Multipliers and convolution spaces for the Hankel space and its dual on the half space $[0,+\infty[\times\mathbb{R}^n$

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Abstract

We define the Hankel space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$; $\mu\geqslant-\frac{1}{2}$, and its dual $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$. First, we characterize the space $\mathscr{M}_\mu([0,+\infty[\times\mathbb{R}^n)$ of multipliers of the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. Next, we define a subspace $\mathbb{O}'_\mu([0,+\infty[\times\mathbb{R}^n)$ of the dual $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ which permits to define and study a convolution product $*$ on $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ and we give nice properties.

2010 Mathematics Subject Classification. **42B10**. 42B15
Keywords. Hankel space, Fourier-Hankel transform, multipliers, convolution product.

1 Introduction.

We define the Hankel space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, $\mu\geqslant-\frac{1}{2}$ to be the space of infinitely differentiable functions f on $]0,+\infty[\times\mathbb{R}^n$, such that for all $(k_1,\alpha),(k_2,\beta)\in\mathbb{N}\times\mathbb{N}^n$, the function

$$(r,x)\mapsto r^{k_1}x^\alpha\big(\frac{\partial}{\partial r^2}\big)^{k_2}D_x^\beta\big(r^{-\mu-\frac{1}{2}}f(r,x)\big)$$

is bounded on $[0,+\infty[\times\mathbb{R}^n$. Where

- $\dfrac{\partial}{\partial r^2}\ =\ \dfrac{1}{r}\dfrac{\partial}{\partial r}$.
- $D_x^\beta\ =\ \big(\dfrac{\partial}{\partial x_1}\big)^{\beta_1}...\big(\dfrac{\partial}{\partial x_n}\big)^{\beta_n}$; $\beta\ =\ \big(\beta_1,\beta_2,...\beta_n\big)$.
- $x^\alpha\ =\ x_1^{\alpha_1}...x_n^{\alpha_n}$; $\alpha\ =\ \big(\alpha_1,\alpha_2,...\alpha_n\big)$.

The space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ is equipped with a topology for which it is a Fréchet one [2, 10].
Our investigation in this work is to determine the space of multipliers of $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and a convolution space for the dual space $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ of $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$.

More precisely, in the second section we define a family of norms N_m^μ, $m\in\mathbb{N}$ and a distance d_μ on the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and we recall some properties. Next, we give the classical description of the element of $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$. Also, we define the Fourier-Hankel transform $\mathscr{H}_\mu$ that will be a topological isomorphism from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ onto itself and from $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ onto itself.

The spaces $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ play for the Fourier-Hankel transform $\mathscr{H}_\mu$ the same role that play the Schwartz space's $\mathscr{S}_e(\mathbb{R}\times\mathbb{R}^n)$ (the space of infinitely differentiable functions on $\mathbb{R}\times\mathbb{R}^n$ rapidly decreasing together with all their derivatives, even with respect to the first variable) and its dual
$\mathscr{S}'_e(\mathbb{R}\times\mathbb{R}^n)$ for the usual Fourier transform $\mathscr{F}$ [7].

Tbilisi Mathematical Journal 9(1) (2016), pp. 197–220.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 02 August 2015.
Accepted for publication: 10 January 2016.

The second section is devoted to define and study the space of multipliers $\mathscr{M}_\mu([0,+\infty[\times\mathbb{R}^n)$. This space is formed by the infinitely differentiable functions θ on $[0,+\infty[\times\mathbb{R}^n$ such that the mapping

$$\varphi \mapsto \theta\varphi$$

is continuous from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ into itself. Then we give a nice characterization of the elements of $\mathscr{M}_\mu([0,+\infty[\times\mathbb{R}^n)$.

In the last section, using the fact that the Fourier-Hankel transform is an isomorphism from $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ onto itself; we define a subspace $\mathbb{O}'_\mu([0,+\infty[\times\mathbb{R}^n)$ of $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ which permits to define the convolution product of an element $T\in\mathbb{O}'_\mu([0,+\infty[\times\mathbb{R}^n)$ and $S\in\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$. We prove in particulier that for every $T\in\mathbb{O}'_\mu([0,+\infty[\times\mathbb{R}^n)$; the mapping

$$S\mapsto T*S,$$

is continuous from $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ into itself and we have

$$\mathscr{H}_\mu(T*S)=\lambda_0^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(T)(\lambda_0,\lambda)\mathscr{H}_\mu(S)$$

in $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$.

2 The multipliers of the Hankel space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$.

Through out this paper, μ is a real number; $\mu\geqslant-\frac{1}{2}$. For all $m\in\mathbb{N}$, we define the norm N_m^μ on the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ by setting

$$N_m^\mu(f)=\sup_{\substack{(r,x)\in[0,+\infty[\times\mathbb{R}^n\\ k_1+k_2+|\alpha|\leqslant m}}\big(1+r^2+|x|^2\big)^{k_1}\left|\Big(\frac{\partial}{\partial r^2}\Big)^{k_2}D_x^\alpha\big(r^{-\mu-\frac{1}{2}}f\big)(r,x)\right|. \tag{2.1}$$

Where $|x|^2=x_1^2+...+x_n^2;\quad x=(x_1,...,x_n)\in\mathbb{R}^n$.
And the distance

$$d_\mu(f,g)=\sum_{m=0}^{\infty}\frac{1}{2^m}\frac{N_m^\mu(f-g)}{1+N_m^\mu(f-g)}.$$

It is well known that a sequence $(f_k)_k$ converges to zero in $(\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n),d_\mu)$ if and only if

$$\forall m\in\mathbb{N},\quad \lim_{k\to+\infty}N_m^\mu(f_k)=0.$$

Moreover, the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ is a Fréchet space when endowed with the topology generated by $(N_m^\mu)_{m\in\mathbb{N}}$.

Definition 2.1. *A function θ defined on $[0,+\infty[\times\mathbb{R}$ is said to be a multiplier of the Hankel space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ if the mapping*

$$\varphi\longmapsto\theta\varphi$$

is continuous from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ into itself.

The space formed by the multipliers of $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ will be denoted by $\mathscr{M}_\mu([0,+\infty[\times\mathbb{R}^n)$.

In this section, we give a nice characterization of the space $\mathscr{M}_\mu([0,+\infty[\times\mathbb{R}^n)$; also we define a topology on this space and we establish some interesting results.

Lemma 2.2. *For all $a,b\in\mathbb{R}$, we have*

$$\frac{1+a^2}{1+b^2}\leqslant 2(1+|a-b|^2).$$

Proof. The result is an immediate consequence of the Peetre's inequality [1, 8], that is if t is a real number and x,y are vectors in $\mathbb{R}^n$, then

$$\left(\frac{1+|x|^2}{1+|y|^2}\right)^{|t|}\leqslant 2^{|t|}(1+|x-y|^2)^{|t|}.$$

Q.E.D.

Lemma 2.3. *Let f be an infinitely differentiable function on $\mathbb{R}$, $supp(f)=[\frac{1}{2},\frac{3}{2}]$ and $f(1)=1$. Let $((r_k,x_k))_k$ be a sequence in $[0,+\infty[\times\mathbb{R}^n$, such that*

$$|(r_0,x_0)|^2>1 \quad \text{and} \quad |(r_{k+1},x_{k+1})|^2>|(r_k,x_k)|^2+1.$$

Then, the function φ_0 defined by

$$\varphi_0(r,x)=r^{\mu+\frac{1}{2}}\sum_{k=0}^{\infty}\frac{f(|(r,x)|^2-|(r_k,x_k)|^2+1)}{\Big(|(r_k,x_k)|^2+1\Big)^k}$$

belongs to the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$.

Proof. Let $\rho\in\mathbb{R}$, $\rho>1$.
For all $(r,x)\in B(0,\rho)=\left\{(r,x)\in[0,+\infty[\times\mathbb{R}^n,\ r^2+|x|^2<\rho^2\right\}$, we have

$$r^{-\mu-\frac{1}{2}}\varphi_0(r,x)=\sum_{k=0}^{k_0}\frac{f(|(r,x)|^2-|(r_k,x_k)|^2+1)}{\Big(|(r_k,x_k)|^2+1\Big)^k},$$

where $k_0=[|-\frac{1}{2}+\rho^2|]+1$, because $supp(f)=[\frac{1}{2},\frac{3}{2}]$.

Consequently, the function

$$(r,x)\longmapsto r^{-\mu-\frac{1}{2}}\varphi_0(r,x)$$

is infinitely differentiable on $[0,+\infty[\times\mathbb{R}^n$. Moreover, for all $j,m\in\mathbb{N}$ and $\alpha\in\mathbb{N}^n$, we get

$$\begin{aligned}\big(1+r^2+|x|^2\big)^m&\left|\Big(\frac{\partial}{\partial r^2}\Big)^jD_x^\alpha\big(r^{-\mu-\frac{1}{2}}\varphi_0\big)(r,x)\right|\leqslant\\ &2^j\big(1+r^2+|x|^2\big)^mP_{j,\alpha}(x)\sum_{k=0}^{\infty}\frac{f^{j+|\alpha|}(r^2+|x|^2-r_k^2-|x_k|^2+1)}{\big(r_k^2+|x_k|^2+1\big)^k}.\end{aligned}$$

Where $P_{j,\alpha}$ is a real polynomial on $\mathbb{R}^n$. Thus, there exist an integer l and a positive constant $C_{j,\alpha}$ such that

$$\big(1+r^2+|x|^2\big)^m\left|\big(\frac{\partial}{\partial r^2}\big)^j D_x^\alpha\big(r^{-\mu-\frac{1}{2}}\varphi_0\big)(r,x)\right| \leqslant$$

$$C_{j,\alpha}\sum_{k=0}^{\infty}(1+r^2+|x|^2)^l\,\frac{\left|f^{j+|\alpha|}(r^2+|x|^2-r_k^2-|x_k|^2+1)\right|}{\big(r_k^2+|x_k|^2+1\big)^k}.$$

However, Lemma 2.2 involves

$$\begin{aligned}(1+r^2+|x|^2)^l &\leqslant 2^l(1+r_k^2+|x_k|^2)^l\left(1+\left[\sqrt{r^2+|x|^2}-\sqrt{r_k^2+|x_k|^2}\right]^2\right)^l\\ &\leqslant 2^l(1+r_k^2+|x_k|^2)^l\left(1+\left|r^2+|x|^2-r_k^2-|x_k|^2\right|\right)^l\\ &\leqslant 2^l(1+r_k^2+|x_k|^2)^l\left(2+\big(r^2+|x|^2-r_k^2-|x_k|^2\big)^2\right)^l.\end{aligned}$$

Consequently,

$$\big(1+r^2+|x|^2\big)^m\left|\big(\frac{\partial}{\partial r^2}\big)^j D_x^\alpha\big(r^{-\mu-\frac{1}{2}}\varphi_0\big)(r,x)\right| \leqslant$$

$$2^l C_{j,\alpha}\sum_{k=0}^{\infty}\left(2+\big(r^2+|x|^2-r_k^2+|x_k|^2\big)^2\right)^l\left|\frac{f^{j+|\alpha|}(r^2+|x|^2-r_k^2-|x_k|^2+1)}{\big(r_k^2+|x_k|^2+1\big)^{k-l}}\right|.$$

On the other hand from the hypothesis, for all $k\in\mathbb{N}$, we have

$$r_k^2+|x_k|^2>k+1,$$

finally, for all $(r,x)\in[0,+\infty[\times\mathbb{R}^n$ we get

$$\big(1+r^2+|x|^2\big)^m\left|\big(\frac{\partial}{\partial r^2}\big)^j D_x^\alpha\big(r^{-\mu-\frac{1}{2}}\varphi_0\big)(r,x)\right| \leqslant 2^l C_{j,\alpha}N_{\alpha,j,l}(f)\sum_{k=0}^{\infty}\frac{1}{(k+2)^{k-l}},$$

where $N_{\alpha,j,l}(f)=\sup_{t\in\mathbb{R}}(2+(t-1)^2)^l|f^{j+|\alpha|}(t)|$. Q.E.D.

Theorem 2.4. *The following assumptions are equivalent*

i) The function θ *is infinitely differentiable on* $[0,\infty[\times\mathbb{R}^n$ *and for all* $(k,\alpha)\in\mathbb{N}\times\mathbb{N}^n$ *the function*

$$\big(\frac{\partial}{\partial r^2}\big)^k D_x^\alpha(\theta)$$

is slowly increasing, i.e there exists $m_{k,\alpha}\in\mathbb{N}$ *such that the function*

$$(r,x)\mapsto(1+r^2+|x|^2)^{-m_{k,\alpha}}\big(\frac{\partial}{\partial r^2}\big)^k D_x^\alpha(\theta)(r,x)$$

is bounded on $[0,\infty[\times\mathbb{R}^n$.

ii) The function θ *is a multiplier of the space* $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$.

iii) The function θ *is infinitely differentiable on* $]0,\infty[\times\mathbb{R}^n$ *and for all* $(k,\alpha)\in\mathbb{N}\times\mathbb{N}^n$*, the mapping* $\varphi\mapsto(\frac{\partial}{\partial r^2})^k D_x^\alpha(\theta)\varphi$ *is a continuous endomorphism of* $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$.

Proof. **.** Suppose that *i*) is satisfied. Let φ be in $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. It is clear that $\theta\varphi$ is an infinitely differentiable function on $]0,\infty[\times\mathbb{R}^n$ and for all $(k,\alpha)\in\mathbb{N}\times\mathbb{N}^n$,

$$\big(\frac{\partial}{\partial r^2}\big)^k D_x^\alpha\big(r^{-\mu-\frac{1}{2}}\theta\varphi\big)(r,x)=\sum_{j=0}^{k}\sum_{\beta+\gamma=\alpha}\frac{k!}{j!(k-j)!}\frac{\alpha!}{\beta!\gamma!}\big(\frac{\partial}{\partial r^2}\big)^j D_x^\gamma\theta(r,x)\big(\frac{\partial}{\partial r^2}\big)^{k-j}D_x^\beta\big(r^{-\mu-\frac{1}{2}}\varphi\big)(r,x).$$

Where $\alpha!=\alpha_1!...\alpha_n!$, $\alpha=(\alpha_1,...,\alpha_n)\in\mathbb{N}^n$. Let $m\in\mathbb{N}$ and $(k_1,k_2,\alpha)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}^n$ such that $k_1+k_2+|\alpha|\leqslant m$. From the hypothesis there exist $l\in\mathbb{N}$ and C_m such that for all $(j,\gamma)\in\mathbb{N}\times\mathbb{N}^n$, $j+|\gamma|\leqslant m$, we have

$$\left|\big(\frac{\partial}{\partial r^2}\big)^j D_x^\gamma\theta(r,x)\right|\leqslant C_m(1+r^2+|x|^2)^l.$$

So,

$$\begin{aligned}&\left|(1+r^2+|x|^2)^{k_1}\big(\frac{\partial}{\partial r^2}\big)^{k_2}D_x^\alpha(r^{-\mu-\frac{1}{2}}\theta\varphi)(r,x)\right|\\ &\leqslant C_m(1+r^2+|x|^2)^{l+k_1}\sum_{j=0}^{k_2}\sum_{\beta+\gamma=\alpha}\frac{k_2!}{j!(k_2-j)!}\frac{\alpha!}{\beta!\gamma!}\left|\big(\frac{\partial}{\partial r^2}\big)^{k_2-j}D_x^\beta(r^{-\mu-\frac{1}{2}}\varphi)(r,x)\right|\\ &\leqslant C_m N^\mu_{m+l}(\varphi)\sum_{j=0}^{k_2}\left(\frac{k_2!}{j!(k_2-j)!}\right)\left(\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}\right)=2^{k_2}2^{|\alpha|}C_m N^\mu_{m+l}(\varphi)\\ &\leqslant 2^m C_m N^\mu_{m+l}(\varphi).\end{aligned}$$

This inequality shows that for every $\varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, the function $\theta\varphi$ belongs to the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and that the mapping $\varphi\longmapsto\theta\varphi$ is continuous from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ into itself.

. Suppose that θ is a multiplier of the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. Let ψ be the element of $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ defined by

$$\psi(r,x)=r^{\mu+\frac{1}{2}}e^{-r^2-|x|^2}.$$

From the hypothesis the function

$$\varphi(r,x)=\theta(r,x)\psi(r,x),$$

belongs to the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and we have

$$\theta(r,x)=r^{-\mu-\frac{1}{2}}e^{r^2+|x|^2}\varphi(r,x), \tag{2.2}$$

this shows that the function θ is infinitely differentiable on $]0,+\infty[\times\mathbb{R}^n$.

Now, the partial differential operators $\Box f(r,x) = r^{\mu+\frac{1}{2}}(\frac{\partial}{\partial r^2})(r^{-\mu-\frac{1}{2}}f)(r,x)$ and $\frac{\partial}{\partial x_j}$; $1 \leqslant j \leqslant n$, are continuous from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ into itself, and for every $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$, we have

$$(\frac{\partial}{\partial r^2})^k D_x^\alpha(\theta)\varphi = \sum_{j=0}^{k}\sum_{\beta+\gamma=\alpha}(-1)^{k-j}(-1)^{|\beta|}\frac{k!}{j!(k-j)!}\frac{\alpha!}{\beta!\gamma!}(\frac{\partial}{\partial r^2})^j D_x^\gamma\left(\Box^j\left(\theta\Box^{k-j}D_x^\beta\varphi\right)\right).$$

Since θ is a multiplier of the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, the last equality shows that for all $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$, the mapping

$$\varphi \longmapsto (\frac{\partial}{\partial r^2})^k D_x^\alpha(\theta)\varphi,$$

is continuous from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ into itself.

. Suppose that the function θ satisfies the assertion iii). From the relation (2.2), and for every $k \in \mathbb{N}$,

$$(\frac{\partial}{\partial r^2})^k(\theta)(r,x) = e^{r^2+|x|^2}\sum_{j=0}^{k}C_k^j 2^j(\frac{\partial}{\partial r^2})^{k-j}(r^{-\mu-\frac{1}{2}}\varphi)(r,x).$$

Let us prove that for every $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$ the function $(\frac{\partial}{\partial r^2})^k D_x^\alpha(\theta)$ is slowly increasing. In fact, suppose that there exists $(k_0,\alpha_0) \in \mathbb{N}\times\mathbb{N}^n$, such that the function $(\frac{\partial}{\partial r^2})^{k_0}D_x^{\alpha_0}(\theta)$ is not slowly increasing. Then, there exists a sequence $((r_j,x_j))_j \subset [0,\infty[\times\mathbb{R}^n$ such that

- $r_0^2+|x_0|^2 > 1.$
- $r_{j+1}^2+|x_{j+1}|^2 > 1+r_j^2+|x_j|^2.$
- $\dfrac{(\frac{\partial}{\partial r^2})^{k_0}D_x^{\alpha_0}(\theta)(r_j,x_j)}{(1+r_j^2+|x_j|^2)^j} > 1.$

From Lemma 2.3, the function

$$\varphi_0(r,x) = r^{\mu+\frac{1}{2}}\sum_{k=0}^{\infty}\frac{f(r^2+|x|^2-r_k^2-|x_k|^2+1)}{\left(1+r_k^2+|x_k|^2\right)^k}$$

belongs to the Hankel space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and for all $j \in \mathbb{N}$, we have

$$\left|(\frac{\partial}{\partial r^2})^{k_0}D_x^{\alpha_0}(\theta)(r_j,x_j)r_j^{-\mu-\frac{1}{2}}\varphi(r_j,x_j)\right| > \frac{f(1)}{(1+r_j^2+|x_j|^2)^j}\left|(\frac{\partial}{\partial r^2})^{k_0}D_x^{\alpha_0}(\theta)(r_j,x_j)\right|$$
$$= \frac{\left|(\frac{\partial}{\partial r^2})^{k_0}D_x^{\alpha_0}(\theta)(r_j,x_j)\right|}{(1+r_j^2+|x_j|^2)^j} > 1.$$

This contradicts the hypothesis, because

$$\lim_{j\to+\infty}(\frac{\partial}{\partial r^2})^{k_0}D_x^{\alpha_0}(\theta)(r_j,x_j)r_j^{-\mu-\frac{1}{2}}\varphi(r_j,x_j) = 0.$$

The proof of the theorem is complete. Q.E.D.

Remark 2.1. From Theorem 2.4 i) and ii), we deduce that the space of multipliers $\mathscr{M}_\mu([0,+\infty[\times\mathbb{R}^n)$ is independent of the real parameter μ and will be denoted by $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$.

In the following, we will define and study a topology of the space $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$.

For every $m\in\mathbb{N}$ and $\varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, we denote by $\rho^\mu_{m,\varphi}$ the seminorm defined on $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ by

$$\rho^\mu_{m,\varphi}(\theta)=\sup_{\substack{(r,x)\in[0,+\infty[\times\mathbb{R}^n\\ k+|\alpha|\leqslant m}}\left|r^{-\mu-\frac{1}{2}}\varphi(r,x)\big(\frac{\partial}{\partial r^2}\big)^k D^\alpha_x(\theta)(r,x)\right|.$$

and we define a basic of neighborhoods of zero in $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ by setting

$$\mathscr{W}^\mu(0)=\big\{B^\mu_{m,\varphi,\varepsilon}(0);\ m\in\mathbb{N},\ \varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n),\ \varepsilon>0\big\} \tag{2.3}$$

where

$$B^\mu_{m,\varphi,\varepsilon}(0)=\big\{\theta\in\mathscr{M}([0,+\infty[\times\mathbb{R}^n);\ \rho^\mu_{m,\varphi}(\theta)<\varepsilon\big\}.$$

Then, a sequence $(\theta_k)_k$ converges to zero in $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ if and only if for all $m\in\mathbb{N}$ and $\varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$,

$$\lim_{k\to+\infty}\rho^\mu_{m,\varphi}(\theta_k)=0.$$

Since the mapping $\varphi\longmapsto r^{\nu-\mu}\varphi=\Phi$ is a topological isomorphism from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ into $\mathbb{H}_\nu(]0,+\infty[\times\mathbb{R}^n)$ and using the fact that for all $m\in\mathbb{N}$ and $\theta\in\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$, we have

$$\rho^\mu_{m,\varphi}(\theta)=\rho^\nu_{m,\Phi}(\theta),$$

it follows that the set $\mathscr{W}^\mu(0)$ defined by the relation (2.3) is independent of the real parameter μ and will be denoted by $\mathscr{W}(0)$.

Proposition 2.5. *i) Let θ be an infinitely differentiable function on $[0,+\infty[\times\mathbb{R}^n$, such that for all $m\in\mathbb{N}$ and $\varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$; $\rho^\mu_{m,\varphi}(\theta)$ is finite, then the function θ lies in $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$.*

ii) The family of seminorms defined on $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ by

$$\gamma^\mu_{m,\varphi}(\theta)=N^\mu_m(\theta\varphi);\quad \theta\in\mathscr{M}([0,+\infty[\times\mathbb{R}^n)\text{ and }\varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n) \tag{2.4}$$

generates the same topology as the family $\{\rho^\mu_{m,\varphi};\ m\in\mathbb{N},\ \varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)\}$.

Proof. i) Let $\varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and $m\in\mathbb{N}$. By Leibniz formula, for all $k_1,k_2\in\mathbb{N},\alpha\in\mathbb{N}^\alpha$ such that $k_1+k_2+|\alpha|\leqslant m$, we get

$$\begin{aligned}\big(1+r^2+|x|^2\big)^{k_1}\big(\frac{\partial}{\partial r^2}\big)^{k_2}D^\alpha_x\big(r^{-\mu-\frac{1}{2}}\varphi(r,x)\theta(r,x)\big)&=\sum_{j=0}^{k_2}\sum_{\beta+\gamma=\alpha}\frac{k_2!}{j!(k_2-j)!}\frac{\alpha!}{\beta!\gamma!}\\ \times r^{-\mu-\frac{1}{2}}\big(\frac{\partial}{\partial r^2}\big)^{k_2-j}D^\beta_x\theta(r,x)\big(1+r^2+|x|^2\big)^{k_1}&r^{\mu+\frac{1}{2}}\big(\frac{\partial}{\partial r^2}\big)^{j}D^\gamma_x\big(r^{-\mu-\frac{1}{2}}\varphi\big)(r,x).\end{aligned}$$

Thus, for all $(r,x) \in [0,+\infty[\times\mathbb{R}^n$, we have

$$\begin{aligned}
&\big|\big(1+r^2+|x|^2\big)^{k_1}\big(\frac{\partial}{\partial r^2}\big)^{k_2} D_x^{\alpha}\big(r^{-\mu-\frac{1}{2}}\varphi(r,x)\theta(r,x)\big)\big| \\
&\qquad \leqslant \sum_{j=0}^{k_2}\sum_{\beta+\gamma=\alpha}\frac{k_2!}{j!(k_2-j)!}\frac{\alpha!}{\beta!\gamma!}\rho^{\mu}_{k_2-j+|\beta|,\Phi_{j,\gamma,k_1}}(\theta) \\
&\qquad \leqslant \sum_{j=0}^{k_2}\sum_{\beta+\gamma=\alpha}\frac{k_2!}{j!(k_2-j)!}\frac{\alpha!}{\beta!\gamma!}\rho^{\mu}_{m,\Phi_{j,\gamma,k_1}}(\theta).
\end{aligned} \tag{2.5}$$

Where, Φ_{j,γ,k_1} is the element of $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ given by

$$\Phi_{j,\gamma,k_1}(r,x) = \big(1+r^2+|x|^2\big)^{k_1} r^{\mu+\frac{1}{2}}\big(\frac{\partial}{\partial r^2}\big)^{j} D_x^{\gamma}\big(r^{-\mu-\frac{1}{2}}\varphi\big)(r,x). \tag{2.6}$$

The inequality (2.5) shows that for all $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$; the function $\theta\varphi$ belongs to the Hankel space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. The remainder of this proof is the same as the proof iii) implies i) in Theorem 2.4.

ii) Let $m,k_1,k_2 \in \mathbb{N}, \alpha \in \mathbb{N}^\alpha$ such that $k_1+k_2+|\alpha| \leqslant m$. Let $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and $\Phi_m \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ such that

$$\rho^{\mu}_{m,\Phi_m}(\theta) = \sup\{\rho^{\mu}_{m,\Phi_{j,\gamma,k_1}}(\theta), \quad j \leqslant k_2, \quad \gamma \leqslant \alpha; \quad k_1+k_2+|\alpha| \leqslant m\},$$

where the functions Φ_{j,γ,k_1} are given by the relation (2.6). The inequality (2.5) involves that

$$N^{\mu}_m(\theta\varphi) \leqslant 2^m \rho^{\mu}_{m,\Phi_m}(\theta), \tag{2.7}$$

which means that

$$\gamma^{\mu}_{m,\varphi}(\theta) \leqslant 2^m \rho^{\mu}_{m,\Phi_m}(\theta).$$

. Let θ and φ be two infinitely differentiable functions on $]0,\infty[\times\mathbb{R}^n$. By induction on $|\alpha|$, $\alpha \in \mathbb{N}^n$, we get

$$\varphi(r,x)D_x^{\alpha}\theta(r,x) = \sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}(-1)^{|\beta|}D_x^{\gamma}\big(\theta(r,x)D_x^{\beta}\varphi(r,x)\big). \tag{2.8}$$

And by induction on $k \in \mathbb{N}$, we get also

$$\varphi(r,x)\big(\frac{\partial}{\partial r^2}\big)^k\theta(r,x) = \sum_{p=0}^{k}(-1)^p\frac{k!}{p!(k-p)!}\big(\frac{\partial}{\partial r^2}\big)^{k-p}\big(\theta(r,x)\big(\frac{\partial}{\partial r^2}\big)^p\varphi(r,x)\big). \tag{2.9}$$

Combining the relations (2.8) and (2.9), we deduce that for all $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$

$$\begin{aligned}
&r^{-\mu-\frac{1}{2}}\varphi(r,x)\big(\frac{\partial}{\partial r^2}\big)^k D_x^{\alpha}\theta(r,x) = \\
&\sum_{p=0}^{k}\sum_{\beta+\gamma=\alpha}(-1)^{|\beta|+p}\frac{k!}{p!(k-p)!}\frac{\alpha!}{\beta!\gamma!}\big(\frac{\partial}{\partial r^2}\big)^{k-p}D_x^{\gamma}\Big(\theta(r,x)\big(\frac{\partial}{\partial r^2}\big)^p D_x^{\beta} r^{-\mu-\frac{1}{2}}\varphi(r,x)\Big).
\end{aligned} \tag{2.10}$$

Let $m \in \mathbb{N}$ and $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, from the last equality, it follows that for every $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$; $k+|\alpha| \leqslant m$, we have

$$\begin{aligned}\left|r^{-\mu-\frac{1}{2}}\varphi(r,x)(\frac{\partial}{\partial r^2})^k D_x^\alpha \theta(r,x)\right| &\leqslant \sum_{p=0}^{k}\sum_{\beta+\gamma=\alpha}\frac{k!}{p!(k-p)!}\frac{\alpha!}{\beta!\gamma!}N^\mu_{k-p+|\alpha|}(\theta\Phi_{p,\beta})\\ &\leqslant \sum_{p=0}^{k}\sum_{\beta+\gamma=\alpha}\frac{k!}{p!(k-p)!}\frac{\alpha!}{\beta!\gamma!}N^\mu_{m}(\theta\Phi_{p,\beta})\end{aligned}$$

where

$$\Phi_{p,\beta}(r,x) = r^{\mu+\frac{1}{2}}(\frac{\partial}{\partial r^2})^p D_x^\beta r^{-\mu-\frac{1}{2}}\varphi(r,x) \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n).$$

Now, let Φ_m be an element of $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, such that

$$\sup\{N^\mu_m(\theta\Phi_{p,\beta}),\ p+|\beta| \leqslant m\} = N^\mu_m(\theta\Phi_m).$$

Then,

$$\rho^\mu_{m,\varphi}(\theta) \leqslant 2^m \gamma_{m,\Phi_m}(\theta).$$

The proof of the proposition is complete. Q.E.D.

Let $\mathscr{C}^\infty([0,+\infty[\times\mathbb{R}^n)$ be the space of infinitely differential functions on $[0,+\infty[\times\mathbb{R}^n$ equipped with the family of seminorms $\{P_{m,l};\ (m,l) \in \mathbb{N}^2\}$ defined by

$$P_{m,l}(f) = \sup_{\substack{r^2+|x|^2\leqslant l^2\\ k+|\alpha|\leqslant m}} \left|(\frac{\partial}{\partial r})^k D_x^\alpha f(r,x)\right|$$

and the distance

$$d(f,g) = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{2^{m+l}}\frac{P_{m,l}(f-g)}{1+P_{m,l}(f-g)}.$$

Then, we have the following continuous embedding

Lemma 2.6. $\mathscr{M}([0,+\infty[\times\mathbb{R}^n) \hookrightarrow \mathscr{C}^\infty([0,+\infty[\times\mathbb{R}^n)$.

Proof. Let $\psi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$; $\psi(r,x) = r^{\mu+\frac{1}{2}}e^{-r^2-|x|^2}$. Let $m \in \mathbb{N}$. From the relation (2.10), for every $\theta \in \mathscr{M}([0,+\infty[\times\mathbb{R}^n)$, $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$, $k+|\alpha| \leqslant m$, we have

$$\begin{aligned}&(\frac{\partial}{\partial r^2})^k D_x^\alpha \theta(r,x) = e^{r^2+|x|^2}\\ &\sum_{p=0}^{k}\sum_{\beta+\gamma=\alpha}(-1)^{|\beta|+p}\frac{k!}{p!(k-p)!}\frac{\alpha!}{\beta!\gamma!}(\frac{\partial}{\partial r^2})^{k-p}D_x^\gamma\Big(\theta(r,x)(\frac{\partial}{\partial r^2})^p D_x^\beta r^{-\mu-\frac{1}{2}}\psi(r,x)\Big).\end{aligned}$$

However, for all $k \in \mathbb{N}$, there exist $k+1$ real polynomials, $Q_j,\ 0 \leqslant j \leqslant k$, such that

$$(\frac{\partial}{\partial r})^k = \sum_{j=0}^{k} Q_j(r)(\frac{\partial}{\partial r^2})^j$$

with $degree(Q_j) \leqslant j$. Hence,

$$\begin{aligned}
&(\frac{\partial}{\partial r^2})^k D_x^\alpha \theta(r,x) = e^{r^2+|x|^2} \sum_{p=0}^{k} \sum_{\beta+\gamma=\alpha} (-1)^{|\beta|+p} \frac{k!}{p!(k-p)!} \frac{\alpha!}{\beta!\gamma!} \\
&\times \Big\{ \sum_{j=0}^{k-p} Q_j(r) (\frac{\partial}{\partial r^2})^j D_x^\gamma \big(\theta(r,x) (\sum_{i=0}^{p} Q_i(r) (\frac{\partial}{\partial r^2})^i D_x^\beta r^{-\mu-\frac{1}{2}} \psi(r,x)) \big) \Big\} = \\
&e^{r^2+|x|^2} \sum_{p=0}^{k} \sum_{\beta+\gamma=\alpha} (-1)^{|\beta|+p} \frac{k!}{p!(k-p)!} \frac{\alpha!}{\beta!\gamma!} \sum_{j=0}^{k-p} Q_j(r) (\frac{\partial}{\partial r^2})^j D_x^\gamma \big(r^{-\mu-\frac{1}{2}} \psi_{p,\beta}(r,x) \big)
\end{aligned}$$

with

$$\psi_{p,\beta}(r,x) = r^{\mu+\frac{1}{2}} (\sum_{i=0}^{p} Q_i(r) (\frac{\partial}{\partial r^2})^i D_x^\beta r^{-\mu-\frac{1}{2}} \psi(r,x) \big).$$

Now, for all $0 \leqslant j \leqslant k-p$, there exists $C_j > 0$, such that

$$\big| Q_j(r) \big| \leqslant C_j (1+r^2)^j \leqslant C_j (1+r^2+|x|^2)^j$$

and consequently

$$\begin{aligned}
&\big| (\frac{\partial}{\partial r^2})^k D_x^\alpha \theta(r,x) \big| \leqslant e^{r^2+|x|^2} \times \\
&\sum_{p=0}^{k} \sum_{\beta+\gamma=\alpha} \frac{k!}{p!(k-p)!} \frac{\alpha!}{\beta!\gamma!} \sum_{j=0}^{k-p} C_j (1+r^2+|x|^2)^j \big| (\frac{\partial}{\partial r^2})^j D_x^\gamma \big(r^{-\mu-\frac{1}{2}} \psi_{p,\beta}(r,x) \big) \big| \\
&\leqslant e^{r^2+|x|^2} \sum_{p=0}^{k} \sum_{\beta+\gamma=\alpha} \frac{k!}{p!(k-p)!} \frac{\alpha!}{\beta!\gamma!} (\sum_{j=0}^{k-p} C_j) N_m^\mu (\theta \psi_{p,\beta}).
\end{aligned}$$

Let $\psi_m \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ such that

$$\sup\{ N_m^\mu(\theta\psi_{p,\beta}),\ p+|\beta| \leqslant m \} = N_m^\mu(\theta\psi_m).$$

Then, for all $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$ such that $k+|\alpha| \leqslant m$

$$\big| (\frac{\partial}{\partial r^2})^k D_x^\alpha \theta(r,x) \big| \leqslant C_m 2^m e^{r^2+|x|^2} N_m^\mu(\theta\psi_m),$$

where, $C_m = \sum_{j=0}^m C_j$. This equality shows that for all $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$

$$P_{l,m}(\theta) \leqslant 2^m C_m e^{l^2} \gamma^\mu_{m,\psi_m}(\theta).$$

Q.E.D.

Proposition 2.7. *The space $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ is Hausdorff and complete.*

Proof. • Let $\theta \in \mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ such that $\theta \neq 0$. Let $\varphi(r,x) = r^{\mu+\frac{1}{2}}e^{-r^2-|x|^2}$, then φ belongs to the Hankel space and we have

$$\rho^{\mu}_{0,\varphi}(\theta) = \sup_{(r,s)\in[0,\infty[\times\mathbb{R}^n} e^{-r^2-|x|^2}|\theta(r,x)| > 0,$$

this shows that the space $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ is separated
• Let $(\theta_k)_k$ be a Cauchy sequence in $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$. This means that for all $m \in \mathbb{N}$, $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$,

$$\rho^{\mu}_{m,\varphi}(\theta_k - \theta_{k'}) \xrightarrow[k,k'\to\infty]{} 0.$$

From Lemma 2.6 $(\theta_k)_k$ is a Cauchy's sequence in $\mathscr{C}^\infty([0,+\infty[\times\mathbb{R}^n)$ which is complete. Consequently, there exists $\theta \in \mathscr{C}^\infty([0,+\infty[\times\mathbb{R}^n)$ such that for all $m,l \in \mathbb{N}$

$$P_{m,l}(\theta_k - \theta) \xrightarrow[k\to\infty]{} 0.$$

Let $\varepsilon > 0$, for all $m \in \mathbb{N}$, $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ there exists $k_0 = k_0(m,\varphi,\varepsilon) \in \mathbb{N}$ such that

$$\forall k,k' > k_0; \quad \rho^{\mu}_{m,\varphi}(\theta_k - \theta_{k'}) < \varepsilon,$$

this means that for all $(r,x) \in [0,\infty[\times\mathbb{R}^n$ and $(p,\alpha) \in \mathbb{N}\times\mathbb{N}^n$; $p+|\alpha| \leqslant m$;

$$\left|r^{-\mu-\frac{1}{2}}\varphi(r,x)(\frac{\partial}{\partial r^2})^p D^{\alpha}_x(\theta_k - \theta_{k'})(r,x)\right| < \varepsilon.$$

and consequently

$$\left|r^{-\mu-\frac{1}{2}}\varphi(r,x)(\frac{\partial}{\partial r^2})^p D^{\alpha}_x(\theta_k - \theta)(r,x)\right| < \varepsilon.$$

This inequality shows that the function θ belongs to $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ and that for all $(m,\varphi) \in \mathbb{N}\times\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$,

$$\rho^{\mu}_{m,\varphi}(\theta_k - \theta) \xrightarrow[k\to\infty]{} 0.$$

Q.E.D.

In the following, we shall study the continuity of some operators defined on $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$.

Proposition 2.8.

i) The bilinear map

$$\mathscr{M}([0,+\infty[\times\mathbb{R}^n)\times\mathscr{M}([0,+\infty[\times\mathbb{R}^n) \to \mathscr{M}([0,+\infty[\times\mathbb{R}^n)$$
$$(\theta,\vartheta) \mapsto \theta\vartheta$$

is separately continuous.

ii) For every $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$, the map $\theta \mapsto (\frac{\partial}{\partial r^2})^k D^{\alpha}_x\theta(r,x)$ is continuous from $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ into itself.

Proof.

i) Fix $\theta \in \mathscr{M}([0,+\infty[\times\mathbb{R}^n)$. Let $\vartheta \in \mathscr{M}([0,+\infty[\times\mathbb{R}^n)$, $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and let $k,m \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$ such that $k+|\alpha| \leqslant m$.

It is clear that $\theta\vartheta$ is an infinitely differentiable function on $]0,\infty[\times\mathbb{R}^n$ and by applying Leibniz formula we get for all $(r,x) \in [0,\infty[\times\mathbb{R}^n$,

$$r^{-\mu-\frac{1}{2}}\varphi(r,x)(\frac{\partial}{\partial r^2})^k D_x^\alpha(\theta\vartheta)(r,x) = \sum_{j=0}^{k}\sum_{\beta+\gamma=\alpha}\frac{k!}{j!(k-j)!}\frac{\alpha!}{\beta!\gamma!}r^{-\mu-\frac{1}{2}}\varphi(r,x) \times(\frac{\partial}{\partial r^2})^j D_x^\beta\theta(r,x)(\frac{\partial}{\partial r^2})^{k-j}D_x^\gamma\vartheta(r,x).$$

From Theorem 2.4, there exist $C_{j,\beta} > 0$ and $m_{j,\beta} \in \mathbb{N}$ such that

$$\left|(\frac{\partial}{\partial r^2})^j D_x^\beta\theta(r,x)\right| \leqslant C_{j,\beta}(1+r^2+|x|^2)^{m_{j,\beta}}.$$

Thus, we have

$$\begin{aligned}&\left|r^{-\mu-\frac{1}{2}}\varphi(r,x)(\frac{\partial}{\partial r^2})^k D_x^\alpha(\theta\vartheta)(r,x)\right|\\ &\leqslant \sum_{j=0}^{k}\sum_{\beta+\gamma=\alpha}\frac{k!}{j!(k-j)!}\frac{\alpha!}{\beta!\gamma!}\left|r^{-\mu-\frac{1}{2}}\Phi_{j,\beta}(r,x)(\frac{\partial}{\partial r^2})^{k-j}D_x^\gamma\vartheta(r,x)\right|\\ &\leqslant \sum_{j=0}^{k}\sum_{\beta+\gamma=\alpha}\frac{k!}{j!(k-j)!}\frac{\alpha!}{\beta!\gamma!}\rho^\mu_{k-j+|\gamma|,\Phi_{j,\beta}}(\vartheta)\\ &\leqslant \sum_{j=0}^{k}\sum_{\beta+\gamma=\alpha}\frac{k!}{j!(k-j)!}\frac{\alpha!}{\beta!\gamma!}\rho^\mu_{m,\Phi_{j,\beta}}(\vartheta),\end{aligned}$$

where $\Phi_{j,\beta}$ is the element of $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ given by

$$\Phi_{j,\beta}(r,x) = C_{j,\beta}(1+r^2+|x|^2)^{m_{j,\beta}}\varphi(r,x).$$

Let $\Phi_m \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ such that

$$\rho^\mu_{m,\Phi_m}(\vartheta) = \sup\{\rho^\mu_{m,\Phi_{j,\beta}}(\vartheta),\ j+|\alpha| \leqslant m\}.$$

Then, the last inequality involves that

$$\rho^\mu_{m,\varphi}(\theta\vartheta) \leqslant 2^m\rho^\mu_{m,\Phi_m}(\vartheta).$$

ii) Let $m \in \mathbb{N}$, $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$. Then for all $\theta \in \mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ and $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, we have

$$\rho^\mu_{m,\varphi}((\frac{\partial}{\partial r^2})^k D_x^\alpha\theta) \leqslant \rho^\mu_{m+k+|\alpha|,\varphi}(\theta).$$

Which completes the proof. Q.E.D.

Proposition 2.9. *The bilinear mapping*

$$\mathscr{M}([0,+\infty[\times\mathbb{R}^n)\times\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)\to\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$$
$$(\theta,\varphi)\mapsto\theta\varphi$$

is separately continuous.

Proof. • From Definition 2.1, it follows that for every $\theta\in\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ the mapping $\varphi\mapsto\theta\varphi$ is continuous from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ into itself.
• Let $\varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. The continuity of the mapping $\theta\mapsto\theta\varphi$ from $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$ into $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ follows from the relation (2.7). Q.E.D.

Proposition 2.10. *The mapping* $\varphi\mapsto r^{-\mu-\frac{1}{2}}\varphi$ *is continuous from* $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ *into* $\mathscr{M}([0,+\infty[\times\mathbb{R}^n)$.

Proof. Let $\varphi,\ \varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and $m,k\in\mathbb{N}$, $\alpha\in\mathbb{N}^n$ such that $k+|\alpha|\leqslant m$, we have for all $(r,x)\in\,]0,\infty[\times\mathbb{R}^n$

$$\left|r^{-\mu-\frac{1}{2}}\varphi(r,x)(\frac{\partial}{\partial r^2})^kD_x^\alpha(r^{-\mu-\frac{1}{2}}\varphi)(r,x)\right|\leqslant N_0^\mu(\varphi)N_m^\mu(\varphi),$$

which implies that

$$\rho_{m,\varphi}^\mu(r^{-\mu-\frac{1}{2}}\varphi)\leqslant N_0^\mu(\varphi)N_m^\mu(\varphi).$$

Q.E.D.

3 The convolution space of the dual $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$.

Let $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ be the topological dual of the Hankel space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. To give the usual characterization of the dual $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ we use the fact that for all $\varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, the family

$$\mathcal{V}_\mu(\varphi)=\{V_{m,\varepsilon,\mu}(\varphi),\ m\in\mathbb{N},\varepsilon>0\}$$

is a basis of neighborhoods of φ in $(\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n),d_\mu)$. Where

$$V_{m,\varepsilon,\mu}(\varphi)=\{\psi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n);\ N_m^\mu(\varphi-\psi)<\varepsilon\}.$$

Thus, we have

Proposition 3.1. *A linear mapping*

$$T:\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)\longrightarrow\mathbb{C}$$

belongs to $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ *if and only if there exist a positive constant C and an integer m such that for all* $\varphi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$;

$$|<T,\varphi>|\leqslant CN_m^\mu(\varphi).\qquad(3.1)$$

The main result of this section consists to define a subspace of the dual $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ which permits to define and study the convolution product on $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$. For this we shall define the Hankel translation operators, the convolution product and the Fourier-Hankel transform and we recall some properties, see [6].

Definition 3.2. *1. For every* $(r,x) \in [0,+\infty[\times\mathbb{R}^n$; *the Hankel translation operator* $\tau^{\mu}_{(r,x)}$ *is defined on* $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ *by*

$$\tau^{\mu}_{(r,x)}(\varphi)(s,y) = \begin{cases} \displaystyle\int_{|r-s|}^{r+s} \varphi(t,x+y)\mathscr{W}_\mu(r,s,t)\frac{t^{\mu+\frac{1}{2}}}{2^{\mu}\Gamma(\mu+1)}dt; \ \mu > -\frac{1}{2} \\ \displaystyle\sqrt{\frac{2}{\pi}}\Big[\frac{\varphi(r+s,x+y)+\varphi(r-s,x+y)}{2}\Big]; \qquad \mu = -\frac{1}{2}. \end{cases}$$

2. *The convolution product of* φ, $\psi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, *is given by*

$$\varphi * \psi(r,x) = \int_0^{\infty}\int_{\mathbb{R}^n} \tau^{\mu}_{(r,-x)}(\check{\varphi})(s,y)\psi(s,y)\frac{dsdy}{(2\pi)^{n/2}}. \tag{3.2}$$

Where $\mathscr{W}_\mu$ *is the Hankel kernel given by*

$$\begin{cases} \dfrac{(rs)^{-\mu+\frac{1}{2}}\Gamma(\mu+1)\left[(r+s)^2-t^2\right]^{\mu-\frac{1}{2}}\left[t^2-(r-s)^2\right]^{\mu-\frac{1}{2}}}{2^{2\mu-1}\sqrt{\pi}\Gamma(\mu+\frac{1}{2})t^{2\mu}}; \ |r-s| < t < r+s \\ 0; \qquad \textit{otherwise}, \end{cases}$$

and $\check{\varphi}(s,y) = \varphi(s,-y)$.
To define the Fourier Hankel transform, we introduce the function $\varphi^{\mu}_{\lambda_0,\lambda}$, $(\lambda_0,\lambda) \in]0,\infty[\times\mathbb{R}^n$ to be

$$\varphi^{\mu}_{\lambda_0,\lambda}(r,x) = \mathscr{J}_\mu(r\lambda_0)e^{-i\langle\lambda|x\rangle}. \tag{3.3}$$

Where

- $\mathscr{J}_\mu$ is the modified Bessel function defined by

$$\mathscr{J}_\mu(z) = \sqrt{z}J_\mu(z).$$

  And J_μ is the Bessel function of the first kind and index μ (see [4, 3, 5, 9]).

- $\langle.|.\rangle$ is the usual inner product on $\mathbb{R}^n$, $\langle\lambda|x\rangle = \sum_{j=1}^{n}\lambda_j x_j$.

Definition 3.3. *The Fourier-Hankel transform* $\mathscr{H}_\mu$ *is defined on* $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ *by, for all* $(\lambda_0,\lambda) \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$;

$$\mathscr{H}_\mu(\varphi)(\lambda_0,\lambda) = \int_0^{\infty}\int_{\mathbb{R}^n}\varphi(r,x)\varphi^{\mu}_{\lambda_0,\lambda}(r,x)\frac{drdx}{(2\pi)^{\frac{n}{2}}}.$$

It was shown in [6] that

- $\mathscr{H}_\mu$ is a topological isomorphism from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ onto itself and that the inverse mapping is given by

$$\mathscr{H}_\mu^{-1}(f)(r,x) = \int_0^{\infty}\int_{\mathbb{R}^n} f(\lambda_0,\lambda)\overline{\varphi^{\mu}_{\lambda_0,\lambda}(r,x)}\frac{d\lambda_0 d\lambda}{(2\pi)^{\frac{n}{2}}}.$$

- For every $\psi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, $(r,x) \in]0,\infty[\times\mathbb{R}^n$ the function $\tau^\mu_{(r,x)}(\psi)$ belongs to $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and we have
$$\mathscr{H}_\mu(\tau^\mu_{(r,x)}(\psi))(\lambda_0,\lambda) = \lambda_0^{-\mu-\frac{1}{2}}\overline{\varphi^\mu_{\lambda_0,\lambda}(r,x)}\mathscr{H}_\mu(\psi)(\lambda_0,\lambda) \tag{3.4}$$

- For every $\varphi,\psi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, the function $\varphi*\psi$ belongs to the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and we have
$$\mathscr{H}_\mu(\varphi*\psi)(\lambda_0,\lambda) = \lambda_0^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(\varphi)(\lambda_0,\lambda)\mathscr{H}_\mu(\psi)(\lambda_0,\lambda),$$

The precedent result allows us to define the Fourier-Hankel transform $\mathscr{H}_\mu$ on $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ by
$$< \mathscr{H}_\mu(T),\varphi >=< T,\mathscr{H}_\mu(\varphi) >, \ \varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n).$$
Then, $\mathscr{H}_\mu$ becomes a topological isomorphism from $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ onto itself.

Next, we establish other properties for the translation operator and the convolution product that we use later.

Proposition 3.4. *For every $(r,x) \in]0,\infty[\times\mathbb{R}^n$, the Hankel translation operator $\tau^\mu_{(r,x)}$ is continuous from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ into itself. Moreover, for all $m \in \mathbb{N}$, there exist $m_1,m_2 \in \mathbb{N}$ and $C>0$ such that*
$$\forall\psi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n), \quad N^\mu_m\left(\tau^\mu_{(r,x)}(\psi)\right) \leqslant C(1+r^2+|x|^2)^{m_1}N^\mu_{m_2}(\psi). \tag{3.5}$$

Proof. From the relation (3.4), we have, for every $\psi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$
$$\tau^\mu_{(r,x)}(\psi)(s,y) = \mathscr{H}_\mu^{-1}\left(\lambda_0^{-\mu-\frac{1}{2}}\overline{\varphi^\mu_{\lambda_0,\lambda}(r,x)}\mathscr{H}_\mu(\psi)\right)(s,y).$$
Since the transform $\mathscr{H}_\mu^{-1}$ is continuous from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ onto itself, for every $m \in \mathbb{N}$, there exist $m' \in \mathbb{N}$ and $C>0$ such that
$$\begin{aligned} N^\mu_m\left(\tau^\mu_{(r,x)}(\psi)\right) &= N^\mu_m\left(\mathscr{H}_\mu^{-1}\left(\lambda_0^{-\mu-\frac{1}{2}}\overline{\varphi^\mu_{\lambda_0,\lambda}(r,x)}\mathscr{H}_\mu(\psi)\right)\right) \\ &\leqslant CN^\mu_{m'}\left(\lambda_0^{-\mu-\frac{1}{2}}\overline{\varphi^\mu_{\lambda_0,\lambda}(r,x)}\mathscr{H}_\mu(\psi)\right). \end{aligned}$$

Q.E.D.

Let
$$\begin{aligned} f(\lambda_0,\lambda) &= \lambda_0^{-\mu-\frac{1}{2}}\overline{\varphi^\mu_{\lambda_0,\lambda}(r,x)}\mathscr{H}_\mu(\psi)(\lambda_0,\lambda) \\ &= \frac{r^{\mu+1/2}}{2^\mu\Gamma(\mu+1)}j_\mu(\lambda_0 r)e^{i\langle\lambda|x\rangle}\mathscr{H}_\mu(\psi)(\lambda_0,\lambda), \end{aligned}$$

where j_μ is the modified Bessel function defined by

$$j_\mu(s) = 2^\mu \Gamma(\mu+1)\frac{J_\mu(s)}{s^\mu}$$
$$= \begin{cases} \dfrac{2\Gamma(\mu+1)}{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})} \displaystyle\int_0^1 \left(1-t^2\right)^{\mu-\frac{1}{2}} \cos(st)dt; & \mu > -\frac{1}{2} \\ \cos(s); & \mu = -\frac{1}{2} \end{cases}$$

It is clear that for every $k \in \mathbb{N}$,

$$\left(\frac{\partial}{\partial \lambda_0^2}\right)^k (j_\mu(\lambda_0 r)) = \frac{(-r^2)^k}{2^k \Gamma(\mu+k+1)} j_{\mu+k}(\lambda_0 r). \tag{3.6}$$

Thus, from Leibniz formula, we have

$$\left(\frac{\partial}{\partial \lambda_0^2}\right)^k \left(\lambda_0^{-\mu-1/2} f(\lambda_0,\lambda)\right) =$$
$$\frac{r^{\mu+1/2}}{2^\mu \Gamma(\mu+1)} e^{i\langle\lambda|x\rangle} \sum_{l=0}^k C_k^l \left(\frac{\partial}{\partial \lambda_0^2}\right)^l (j_\mu(\lambda_0 r)) \left(\frac{\partial}{\partial \lambda_0^2}\right)^{k-l} \left(\lambda_0^{-\mu-1/2} \mathscr{H}_\mu(\psi)(\lambda_0,\lambda)\right),$$

and from the relation (3.6), for every $\alpha \in \mathbb{N}^n$

$$D_\lambda^\alpha \left(\frac{\partial}{\partial \lambda_0^2}\right)^k \left(\lambda_0^{-\mu-1/2} f(\lambda_0,\lambda)\right) = \frac{r^{\mu+1/2}}{2^\mu \Gamma(\mu+1)} \times$$
$$\sum_{l=0}^k C_k^l (-1)^l \frac{r^{2l}}{2^l \Gamma(\mu+l+1)} j_{\mu+l}(\lambda_0 r) D_\lambda^\alpha \left(e^{i\langle\lambda|x\rangle} \left(\frac{\partial}{\partial \lambda_0^2}\right)^{k-l} \left(\lambda_0^{-\mu-1/2} \mathscr{H}_\mu(\psi)(\lambda_0,\lambda)\right)\right)$$
$$= \frac{r^{\mu+1/2}}{2^\mu \Gamma(\mu+1)} \sum_{l=0}^k C_k^l (-1)^l \frac{r^{2l}}{2^l \Gamma(\mu+l+1)} j_{\mu+l}(\lambda_0 r) \times$$
$$\sum_{\beta \leqslant \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} (ix)^\beta e^{i\langle\lambda|x\rangle} D_\lambda^{\alpha-\beta} \left(\left(\frac{\partial}{\partial \lambda_0^2}\right)^{k-l} \left(\lambda_0^{-\mu-1/2} \mathscr{H}_\mu(\psi)(\lambda_0,\lambda)\right)\right).$$

Let $k_1, k_2 \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$ such that $k_1 + k_2 + |\alpha| \leqslant m'$. For every $(\lambda_0,\lambda) \in [0,+\infty[\times\mathbb{R}^n$,

$$\left| (1+\lambda_0^2+|\lambda|^2)^{k_1} D_\lambda^\alpha \left(\frac{\partial}{\partial \lambda_0^2}\right)^{k_2} \left(\lambda_0^{-\mu-1/2} f(\lambda_0,\lambda)\right) \right|$$
$$\leqslant C_1 2^{k_2+|\alpha|} (1+r^2+|x|^2)^{2m'+[\mu+1/2]+1} N_{m'}^\mu(\mathscr{H}_\mu(\psi))$$
$$\leqslant C_2 (1+r^2+|x|^2)^{2m'+[\mu+1/2]+1} N_{m''}^\mu(\psi).$$

Which completes the proof.

Proposition 3.5. *For every $T \in \mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ and $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, the function defined by*

$$T * \varphi(r,x) = \langle T, \tau^\mu_{(r,-x)}(\check{\varphi})\rangle$$

is continuous on $[0,\infty[\times\mathbb{R}^n$ and slowly increasing.

Proof. Let $T \in \mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ and $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. From Proposition 3.4 and for every $(r,x) \in [0,\infty[\times\mathbb{R}^n$, the function $\tau^\mu_{(r,x)}(\check{\varphi})$ belongs to the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. Hence, the convolution product $T*\varphi$ is well defined. Let $((r_k,x_k))_k \subset [0,\infty[\times\mathbb{R}^n$ such that $\lim_{k\to\infty}(r_k,x_k) = (r,x)$.

Let us prove that the sequence $\left(\tau^\mu_{(r_k,-x_k)}(\check{\varphi})\right)_k$ converges to $\tau^\mu_{(r,-x)}(\check{\varphi})$ in $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$.

Since the Fourier-Hankel transform is a topological isomorphism from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ onto itself, it is enough to show that

$$\lim_{k\to\infty} \mathscr{H}_\mu\left(\tau^\mu_{(r_k,-x_k)}(\check{\varphi})\right) = \mathscr{H}_\mu\left(\tau^\mu_{(r,-x)}(\check{\varphi})\right)$$

in $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$.

By relation (3.4), for every $(\lambda_0,\lambda) \in [0,\infty[\times\mathbb{R}^n$,

$$\begin{aligned}\mathscr{H}_\mu\left(\tau^\mu_{(r_k,-x_k)}(\check{\varphi})\right)(\lambda_0,\lambda) &= \lambda_0^{-\mu-\frac{1}{2}}\varphi^\mu_{\lambda_0,\lambda}(r_k,x_k)\mathscr{H}_\mu(\check{\varphi})(\lambda_0,\lambda)\\ &= \lambda_0^{-\mu-\frac{1}{2}}\mathscr{J}_\mu(\lambda_0 r_k)e^{-i\langle\lambda,x_k\rangle}\mathscr{H}_\mu(\check{\varphi})(\lambda_0,\lambda)\\ &= \frac{r_k^{\mu+1/2}}{2^\mu\Gamma(\mu+1)} j_\mu(r_k\lambda_0)e^{-i\langle\lambda,x_k\rangle}\mathscr{H}_\mu(\check{\varphi})(\lambda_0,\lambda).\end{aligned}$$

Thus, for every $(\lambda_0,\lambda) \in [0,\infty[\times\mathbb{R}^n$,

$$\begin{aligned}&\mathscr{H}_\mu\left(\tau^\mu_{(r_k,-x_k)}(\check{\varphi}) - \tau^\mu_{(r,-x)}(\check{\varphi})\right)(\lambda_0,\lambda) =\\ &\left(r_k^{\mu+1/2} j_\mu(r_k\lambda_0)e^{-i\langle\lambda,x_k\rangle} - r^{\mu+1/2} j_\mu(r\lambda_0)e^{-i\langle\lambda,x\rangle}\right)\times\frac{\mathscr{H}_\mu(\check{\varphi})(\lambda_0,\lambda)}{2^\mu\Gamma(\mu+1)}.\end{aligned}$$

By standard computation and using the relation (3.6), we deduce that for every $(k_1,k_2,\alpha) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}^n$,

$$\lim_{k\to\infty}\sup_{(\lambda_0,\lambda)\in[0,\infty[\times\mathbb{R}^n}\left(1+\lambda_0^2+|\lambda|^2\right)^{k_1}\left|\left(\frac{\partial}{\partial\lambda_0^2}\right)^{k_2}D_x^\alpha\left(\mathscr{H}_\mu\left(\tau^\mu_{(r_k,-x_k)}(\check{\varphi}) - \tau^\mu_{(r,-x)}(\check{\varphi})\right)(\lambda_0,\lambda)\right)\right| = 0,$$

which means that $\left(\tau^\mu_{(r_k,-x_k)}(\check{\varphi})\right)$ converges to $\left(\tau^\mu_{(r,-x)}(\check{\varphi})\right)$ in $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. Since $T \in \mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$, then

$$\lim_{k\to\infty}\langle T,\tau^\mu_{(r_k,-x_k)}(\check{\varphi}\rangle = \langle T,\tau^\mu_{(r,-x)}(\check{\varphi}\rangle,$$

and consequently, the function $T*\varphi$ is continuous on $[0,\infty[\times\mathbb{R}^n$.

Moreover, from relation (3.1), there exist $m \in \mathbb{N}$ and $C_1 > 0$ such that for every $(r,x) \in [0,\infty[\times\mathbb{R}^n$,

$$|T*\varphi(r,x)| \leqslant C_1 N^\mu_m(\tau^\mu_{(r,-x)}(\check{\varphi})),$$

and by relation (3.5)

$$|T*\varphi(r,x)| \leqslant C_2\left(1+r^2+|x|^2\right)^{m_1} N^\mu_{m_2}(\varphi),$$

so the function $T*\varphi$ is slowly increasing and the proof is complete. Q.E.D.

Lemma 3.6. *Let* $\varphi, \psi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. *Then, for every* $X>0$, *the sequence* $(\theta_{X,N})_N$, $N=(N_0,...,N_n)\in\mathbb{N}^{n+1}$, *defined by*

$$\theta_{X,N}(s,y)=\frac{X}{N_0}\frac{2X}{N_1}...\frac{2X}{N_n}\sum_{k_0=0}^{N_0-1}...\sum_{k_n=0}^{N_n-1}\tau^{\mu}_{(\frac{k_0X}{N_0},-X+\frac{2X}{N_1}k_1,...,-X+\frac{2X}{N_n}k_n)}(\varphi)(s,y)$$
$$\psi(\frac{k_0X}{N_0},-X+\frac{2X}{N_1}k_1,...,-X+\frac{2X}{N_n}k_n)$$

converges in $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ *to the function*

$$\theta_X(s,y)=\int_0^X\int_{[-X,X]^n}\tau^{\mu}_{(r,x)}(\varphi)(s,y)\psi(r,x)drdx.$$

Proof. From Proposition 3.4 the function $\theta_{X,N}$ belongs to the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. Now, for every $(s,y)\in]0,\infty[\times\mathbb{R}^n$, we have

$$\theta_{X,N}(s,y)-\theta_X(s,y)=\sum_{k_0=0}^{N_0-1}...\sum_{k_n=0}^{N_n-1}\int_{\frac{k_0X}{N_0}}^{\frac{k_0+1}{N_0}X}...\int_{-X+\frac{k_nX}{N_n}}^{-X+\frac{k_n+1}{N_n}X}$$
$$\left(\tau_{\frac{k_0X}{N_0},...,-X+\frac{2k_n}{N_n}X}(\varphi)(s,y)\Psi\left(\frac{k_0X}{N_0},...,-X+\frac{2k_n}{N_n}X\right)-\tau_{(r,x)}(\varphi)(s,y)\Psi(r,x)\right)drdx.$$

Let $(k_1,k_2,\alpha)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}^n$, then

$$(1+s^2+|y|^2)^{k_1}(\frac{\partial}{\partial s^2})D_y^{\alpha}\left(s^{-\mu-\frac{1}{2}}(\theta_{X,N}-\theta_X)(s,y)\right)$$
$$=\sum_{k_0=0}^{N_0-1}...\sum_{k_n=0}^{N_n-1}\int_{\frac{k_0X}{N_0}}^{\frac{k_0+1}{N_0}X}...\int_{-X+\frac{k_nX}{N_n}}^{-X+\frac{k_n+1}{N_n}X}(1+s^2+|y|^2)^{k_1}(\frac{\partial}{\partial s^2})D_y^{\alpha}$$
$$\left(s^{-\mu-\frac{1}{2}}\left(\tau_{\frac{k_0X}{N_0},...,-X+\frac{2k_n}{N_n}X}(\varphi)(s,y)\Psi\left(\frac{k_0X}{N_0},...,-X+\frac{2k_n}{N_n}X\right)-\tau_{(r,x)}(\varphi)(s,y)\Psi(r,x)\right)\right)drdx$$
$$=\sum_{k_0=0}^{N_0-1}...\sum_{k_n=0}^{N_n-1}\int_{\frac{k_0X}{N_0}}^{\frac{k_0+1}{N_0}X}...\int_{-X+\frac{k_nX}{N_n}}^{-X+\frac{k_n+1}{N_n}X}\left(F(\frac{k_0X}{N_0},...,-X+\frac{2k_n}{N_n}X)-F(r,x,s,y)\right)drdx$$

where $F:([0,+\infty[\times\mathbb{R}^n)^2\to\mathbb{C}$ is defined by

$$F(r,x,s,y)=(1+s^2+|y|^2)^{k_1}(\frac{\partial}{\partial s^2})D_y^{\alpha}\left(\tau_{(r,x)}(\varphi)(s,y)\Psi(r,x)\right).$$

The function F is continuous on $([0,+\infty[\times\mathbb{R}^n)^2$. Moreover,

$$(1+r^2+s^2+|x|^2+|y|^2)|F(r,x,s,y)|$$
$$\leqslant(1+s^2+|y|^2)^{k_1+1}(\frac{\partial}{\partial s^2})^{k_2}D_y^{\alpha}\left(s^{-\mu-\frac{1}{2}}\tau_{(r,x)}(\varphi)(s,y)\right)(1+r^2+|x|^2)|\Psi(r,x)|$$
$$\leqslant(1+s^2+|y|^2)^{k_1+1}(\frac{\partial}{\partial s^2})^{k_2}D_y^{\alpha}\left(s^{-\mu-\frac{1}{2}}\tau_{(r,x)}(\varphi)(s,y)\right)(1+r^2+|x|^2)^{2+[\mu+\frac{1}{2}]}|r^{-\mu-\frac{1}{2}}\Psi(r,x)|$$
$$\leqslant N^{\mu}_{k_1+k_2+|\alpha|+1}\left(\tau^{\mu}_{(r,x)}(\varphi)\right)(1+r^2+|x|^2)^{2+[\mu+\frac{1}{2}]}|r^{-\mu-\frac{1}{2}}\Psi(r,x)|$$

and by Proposition 3.4, we get

$$\begin{aligned}&(1+r^2+s^2+|x|^2+|y|^2)|F(r,x,s,y)|\\ &\leqslant CN^{\mu}_{m_1}(\varphi)(1+r^2+|x|^2)^{m_2+[\mu+\frac{1}{2}]}|(r^{-\mu-\frac{1}{2}}\Psi(r,x))|\\ &\leqslant CN^{\mu}_{m_1}(\varphi)N^{\mu}_{m_2+[\mu+\frac{1}{2}]}(\Psi).\end{aligned}$$

The last inequality shows that

$$\lim_{r^2+s^2+|y|^2+|x|^2\to+\infty} F(r,x,s,y)=0$$

and consequently, the function F is uniformly continuous.

Let $\varepsilon>0$, there exists $\alpha>0$ such that for $|r-r'|<\alpha$, $|x_j-x'_j|<\alpha$, $1\leqslant j\leqslant n$; we have for every $(s,y)\in[0,+\infty[\times\mathbb{R}^n$,

$$|F(r,x,s,y)-F(r',x',s,y)|\leqslant\varepsilon.$$

So for $(N_0,...,N_n)\in(\mathbb{N}^*)^{n+1}$, such that $\frac{X}{N_0}<\alpha$, $\frac{2X}{N_j}<\alpha$, $1\leqslant j\leqslant n$, we get for every $(s,y)\in[0,+\infty[\times\mathbb{R}^n$,

$$\begin{aligned}&\left|(1+s^2+|y|^2)^{k_1}(\frac{\partial}{\partial s^2})D^{\alpha}_y\left(s^{-\mu-\frac{1}{2}}(\theta_{X,N}-\theta_X)(s,y)\right)\right|\\ &\qquad\leqslant\varepsilon\sum_{k_0=0}^{N_0-1}...\sum_{k_n=0}^{N_n-1}\int_{\frac{k_0X}{N_0}}^{\frac{k_0+1}{N_0}X}...\int_{-X+\frac{k_nX}{N_n}}^{-X+\frac{k_n+1}{N_n}X}drdx_1...dx_n\\ &\qquad\leqslant\varepsilon\sum_{k_0=0}^{N_0-1}...\sum_{k_n=0}^{N_n-1}\frac{X}{N_0}\frac{2X}{N_1}...\frac{2X}{N_n}=\varepsilon 2^nX^{n+1}.\end{aligned}$$

This proves that for every $(k_1,k_2,\alpha)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}^n$,

$$\sup_{(s,y)\in[0,+\infty[\times\mathbb{R}^n}\left|(1+s^2+|y|^2)^{k_1}(\frac{\partial}{\partial s^2})D^{\alpha}_y\left(s^{-\mu-\frac{1}{2}}(\theta_{X,N}-\theta_X)(s,y)\right)\right|\xrightarrow[(N_0,...,N_n)\to(+\infty,...,+\infty)]{}0.$$

Which achieves the proof. Q.E.D.

Theorem 3.7. *For all* $\varphi,\psi\in\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ *and* $T\in\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$*, we have*

$$\int_0^\infty\int_{\mathbb{R}^n}\langle T,\tau^{\mu}_{(r,x)}(\varphi)\rangle\psi(r,x)drdx=\langle T,\int_0^\infty\int_{\mathbb{R}^n}\tau^{\mu}_{(r,x)}(\varphi)(.,.)\psi(r,x)drdx\rangle.$$

Proof. From Proposition 3.5, the integral

$$\int_0^\infty\int_{\mathbb{R}^n}\langle T,\tau^{\mu}_{(r,x)}(\varphi)\rangle\psi(r,x)drdx=\int_0^\infty\int_{\mathbb{R}^n}T*\check{\varphi}(r,-x)\psi(r,x)drdx,$$

is well defined. Since the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ is stable under convolution product, the function

$$(s,y)\mapsto\int_0^\infty\int_{\mathbb{R}^n}\tau^{\mu}_{(r,x)}(\varphi)(s,y)\psi(r,x)drdx=\check{\varphi}*\psi(s,-y)$$

belongs to $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, and then $\langle T, \int_0^\infty \int_{\mathbb{R}^n} \tau^\mu_{(r,x)}(\varphi)(.,.)\psi(r,x)drdx\rangle$ is also well defined.

Let $X>0$, by Lemma 3.6, the function

$$\theta_X(s,y) = \int_0^X \int_{[-X,X]^n} \tau^\mu_{(r,x)}(\varphi)(s,y)\psi(r,x)drdx$$

belongs to the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$. It follows that the function

$$(s,y) \mapsto \int\int_{([0,X]\times[-X,X]^n)^c} \tau^\mu_{(r,x)}(\varphi)(s,y)\psi(r,x)drdx$$

lies in $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and we have

$$\begin{aligned}&\langle T, \int_0^\infty \int_{\mathbb{R}^n} \tau^\mu_{(r,x)}(\varphi)(.,.)\psi(r,x)drdx\rangle = \\ &\langle T, \int_0^X \int_{[-X,X]^n} \tau^\mu_{(r,x)}(\varphi)(.,.)\psi(r,x)drdx\rangle + \langle T, \int\int_{([0,X]\times[-X,X]^n)^c} \tau^\mu_{(r,x)}(\varphi)(s,y)\psi(r,x)drdx\rangle.\end{aligned} \tag{3.7}$$

Let $F_X(s,y) = \int\int_{([0,X]\times[-X,X]^n)^c} \tau^\mu_{(r,x)}(\varphi)(s,y)\psi(r,x)drdx.$

Then, for every $m\in\mathbb{N}$,

$$N^\mu_m(F_X) \leqslant \int\int_{([0,X]\times[-X,X]^n)^c} N^\mu_m(\tau^\mu_{(r,x)}(\varphi))|\psi(r,x)|drdx$$

and from (3.5), we get

$$N^\mu_m(F_X) \leqslant CN_{m_2}(\varphi)\int\int_{([0,X]\times[-X,X]^n)^c}(1+r^2+|x|^2)^{m_1}|\psi(r,x)|drdx.$$

the last inequality shows that

$$\lim_{X\to\infty} F_X = 0, \text{ in } \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n),$$

and by relation (3.7), we get

$$\begin{aligned}\langle T, \int_0^\infty \int_{\mathbb{R}^n} \tau^\mu_{(r,x)}(\varphi)(.,.)\psi(r,x)drdx\rangle &= \lim_{X\to\infty}\langle T, \int_0^X \int_{[-X,X]^n} \tau^\mu_{(r,x)}(\varphi)(.,.)\psi(r,x)drdx\rangle \\ &= \lim_{X\to\infty}\langle T,\theta_X\rangle.\end{aligned}$$

Let $\theta_{N,X}, N=(N_0,...,N_n)$ be the sequence defined in Lemma 3.6, then

$$\begin{aligned}&\langle T,\theta_X\rangle = \lim_{N\to(\infty,...,\infty)}\langle T,\theta_{N,X}\rangle \\ &= \lim_{N\to(\infty,...,\infty)} \frac{X}{N_0}\frac{2X}{N_1}\cdots\frac{2X}{N_n}\sum_{k_0=0}^{N_0-1}\cdots\sum_{k_n=0}^{N_n-1}\psi\left(\frac{k_0X}{N_0}, -X+\frac{2X}{N_1}k_1,...,-X+\frac{2X}{N_n}k_n\right) \\ &\langle T, \tau^\mu_{(\frac{k_0X}{N_0},-X+\frac{2X}{N_1}k_1,...,-X+\frac{2X}{N_n}k_n)}(\varphi)(.,.)\rangle \\ &= \int_0^X\int_{[-X,X]^n}\langle T,\tau^\mu_{(r,x)}(\varphi)\rangle\psi(r,x)drdx.\end{aligned}$$

Finally,

$$\langle T, \int_0^\infty \int_{\mathbb{R}^n} \tau^\mu_{(r,x)}(\varphi)(.,.)\psi(r,x)drdx\rangle = \lim_{X\to\infty}\int_0^X \int_{[-X,X]^n} \langle T, \tau^\mu_{(r,x)}(\varphi)\rangle\psi(r,x)drdx$$
$$= \int_0^\infty \int_{\mathbb{R}^n} \langle T, \tau^\mu_{(r,x)}(\varphi)\rangle\psi(r,x)drdx.$$

Q.E.D.

Proposition 3.8. *For every $T \in \mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ and every $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$; we have*

$$\mathscr{H}_\mu(T_{T*\varphi}) = \lambda^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(\varphi)\mathscr{H}_\mu(T).$$

Where $T_{T*\varphi}$ is the element of $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$, defined by

$$\langle T_{T*\varphi}, \psi\rangle = \int_0^{+\infty}\int_{\mathbb{R}^n} T*\varphi(r,x)\psi(r,x)\frac{drdx}{(2\pi)^{\frac{n}{2}}}.$$

Proof. From Proposition 3.5, for $T \in \mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ and $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, the function $T*\varphi$ is continuous on $[0,+\infty[\times\mathbb{R}^n$, and slowly increasing. Thus, $T_{T*\varphi}$ is an element of $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ and for every $\psi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, we have

$$\langle \mathscr{H}_\mu(T_{T*\varphi}), \psi\rangle = \langle T_{T*\varphi}, \mathscr{H}_\mu(\psi)\rangle = \int_0^{+\infty}\int_{\mathbb{R}^n} \langle T, \tau^\mu_{(r,-x)}(\check{\varphi})\rangle\mathscr{H}_\mu(\psi)(r,x)drdx.$$

Applying Theorem 3.7, we obtain

$$\langle \mathscr{H}_\mu(T_{T*\varphi}), \psi\rangle = \langle T, \int_0^{+\infty}\int_{\mathbb{R}^n} \tau^\mu_{(r,-x)}(\check{\varphi})(.,.)\mathscr{H}_\mu(\psi)(r,x)\frac{drdx}{(2\pi)^{n/2}}\rangle. \tag{3.8}$$

Now, for every $(s,y) \in [0,+\infty[\times\mathbb{R}^n$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \tau^\mu_{(r,-x)}(\varphi)(s,y)\mathscr{H}_\mu(\psi)(r,x)\frac{drdx}{(2\pi)^{n/2}}$$
$$= \int_0^{+\infty}\int_{\mathbb{R}^n} \tau^\mu_{(s,-y)}(\check{\varphi})(r,x)\mathscr{H}_\mu(\psi)(r,x)\frac{drdx}{(2\pi)^{n/2}}$$
$$= \int_0^{+\infty}\int_{\mathbb{R}^n} \mathscr{H}_\mu(\tau^\mu_{(s,-y)}(\varphi))(r,x)\psi(r,x)\frac{drdx}{(2\pi)^{n/2}}.$$

By means of relation (3.4), we obtain

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \tau_{(r,-x)}(\check{\varphi})(s,y)\mathscr{H}_\mu(\psi)(r,x)\frac{drdx}{(2\pi)^{n/2}}$$
$$= \int_0^{+\infty}\int_{\mathbb{R}^n} r^{-\mu-\frac{1}{2}}\varphi^\mu_{s,y}(r,x)\mathscr{H}_\mu(\varphi)(r,x)\psi(r,x)\frac{drdx}{(2\pi)^{1/2}}$$
$$= \mathscr{H}_\mu(r^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(\varphi)\psi)(s,y).$$

Replacing in (3.8), it follows that for $\varphi, \psi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and $T \in \mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$,

$$\begin{aligned}\langle \mathscr{H}_\mu(T_{T*\varphi}), \psi\rangle &= \langle T, \mathscr{H}_\mu(r^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(\varphi)\psi)\rangle \\ &= \langle r^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(\varphi)\mathscr{H}_\mu(T), \psi\rangle.\end{aligned}$$

This completes the proof. Q.E.D.

We denote by $\mathbb{M}(]0,\infty[\times\mathbb{R}^n)$ the subspace of $\mathscr{M}(]0,\infty[\times\mathbb{R}^n)$ consisting of functions f such that for every $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$, there is $m = m(k,\alpha) \in \mathbb{N}$, for which the function

$$(r,x) \mapsto (1+r^2+|x|^2)^m \left(\frac{\partial}{\partial r^2}\right)^k D_x^\alpha(f(r,x)),$$

is bounded on $[0,+\infty[\times\mathbb{R}^n$.

$\mathbb{M}(]0,\infty[\times\mathbb{R}^n)$ is equipped with the topology induced by $\mathscr{M}(]0,\infty[\times\mathbb{R}^n)$.

Definition 3.9. *We define the space $\mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$ to be the subspace of $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ formed by the distributions T such that $\mathscr{H}_\mu(T)$ is an infinitely differentiable function on $[0,+\infty[\times\mathbb{R}^n$, verifying for every $(k,\alpha) \in \mathbb{N}\times\mathbb{N}^n$, there exists $m = m(k,\alpha) \in \mathbb{N}$, such that the function*

$$(r,x) \mapsto (1+r^2+|x|^2)^m \left(\frac{\partial}{\partial r^2}\right)^k D_x^\alpha(r^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(T))(r,x)$$

is bounded on $[0,+\infty[\times\mathbb{R}^n$.

The space $\mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$ is endowed with the topology generated by the family

$$Q^\mu_{m,\varphi}(T) = \gamma^\mu_{m,\varphi}(r^{\mu+\frac{1}{2}}\mathscr{H}_\mu(T)), \quad \forall\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n),$$

where, $\gamma^\mu_{m,\varphi}$ is defined by relation (2.4).

Remark 3.1. It is clear from Definition 3.9, that for every $T \in \mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$, the function

$$(r,x) \mapsto r^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(T)(r,x),$$

is a multiplier of the space $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$.

Lemma 3.10. *For every $T \in \mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$, the mapping $\varphi \mapsto T*\varphi$ is continuous from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ into itself.*

Proof. From Proposition 3.8 and Definition 3.9, for every $T \in \mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$ and every $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$, we have

$$\mathscr{H}_\mu(T*\varphi)(\lambda_0,\lambda) = \lambda_0^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(T)(\lambda_0,\lambda)\mathscr{H}_\mu(\varphi)(\lambda_0,\lambda).$$

Now, from Remark 3.1, the mapping

$$\psi \mapsto \lambda_0^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(T)\psi$$

is continuous from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ into itself, then the result follows from the fact that $\mathscr{H}_\mu$ is a topological isomorphism from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ onto itself. Q.E.D.

Proposition 3.11. *The Hankel transform* $\mathscr{H}_\mu$ *is a topological isomorphism from* $\mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$ *into* $r^{\mu+\frac{1}{2}}\mathbb{M}(]0,\infty[\times\mathbb{R}^n)$.

Where $r^{\mu+\frac{1}{2}}\mathbb{M}(]0,\infty[\times\mathbb{R}^n)$ denotes the space of functions f such that

$$f(r,x) = r^{\mu+\frac{1}{2}}g(r,x),$$

with $g \in \mathbb{M}(]0,\infty[\times\mathbb{R}^n)$, equipped with the family of semi norms

$$\widetilde{\gamma}^\mu_{m,\varphi}(f) = \gamma^\mu_{m,\varphi}(r^{-\mu-\frac{1}{2}}f).$$

Proof. • It is clear from Definition 3.9 that $\mathscr{H}_\mu$ is an injective mapping from $\mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$ into $r^{\mu+\frac{1}{2}}\mathbb{M}(]0,\infty[\times\mathbb{R}^n)$.

• Let $g \in r^{\mu+\frac{1}{2}}\mathbb{M}(]0,\infty[\times\mathbb{R}^n)$, there exists $T \in \mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ such that for every $(r,x) \in [0,+\infty[\times\mathbb{R}^n$,

$$\mathscr{H}_\mu(T)(r,x) = g(r,x) = r^{\mu+\frac{1}{2}}f(r,x),$$

with $f \in \mathbb{M}(]0,\infty[\times\mathbb{R}^n)$.

This shows that T belongs to $\mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$ and that $\mathscr{H}_\mu$ is a bijective mapping from $\mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$ into $r^{\mu+\frac{1}{2}}\mathbb{M}(]0,\infty[\times\mathbb{R}^n)$.

On the other hand, for $T \in \mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$ and for every $\varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ and $m \in \mathbb{N}$, we have

$$\widetilde{\gamma}^\mu_{m,\varphi}(\mathscr{H}_\mu(T)) = \gamma^\mu_{m,\varphi}(r^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(T)) = Q^\mu_{m,\varphi}(T).$$

Q.E.D.

Remark 3.2. It is clear from Lemma 3.10 that, for every $T \in \mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ and $S \in \mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$ the mapping

$$\varphi \longmapsto < T, S*\varphi >,$$

defines an element of $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$.

Definition 3.12. *For every* $T \in \mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ *and* $S \in \mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$, *we define the convolution product* $T*S$ *by the following brackets*

$$\langle T*S,\varphi\rangle = \langle T, S*\varphi\rangle, \quad \varphi \in \mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n).$$

Proposition 3.13. *For every* $T \in \mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ *and* $S \in \mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$, *we have*

$$\mathscr{H}_\mu(T*S) = \lambda_0^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(S)(\lambda_0,\lambda)\mathscr{H}_\mu(T).$$

Proof. Let φ be in $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$,

$$\begin{aligned}\langle\mathscr{H}_\mu(T*S),\varphi\rangle &= \langle T*S, \mathscr{H}_\mu(\varphi)\rangle \\ &= \langle T, S*\mathscr{H}_\mu(\varphi)\rangle.\end{aligned}$$

Using Proposition 3.8, Remark 3.1 and the fact that the Hankel transform is an isomorphism from $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ onto itself, we get

$$\begin{aligned}\langle \mathscr{H}_\mu(T*S),\varphi\rangle &= \langle T, \mathscr{H}_\mu(\lambda_0^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(S)\varphi)\rangle\\ &= \langle \mathscr{H}_\mu(T), \lambda_0^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(S)\varphi\rangle\\ &= \langle \lambda_0^{-\mu-\frac{1}{2}}\mathscr{H}_\mu(S)\mathscr{H}_\mu(T),\varphi\rangle.\end{aligned}$$

This proves the result. Q.E.D.

Example 3.3. Let δ_μ be defined on $\mathbb{H}_\mu(]0,+\infty[\times\mathbb{R}^n)$ by

$$\langle \delta_\mu,\varphi\rangle = \lim_{(r,x)\to(0,0)} r^{-\mu-\frac{1}{2}}\varphi(r,x).$$

Then, δ_μ belongs to the dual space $\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$ and by standard computation, we have

$$\mathscr{H}_\mu(\delta_\mu) = r^{\mu+\frac{1}{2}}\otimes 1.$$

In particular, δ_μ belongs to the subspace $\mathbb{O}'_\mu(]0,\infty[\times\mathbb{R}^n)$ then, from Proposition 3.11 and Proposition 3.13, for every $T\in\mathbb{H}'_\mu(]0,+\infty[\times\mathbb{R}^n)$, we have

$$\delta_\mu * T = T.$$

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