8^{a\alpha} K^{13}}{3628800^\beta} + \frac{3 \cdot 10^{a\alpha} K^{14}}{39916800^\beta} \Big].
\end{aligned}$$

Proof. The proof is obtained by taking $\varphi(x,y) = c_1 \left(\|x\|_X^a + \|y\|_X^a\right)$, for all $x, y \in X$ and $L = \frac{2^{\alpha\lambda}}{2048^\beta}$ in Theorem 4.2. Q.E.D.

Corollary 4.4. Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_Y$. Let c_2, r, s be positive numbers with $a = r + s \neq \frac{11\beta}{\alpha}$ and $f : X \to Y$ be a mapping satisfying

$$\|D_u f(x,y)\|_Y \leq c_2 \|x\|_X^r \|y\|_X^s$$

for all $x, y \in X$. Then there exists a unique undecic mapping $U : X \to Y$ such that

$$\|f(x) - U(x)\|_Y \le \begin{cases} \frac{c_2\delta_{r,s}}{2048^\beta - 2^{a\alpha}} \|x\|_X^a, & a \in \left(0, \frac{11\beta}{\alpha}\right) \\ \frac{2^{a\alpha} c_2 \delta_{r,s}}{2048^\beta(2^{a\alpha} - 2048^\beta)} \|x\|_X^a, & a \in \left(\frac{11\beta}{\alpha}, \infty\right) \end{cases}$$

for all $x \in X$, where

$$\begin{aligned} \delta_{r,s} = \frac{1}{39916800^\beta} \Big[& K^7 6^{r\alpha} + 11^\beta K^6 5^{r\alpha} + 56^\beta K^6 4^{r\alpha} + 176^\beta K^5 3^{r\alpha} + 385^\beta K^4 2^{r\alpha} \\ & + 627^\beta K^3 + \left(\frac{56^\beta K^7}{39916800^\beta} + \frac{176^\beta K^6}{3628800^\beta} + \frac{385^\beta K^5}{725760^\beta} + \frac{627^\beta}{241920^\beta} + \frac{792^\beta k^3}{120960^\beta} \right) \\ & + 2^{a\alpha} \left(\frac{K^{10}}{120960^\beta} + \frac{176^\beta K^7}{39916800^\beta} + \frac{385^\beta K^6}{3628800^\beta} + \frac{627^\beta K^5}{725760^\beta} + \frac{792^\beta K^4}{241920^\beta} \right) \\ & + 3^{a\alpha} \left(\frac{385^\beta K^7}{39916800^\beta} + \frac{627^\beta K^6}{3628800^\beta} + \frac{792^\beta K^5}{725760^\beta} \right) \\ & + 4^{a\alpha} \left(\frac{K^{11}}{241920^\beta} + \frac{627^\beta K^7}{39916800^\beta} + \frac{792^\beta K^6}{3628800^\beta} \right) \\ & + \frac{5^{a\alpha} 792^\beta K^7}{39916800^\beta} + \frac{6^{a\alpha} K^{12}}{725760^\beta} + \frac{8^{a\alpha} K^{13}}{3628800^\beta} + \frac{10^{a\alpha} K^{14}}{39916800^\beta} \Big]. \end{aligned}$$

Proof. Letting $\varphi(x, y) = c_2 \|x\|_X^r \|y\|_X^s$, for all $x, y \in X$ and $L = \frac{2^{a\alpha}}{2048^\beta}$ in Theorem 4.2, we obtain the required results. Q.E.D.

Corollary 4.5. Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_Y$. Let c_3, r, s be positive numbers with $a = r + s \neq \frac{11\beta}{\alpha}$ and $f : X \to Y$ be a mapping satisfying

$$\|D_u f(x, y)\|_Y \le c_3 \left[\|x\|_X^r \|y\|_X^s + \left(\|x\|_X^{r+s} + \|y\|_X^{r+s} \right) \right]$$

for all $x, y \in X$. Then there exists a unique undecic mapping $U : X \to Y$ such that

$$\|f(x) - U(x)\|_Y \le \begin{cases} \frac{c_3(\delta_{r,s} + \delta_a)}{2048^\beta - 2^{a\alpha}} \|x\|_X^a, & a \in \left(0, \frac{11\beta}{\alpha}\right) \\ \frac{2^{a\alpha} c_3(\delta_{r,s} + \delta_a)}{2048^\beta(2^{a\alpha} - 2048^\beta)} \|x\|_X^a, & a \in \left(\frac{11\beta}{\alpha}, \infty\right) \end{cases}$$

for all $x \in X$, where $\delta_{r,s}$ and δ_a are defined as in Corollaries in 4.4 and 4.3.

Proof. By taking $\varphi(x, y) = c_3 \left[\|x\|_X^r \|y\|_X^s + \left(\|x\|_X^{r+s} + \|y\|_X^{r+s} \right) \right]$, for all $x, y \in X$ and $L = \frac{2^{\alpha\lambda}}{2048^\beta}$ in Theorem 4.2, we arrive at the desired results. Q.E.D.

5 Counter-example

In this section, using the idea of the well-known counter-example provided by Z. Gajda [9], we illustrate a counter-example that the functional equation (1.1) is not stable for $a = \frac{11\beta}{\alpha}$ in Corollary 4.3.

We consider the function

$$\varphi(x) = \begin{cases} x^{11}, & \text{for } |x| < 1 \\ 1, & \text{for } |x| \geq 1. \end{cases} \tag{5.1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sum_{n=0}^{\infty} 2^{-11n} \varphi(2^n x) \tag{5.2}$$

for all $x \in \mathbb{R}$. The function f serves as a counter-example for the fact that the functional equation (1.1) is not stable for $a = \frac{11\beta}{\alpha}$ in Corollary 4.3 in the following theorem.

Theorem 5.1. If the function f defined in (5.2) satisfies the functional inequality

$$|D_u f(x, y)| \leq \frac{39918848 \cdot 2048^3}{2047} \left(|x|^{11} + |y|^{11} \right) \tag{5.3}$$

for all $x, y \in \mathbb{R}$, then there do not exist an undecic mapping $U : \mathbb{R} \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$|f(x) - U(x)| \leq \delta |x|^{11}, \qquad \text{for all } x \in \mathbb{R}.$$

Proof. First, we are going to show that f satisfies (5.3).

$$|f(x)| = \left| \sum_{n=0}^{\infty} 2^{-11n} \varphi(2^n x) \right| \leq \sum_{n=0}^{\infty} \frac{1}{2^{11n}} = \frac{2048}{2047}.$$

Therefore, we see that f is bounded by $\frac{2048}{2047}$ on $\mathbb{R}$.

If $|x|^{11} + |y|^{11} = 0$ or $|x|^{11} + |y|^{11} \geq \frac{1}{2048}$, then

$$|D_u f(x, y)| \leq \frac{(39918848)(2048)}{2047} \leq \frac{(39918848)(2048)^2}{2047} \left(|x|^{11} + |y|^{11} \right).$$

Now, suppose that $0 < |x|^{11} + |y|^{11} < \frac{1}{2048}$. Then there exists a non-negative integer k such that

$$\frac{1}{2048^{k+1}} \leq |x|^{11} + |y|^{11} < \frac{1}{2048^k}. \tag{5.4}$$

Hence $2048^k |x|^{11} < \frac{1}{2048}$, $2048^k |y|^{11} < \frac{1}{2048}$ and $2^n(x + 6y)$, $2^n(x + 5y)$, $2^n(x + 4y)$, $2^n(x + 3y)$, $2^n(x+2y)$, $2^n(x+y)$, $2^n(x)$, $2^n(y)$, $2^n(x-y)$, $2^n(x-2y)$, $2^n(x-3y)$, $2^n(x-4y)$, $2^n(x-5y) \in (-1, 1)$

for all $n = 0, 1, 2, \ldots, k-1$. Hence for $n = 0, 1, 2, \ldots, k-1$,

$$\begin{aligned}
&\varphi\left(2^n(x+6y)\right) - 11\varphi\left(2^n(x+5y)\right) + 55\varphi\left(2^n(x+4y)\right) - 165\varphi\left(2^n(x+3y)\right) \\
&\quad + 330\varphi\left(2^n(x+2y)\right) - 462\varphi\left(2^n(x+y)\right) + 462\varphi\left(2^n x\right) - 330\varphi\left(2^n(x-y)\right) \\
&\quad + 165\varphi\left(2^n(x-2y)\right) - 55\varphi\left(2^n(x-3y)\right) + 11\varphi\left(2^n(x-4y)\right) - \varphi\left(2^n(x-5y)\right) \\
&\qquad\qquad - 39916800\varphi\left(2^n y\right) = 0. \qquad (5.5)
\end{aligned}$$

From the definition of f and the inequality (5.4), we obtain that

$$\begin{aligned}
|D_u f(x,y)| = \Big| &\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n(x+6y)\right) - 11\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n(x+5y)\right) \\
&+ 55\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n(x+4y)\right) - 165\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n(x+3y)\right) \\
&+ 330\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n(x+2y)\right) - 462\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n(x+y)\right) \\
&+ 462\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n x\right) - 330\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n(x-y)\right)
\end{aligned}$$

$$\begin{aligned}
&+ 165\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n(x-2y)\right) - 55\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n(x-3y)\right) \\
&+ 11\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n(x-4y)\right) - \sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n(x-5y)\right) \\
&\qquad - 39916800\sum_{n=0}^{\infty} 2^{-11n}\varphi\left(2^n y\right) \Big| \\
&\le \sum_{n=k}^{\infty} 2^{-11n}\cdot 39918848 \\
&\le 39918848\cdot\frac{2^{11(1-k)}}{2047} \le \frac{39918848\cdot 2048^3}{2047}\left(|x|^{11} + |y|^{11}\right). \qquad (5.6)
\end{aligned}$$

Therefore, f satisfies (5.3) for all $x, y \in \mathbb{R}$. Now, we claim that the functional equation (1.1) is not stable for $a = \frac{11\beta}{\alpha}$ in Corollary 4.3 ($\alpha = \beta = p = 1$). Suppose on the contrary that there exists an undecic mapping $U : \mathbb{R} \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$|f(x) - U(x)| \le \delta\,|x|^{11}, \quad \text{for all } x \in \mathbb{R}.$$

Then there exists a constant $c \in \mathbb{R}$ such that $U(x) = cx^{11}$ for all rational numbers x (see [14]). So we obtain that

$$|f(x)| \leq (\delta + |c|)\,|x|^{11} \tag{5.7}$$

for all $x \in \mathbb{Q}$. Let $m \in \mathbb{N}$ with $m+1 > \delta+|c|$. If x is a rational number in $(0, 2^{-m})$, then $2^n x \in (0,1)$ for all $n = 0, 1, 2, \ldots, m$, and for this x, we get

$$f(x) = \sum_{n=0}^{\infty} 2^{-11n}\varphi(2^n x) \geq \sum_{n=0}^{m} 2^{-11n}\,(2^n x)^{11} = (m+1)x^{11} > (\delta + |c|)x^{11} \tag{5.8}$$

which contradicts (5.7). Hence the functional equation (1.1) is not stable for $a = \frac{11\beta}{\alpha}$ in Corollary 4.3. Q.E.D.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces,* J. Math.Soc. Japan, 2(1950), 64-66.

[2] L. Cadariu and V. Radu, *Fixed points and stability for functional equations in probabilistic metric and random normed spaces*, Fixed Point Theory and Applications, Vol. 2009, Article ID 589143 (2009), 18 pages.

[3] I.S. Chang and H.M. Kim, *On the Hyers-Ulam stability of quadratic functional equations,* J. Ineq. Appl. Math. 33(2002), 1-12.

[4] Y.J. Cho, M. Eshaghi Gordji and S. Zolfaghari, *Solutions and stability of generalized mixed type QC functional equations in random normed spaces*, U. Inequal. Appl. Article ID 403101, doi:10.1155/2010/403101, 16 pages.

[5] P.W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math., 27 (1984), 76-86.

[6] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, (World Scientific Publishing Company, New Jersey, London, Singapore and Hong Kong), 2002.

[7] A. Ebadian and S. Zolfaghari, *Stability of a mixed additive and cubic functional equation in several variables in non-Archimedean spaces*, Ann. Univ. Ferrara. 58 (2012), 291-306.

[8] M. Eshaghi Gordji, A. Ebadian and S. Zolfaghari, *Stability of a functional equation deriving from cubic and quartic functions*, Abs, Appl. Anal. Article ID 801904, (2008), 17 pages.

[9] Z. Gajda, *On stability of additive mappings*, Int. J. Math. Math. Sci. 14(3) (1991), 431-434.

[10] P. Găvruţă, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., 184 (1994), 431-436.

[11] D.H. Hyers, *On the stability of the linear functional equation*, Proc.Nat. Acad.Sci., U.S.A., 27 (1941), 222-224.

[12] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables,* Birkhauser, Basel, 1998.

[13] G. Isac and Th.M. Rassias, *Stability of ψ-additive mappings: applications to nonlinear analysis*, Int. J. Math. Math. Sci., 19(2) (1996), 2192-28.

[14] K.W. Jun and H.M. Kim, *On the stability of Euler-Lagrange type cubic mappings in quasi-Banach spaces*, J. Math. Anal. Appl. 332(2) (2007), 1335-1350.

[15] C. Park, *Fixed points and the stability of an AQCQ-functional equation in non-Archimedean normed spaces*, Abs. Appl. Anal. Vol. 2010, Article ID 849543 (2010), 15 pages.

[16] J.M. Rassias, *On approximately of approximately linear mappings by linear mappings*, J. Funct. Anal. USA, 46 (1982), 126-130.

[17] J.M. Rassias, *On approximately of approximately linear mappings by linear mappings*, Bull. Sci. Math., 108 (4) (1984), 445-446.

[18] J.M. Rassias, *Solution of problem of Ulam*, J.Approx. Theory. USA, 57 (3) (1989), 268-273.

[19] J.M. Rassias, *Solution of the Ulam stability problem for quartic mappings*, Glasnic Matematicki. Serija III, 34(2) (1999), 243-252.

[20] J.M. Rassias, *Solution of the Ulam stablility problem for cubic mappings*, Glasnik Matematicki. Serija III, 36(1) (2001), 63-72.

[21] K. Ravi, J.M. Rassias and R. Kodandan, *Generalized Ulam-Hyers stability of an AQ-functional equation in quasi-β-normed spaces*, Math. AEterna, 1(3-4) (2011), 217-236.

[22] J.M. Rassias and Mohammad Eslamian, *Fixed points and stability of nonic functional equation in quasi-β-normed spaces*, Contemporary Anal. Appl. Math. 3(2) (2015), 293-309.

[23] K. Ravi, M. Arunkumar and J.M. Rassias, *Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, Int. J. Math. Stat. 3(A08) (2008), 36-46.

[24] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc., 72 (1978), 297-300.

[25] S.M. Ulam, *Problems in Modern Mathematics,* Rend. Chap. VI, Wiley, New York, 1960.

[26] T.Z. Xu, J.M. Rassias and W.X. Xu, *A fixed point approach to the stability of a general mixed AQCQ-functional equation in non-Archimedean normed spaces*, Discrete Dynamics in Nature and Society, Vol. 2010, Article ID 812545, (2010), 24 pages.

[27] T.Z. Xu, J.M. Rassias, M.J. Rassias and W.X. Xu, *A fixed point approach to the stability of quintic and sextic functional equations in quasi-β-normed spaces*, J. Inequal. Appl. Vol. 2010, Article ID 423231, (2010), 1-23.

[28] T.Z. Xu, J.M. Rassias and W.X. Xu, *A fixed point approach to the stability of a general mixed additive-cubic functional equation in quasi fuzzy normed spaces*, Int. J. Physical Sci. 6(2) (2011), 12 pages.

[29] T.Z. Xu, J.M. Rassias and W.X. Xu,*A generalized mixed quadratic-quartic functional equation*, Bull. Malays. Math. Sci. Soc., 35(3) (2012), 633-649.

[30] T.Z. Xu and J.M. Rassias, *Approximate septic and octic mappings in quasi-β-normed spaces*, J. Comp. Anal. Appl. 15(6) (2013), 1110-1119.

[31] S. Zolfaghari, *Stability of generalized QCA-functional equation in p-Banach spaces*, Int. J. Nonlinear Anal. Appl. 1 (2010), 84-99.

[32] S. Zolfaghari, A. Ebadian, S. Ostadbashi, M. De La Se and M. Eshaghi Gordji, *A fixed point approach to the Hyers-Ulam stability of an AQ functional equation in probabilistic modular spaces*, Int. J. Nonlinear Anal. Appl. 4(2) (2013), 1-14.