Approximation in weighted rearrangement invariant Smirnov spaces

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Abstract

In the present work, we investigate the approximation problems in weighted rearrangement invariant Smirnov spaces. We prove a direct theorem for polynomial approximation of functions in certain subclasses of weighted rearrangement invariant Smirnov spaces

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1 Introduction and main result

Denote by $\mathcal{M}$ the set of all $\mu-$measurable complex valued functions on $\mathcal{R}$ and let $\mathcal{M}^+$ be the subset of functions from $\mathcal{M}$ whose values lie in $[0,\infty]$. By λ_E we denote the characteristic function of a $\mu-$measurable set $E \subset \mathcal{R}$.

Let a function $\rho : \mathcal{M}^+ \longrightarrow [0,\infty]$ be given. The function ρ is called a *function norm* if it satisfies the following properties for all functions $f,\ g,\ f_n\ (n \in N),$ for all constants $a \geqslant 0$ and for all $\mu-$measurable subsets E of $\mathcal{R}$:

(1) $\rho(f) = 0$ if $f = 0$ $\mu-$a.e. ; $\rho(af) = a\ \rho(f),\ \rho(f+g) \leq \rho(f) + \rho(g),$
(2) $0 \leq g \leq f$ $\mu-$a. e., then $\rho(g) \leq \rho(f),$
(3) $0 \leq f_n \uparrow f$ $\mu-$a.e., then $\rho(f_n) \uparrow \rho(f),$
(4) if $E \subset \mathcal{R}$, $\mu(E) < \infty$, then $\rho(\lambda_E) < \infty,$
(5) for every set $E \subset \mathcal{R}$ with $\mu(E) < \infty$, then $\int\limits_E f d\mu \leq C_E \rho(f),$

where C_E is a constant depending on E and ρ but independent of f. The collection $X = X(\rho)$ of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$ is called a *Banach function space*. For each $f \in X$, the norm of f is defined by

$$\|f\|_X := \rho(|f|).$$

Note that Banach function space X equipped with the norm $\|f\|_X$ is a Banach space [6, pp. 6-7].

If ρ is a function norm, its *associate* norm ρ' is defined on $\mathcal{M}^+$ by

$$\rho'(f) := \sup \left\{ \int\limits_{\mathcal{R}} fgd\mu : f \in \mathcal{M}^+,\ \rho(f) \leq 1 \right\},\ g \in \mathcal{M}^+.$$

If ρ is a function norm, then ρ' is also a function norm [6, pp. 8-9].

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Let ρ be a function norm and ρ' be its associate function norm. The Banach function space $X(\rho')$ determined by the function norm ρ' is called the *associate space* of $X = X(\rho)$ and is denoted by $X^{'}$. Note that every Banach function space coincides with its second associate space $X^{''} = (X')'$ and $\|f\|_X = \|f\|_{X'}$ for all $f \in X$ [6, pp. 10-12].

It is well- known that [6, p. 9]

$$\|g\|_{X'} = \sup \left\{ \int_{\mathcal{R}} |fg| \, d\mu : f \in X, \ \|f\|_X \leq 1 \right\}.$$

Note that for every $f \in X$ and $g \in X'$ the following inequality holds [6, Ch.1, Theorem 2.4]:

$$\int_{\mathcal{R}} |fg| \, d\mu \leq \|f\|_X \ \|g\|_{X'} .$$

Moreover, it is important fact [6, p.10] that for every $f \in X$

$$\|f\|_X = \sup \left\{ \int_{\mathcal{R}} |fg| \, d\mu : g \in X', \ \|g\|_{X'} \leq 1 \right\}.$$

The distribution function μ_f of a measurable function is defined by

$$\mu_f(\lambda) := meas \{ x \in \mathcal{R} : |f(x)| > \lambda \},$$

for $\lambda \geqslant 0$. Two measurable functions f and g are said to be *equimeasurable* if $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geqslant 0$.

Definition 1. A function norm $\rho : \mathcal{M}^+ \longrightarrow [0, \infty]$ is called *rearrangement- invariant* if for every pair of equimeasurable functions $f, \ g \ \in \mathcal{M}^+$ the equality $\rho(f) = \rho(g)$ holds. In this case, the Banach function space generated by ρ is called a *rearrangement invariant space (r.i.space)* [6, p. 59].

These spaces are sufficiently wide; the Lebesgue, Orlicz, Lorentz spaces are examples of rearrangement invariant spaces. For every rearrangement - invariant space X [6, p.78] we have $L_\infty \subset X \subset L_1$. A Banach function space X is rearrangement- invariant if and only if its associate space X' rearrangement-invariant too [6, p. 60].

Note that detailed information on r.i. space can be found in [6], [36], [41] and [42].

Let f be a measurable function. The function f^* defined by

$$f^*(t) := \inf \{ \lambda : \mu_f(\lambda) \leq t \}, \ t \geqslant 0$$

is called the *decreasing rearrangement* of the function f.

Let X be a rearrangement -invariant space. Considering Luxemburg representation theorem [6, pp. 62-64], there is a (not necessarily unique) rearrangement invariant function norm $\overline{\rho}$ over $[0, \infty]$ with the Lebesgue measure m for which

$$\rho(f) = \overline{\rho}(f^*), \ f \ \in \mathcal{M}^+$$

The r.i. space over $[0,\infty]$ with Lebesgue measure m generated by $\overline{\rho}$ is denoted by $\overline{X}$.

For each $x>0$, let us consider the dilation operator H_x defined on $\overline{X}$ by

$$(H_x f)(t) := \begin{cases} f(xt), & xt \in [0,\mu(\mathcal{R})] \\ 0, & xt \notin [0,\mu(\mathcal{R})] \end{cases}, \ t>0.$$

Let $\mathcal{B}(\overline{X})$ be the Banach algebra of the bounded linear operator on $\overline{X}$. According to [6, pp. 165] $H_{1/x} \in \mathcal{B}(\overline{X})$. We denote by $h_X(x)$ the operator norm of $H_{1/x}$, i.e.,

$$h_X(x) := \left\| H_{1/x} \right\|_{\mathcal{B}(\overline{X})}.$$

Let us consider the limits [36], [40]

$$\alpha_X := \lim_{t\to o} \frac{\log h_X(t)}{\log t}, \quad \beta_X := \lim_{t\to\infty} \frac{\log h_X(t)}{\log t}$$

By [36] the above limits exist and $\alpha_X \le \beta_X$. The numbers α_X and β_X are called the *lower* and *upper* Boyd indices of the r.i. space X, respectively. Note that for an arbitrary r.i. space the Boyd indices $\alpha_X,\ \beta_X \in [0,1]$. The Boyd indices are said to be *nontrivial* if $0<\alpha_X \le \beta_X < 1$. Detailed information on properties of the Boyd indices can be found in [6], [8], [42] and [43].

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$. This curve separates the plane into two domains G :=int Γ and G^- :=ext Γ.Without loss of generality, we may asume that $0\in G$. Let $T := \{w \in \mathbb{C} : \ |w| = 1\}$, D :=int T, D^- :=ext T and $w=\varphi(z)$ be the conformal mapping of G^- onto D^- normalized by the conditions

$$\varphi(\infty)=\infty, \ \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0$$

and let $\psi := \varphi^{-1}$ be the inverse mapping of φ.

We denote by $L_p(\Gamma)$, $1<p<\infty$, the set of all measurable complex valued functions such that $|f|^p$is Lebesgue integrable with respect to the arclength on Γ. Let $w=\varphi_1(z)$ indicate a function that maps the domain G conformally onto the disk $|w|<1$. The inverse mapping of φ_1 will be shown by ψ_1. Let Γ_r denote circular images in the domain G, that is, curves in G corresponding to circle $|\varphi_1(z)|=r$ under the mapping $z=\psi_1(w)$.

We use $c,\ c_1,\ c_2,...$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the question of our interest.

Definition 2. The analytic function f in domain G will be called a function of the class $E_p(G)$ if

$$\int_{\Gamma_r} |f(z)|^p \, |dz| < \infty.$$

Definition 3. We shall call the $E_p(G)$ class the *Smirnov class.*

It is evident that any analytic function f belonging to the $E_p(G)$ class will also belong to the $E_1(G)$ class, that is,

$$\int_{\Gamma_r} |f(z)| \, |dz| \le c < \infty,$$

informly in r, $0 < r < 1$. Since $E_p(G) \subset E_1(G)$,every function $f \in E_p(G)$ has a non-tangential limit almost everywhere (a. e.) on Γ, and if we use the same notation for the non-tangential limit of f,then $f \in L^p(\Gamma)$.

$L_p(\Gamma)$ and $E_p(G)$ are Banach spaces with respect to the norm

$$\|f\|_{E_p(G)} := \|f\|_{L_p(\Gamma)} := \left(\int\limits_{\Gamma} |f(z)|\, |dz| \right).$$

Note that the general information about Smirnov classes can be found in the books [10, pp. 168-185] and [15, pp. 438-453].

A measurable function $\omega : \Gamma \to [0, \infty]$ is called a *weighted function* if the preimage $\omega^{-1}\{0, \infty\}$ has measure zero.

Definition 4. The class of measurable functions f defined on Γ and satisfying the condition $|f|\,\omega \in L_p(\Gamma)$, $1 < p < \infty$ is called $\omega-$weighted Lebesgue space $L_p(\Gamma, \omega)$ with the norm

$$\|f\|_{L_p(\Gamma,\omega)} := \|f\omega\|_{L_p(\Gamma)}.$$

Definition 5. The $\omega-$weighted Smirnov class $E_p(G, \omega)$ is defined as

$$E_p(G, \omega) := \{ f \in E_1(G) : f \in L_p(\Gamma, \omega) \}.$$

Let Γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$. For $t \in \Gamma$ and $r > 0$ we denote by $\Gamma(z, \varepsilon)$ the portion of Γ in the open disk of radius r centred at z.

Definition 6. [26]. Let $|\Gamma(z, \varepsilon)|$ denote the length (Lebesgue measure) of $\Gamma(z, \varepsilon)$. The curve Γ is called a *Carleson curve* (or *regular* curve) if

$$C_\Gamma := \sup_{z\in\Gamma} \sup_{\varepsilon>0} \frac{1}{\varepsilon} |\Gamma(z, \varepsilon)| < \infty$$

For instance, convex curves, Ljapunov curves, chord arcs, smooth curves and Lipschitz curves are all regular. We denote by S the set of all Carleson curves in the complex plane.

Definition 7. Let $1 < p < \infty$, $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $A_p(\Gamma)$ be the collection of all weights on Γ satisfying the condition

$$\sup_{t\in\Gamma} \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int\limits_{\Gamma(z,r)} \omega^p(\tau)\, |d\tau| \right)^{1/p} \left(\frac{1}{\varepsilon} \int\limits_{\Gamma(z,r)} \omega^{-q}(\tau)\, |d\tau| \right)^{1/q} < \infty$$

The weight functions which belong to $A_p(\Gamma)$ for some $1 < p < \infty$, are called the *Muckenhoupt weights.*

Let $\Gamma \subset \mathbb{C}$ be a closed rectifiable Jordan curve with the Lebesgue length measure $|d\tau|$ and $X(\Gamma)$ be a r.i. space over Γ and $X'(\Gamma)$ it is associate space.

Let ω be a weighted function. We denote by [40] $X(\Gamma,\omega)$ the linear space of all measurable functions f such that $f\omega \in X(\Gamma)$ and set

$$\|f\|_{X(\Gamma,\omega)} := \|f\omega\|_{X(\Gamma)}.$$

Definition 8. A normed space $X(\Gamma,\omega)$ is called a *weighted rearrangement - invariant space* (w.r.i. space).

Note that according to [6, Section 1.1] and [39] if $\omega \in X(\Gamma,\omega)$ is a Banach function space then its associate space is the Banach function space $X'(\Gamma,\omega^{-1})$ with the norm $\|f\|_{X'(\Gamma,\omega^{-1})} = \left\|f\omega^{-1}\right\|_{X'(\Gamma)}$. If $\omega \in X(\Gamma)$ and $\omega^{-1} \in X'(\Gamma)$, then from Hölder's inequality we obtain

$$L^{\infty} \subset X(\Gamma,\omega) \subset L^{1}(\Gamma).$$

Definition 9. For a weight ω on Γ we denote by $E_X(G,\omega)$ the subclass of analytic functions of $E^1(G)$ whose boundary value functions belong to the w.r.i space $X(\Gamma,\omega)$.

For $\zeta \in \Gamma$ we define the point $\zeta_h \in \Gamma$ by

$$\zeta_h := \psi(\varphi(\zeta)e^{ih}),\ h \in [0,2\pi].$$

Definition 10. Let Γ rectifiable Jordan curve, and $f \in X(\Gamma,\omega)$.Then the function $\Omega^{(2)}_{p(.),\omega}(f,.)$ defined by

$$\Omega^{(2)}_{\Gamma,\ X,\ \omega}(\ f,\delta) := \delta^2 \sup_{t\geq\delta} t^{-2} \sup_{|h|\leq t} \|f(\zeta_h)+f(\zeta_{-h})-2f(\zeta)\|_{X(\Gamma,\omega)}$$

is called *generalized modulus of smoothness in the* $X(\Gamma,\omega)$.

We suppose that $\omega(\delta)$ is a nonnegative, continuous, nondecreasing real function satisfying the conditions $\omega(0)=0$, $\omega(\delta)>0$, $(\delta>0)$ and $\omega(n\delta)\leq cn\omega(\delta)$, where $n \in \mathbb{N}$ and constant $c>0$. We denote by $H^{\omega}_{\Gamma}E_X(G,\omega)$ the class of functions $f \in E_X(G,\omega$ for which

$$\Omega^{(2)}_{\Gamma,\ X,\ \omega}(\ f,\delta) < c\omega(\delta),$$

where some constant c independent of f and δ.

Using the method of proof in [44] it can be shown easily that if $f, f_1 \in H^{\omega}_{\Gamma}E_X(G,\omega)$, the modulus of smoothness $\Omega^{(2)}_{\Gamma,\ X,\ \omega}(\ f,\delta)$ satisfy the following conditions:

$$\begin{aligned}
\Omega^{(2)}_{\Gamma,\ X,\ \omega}(\ f,0) &= 0,\\
\Omega^{(2)}_{\Gamma,\ X,\ \omega}(\ f,\delta) &\geq 0,\\
\lim_{\delta\to 0}\Omega^{(2)}_{\Gamma,\ X,\ \omega}(\ f,\delta) &= 0,\\
\Omega^{(2)}_{\Gamma,\ X,\ \omega}(\ f+f_1,\delta) &\leq \Omega^{(2)}_{\Gamma,\ X,\ \omega}(\ f,\delta)+\Omega^{(2)}_{\Gamma,\ X,\ \omega}(\ f_1,\delta).
\end{aligned}$$

Under different restrictive conditions upon $\Gamma = \partial G$, the direct problems of approximation theory in non-weighted and weighted Smirnov spaces have been investigated by several authors (see, for example, [48], [1], [2], [38], [21], [13], [44], [22], [24]-[26] and [29]). The problems of approximation theory for weighted rearrangement invariant spaces are studied in [17], [19], [23] and [51]. In this work, we prove a direct theorem of approximation theory in weighted rearrangement invariant Smirnov spaces. We investigate approximation problems in the class $H_\Gamma^\omega E_X(G,\omega)$. Similar problems of the approximation theory in different spaces have been studied by several authors (see, for example, [3]-[5], [9], [16], [18], [30]-[35], [37], [46], [47], [49]and [50]).

Main result in our work is the following theorem.

Theorem 1. Let $\Gamma \in S$, α_X, β_X be the nontrivial indices and let $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$. If $f \in H_\Gamma^\omega E_X(G,\omega)$, then for any $n \in N$ there exists an algebraic polynomial $P(\cdot, f)$ of degree at most n such that

$$\|f - P(\cdot, f)\|_{X(\Gamma,\omega)} \leq c_1 \omega\,(1/n) \tag{1}$$

with some constant c_1 independent of n.

Note that Theorem 1 is proved for the first time in the case where Γ is a unit circle.

2 Auxiliary results

Let $f \in L_1(\Gamma)$. Then the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta, \quad z \in G \tag{2}$$

and

$$f^-(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta, \quad z \in G^- \tag{3}$$

are analytic in G and G^- respectively, and $f^-(\infty) = 0$. Thus the limit

$$S_\Gamma(f)(z) := (P.V.)\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta := \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma\cap\{\zeta:\ |\zeta - z|>\varepsilon\}} \frac{f(\zeta)}{\zeta - z}d\zeta$$

exists and is finite for almost all $z \in L$.

The quantity $S_\Gamma(f)(z)$ is called the *Cauchy singular integral* of f at $z \in \Gamma$. The linear operator $S_\Gamma : f \to S_\Gamma f$ is called the *Cauchy singular operator*.

According to the Privalov's theorem [15, p. 431] if one of the functions f^+ or f^- has the non-tangential limits a. e. on Γ, then $S_\Gamma(f)(z)$ exists a. e. on Γ and also the other one has non-tangential limits a. e. on Γ. Conversely, if $S_\Gamma(f)(z)$ exists a. e. on Γ, then the functionsf^+ and f^- have non-tangential limits a. e. on Γ. In both cases, the formulae

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z), \quad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z)$$

and hence

$$f = f^{+} - f^{-}$$

holds a. e. on Γ.

Note that the class of all regular curves is very wide. G. David proved [12] that Γ is a regular curve if and only if for every $f \in L_p(\Gamma)$, $S_\Gamma(f)$ exists and belongs to $L_p(\Gamma)$ and the singular operator $S_\Gamma(f) : L_p(\Gamma) \to L_p(\Gamma)$ is bounded, that is, there exists a constants $c_2(p,\Gamma)$ such that

$$\|S_\Gamma(f)\|_{L_p(\Gamma)} \leq c_2(p,\Gamma)\,\|f\|_{L_p(\Gamma)}$$

for all $f \in L_p(\Gamma)$. In [20], V.Havin proved that if the singular operator $S_\Gamma(f) : L_p(\Gamma) \to L_p(\Gamma)$ is bounded, for every $f \in L_p(\Gamma)$, the functions f^+ and f^- defined by the formulae (2) and (3) belong to Smirnov's classes $E_p(G)$ and $E_p(G^-)$, respectively.

We need the following results.

Lemma 1 [45, p. 208]. In order to represent f according to its boundary values in the form Cauchy integral, it is necessary and sufficient that $f \in E_1(G)$.

The following theorem, given in [7, pp. 117-144] and [14, p. 89] characterizes the weight functions for which S_Γ is bounded in the weighted Lebesgue spaces $L_p(\Gamma,\omega)$.

Theeorem 2. Let Γ be a Carleson curve, $1 < p < \infty$, and let ω be a weight function on Γ. The inequality

$$\|S_\Gamma(f)\|_{L_p(\Gamma,\omega)} \leq c_3(p,\Gamma)\,\|f\|_{L_p(\Gamma,\omega)}$$

holds for every $f \in L_p(\Gamma,\omega)$ if and only if $\omega \in A_p(\Gamma)$.

The following theorem associated with boundedness of the Cauchy singular integral operator S_Γ in weighted rearrangement -invariant spaces holds [40]:

Theorem 3. Let $X(\Gamma)$ be an r.i. space with nontrivial Boyd indices α_X, β_X. If a weight ω belongs to the Muckenhoupt classes $A_{\frac{1}{\alpha_X},}(\Gamma)$ and $A_{\frac{1}{\beta_X},}(\Gamma)$, then the operator S_Γ is bounded in the w.r.i space $X(\Gamma,\omega)$.

2.1 Proof of the main result

Proof of Theorem 1. We set

$$F(z_h) = f(z_h) + f(z_{-h}).$$

From the condition of Theorem 1 we have $f(z) \in X(\Gamma,\omega)$ and S_Γ is bounded in the space $X(\Gamma,\omega)$ and $\|f(z_h) + f(z_{-h}) - 2f(z)\|_{X(\Gamma,\omega)} < \infty$. In this case according to [39] and [40] $\omega \in X(\Gamma)$ and

$1/\omega \in X'(\Gamma)$. Then, from the Hölder inequality for the $X(\Gamma)$ spaces we have

$$\begin{aligned}
\int_\Gamma |F(z_h)|\,|dz| &= \int_\Gamma \frac{|f(z_h)+f(z_{-h})-2f(z)+2f(z)|}{|\omega|}\,|\omega|\,|dz| \\
&\le \int_\Gamma \frac{|f(z_h)+f(z_{-h})-2f(z)|}{|\omega|}\,|\omega|\,|dz| \\
+2\int_\Gamma \frac{|f(z)|}{|\omega|}\,|\omega|\,|dz| &\le \|f(z_h)+f(z_{-h})-2f(z)\|_{X(\Gamma,\omega)}\left(\left\|\frac{1}{\omega}\right\|_{X'(\Gamma)}\right) \\
+2\,\|f\|_{X(\Gamma,\omega)}\left(\left\|\frac{1}{\omega}\right\|_{X'(\Gamma)}\right) &< \infty
\end{aligned}$$

It follows from the last inequality, that $F(z_h) \in L_1(\Gamma)$. Since $F(z_h) \in L_1(\Gamma)$ and S_Γ is bounded in the space $X(\Gamma,\omega)$, singular integral $S_\Gamma(F_h)(z)$ in principal value meaning exists a. e. on Γ. Then the function is approximed with Jackson-Dzydyk polynomial [11, p.440]. We represent the polynomial Jackson-Dzydyk in the form

$$P_n(z) = \frac{1}{2\pi}\int_0^\pi K_n(h)\,[S_\Gamma(F_h)(z)]\,dh + \frac{1}{4\pi}\int_0^\pi K_n(h)\,[f(z_h)+f(z_{-h})]\,dh, \tag{4}$$

where $K_n(h)$ is a kernel, which is trigonometric polynomials of degree not exceeding n and satisfies the conditions [11, p. 428]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1,\ \ (n = 0,1,2,...), \tag{5}$$

$$\int_{-\pi}^{\pi} |K_n(t)|\,dt \le c_4,\ \ (n = 0,1,2,...), \tag{6}$$

$$\int_{-\pi}^{\pi} |t|^k\,|K_n(t)|\,dt \le c_5(k)(n+1)^{-k}, \tag{7}$$

$$\int_{-\pi}^{\pi} (|t| + \frac{1}{n})^k\,|K_n(t)|\,dt \le c_6(k)n^{-k},\ \ (n = 1,2,...). \tag{8}$$

Note that Jackson kernel satisfies the conditions (5)- (8). If $f \in E_X(G,\omega)$, it follows that $f \in E_1(G)$. Then according to Lemma 1, the function $f \in E_1(G)$ can be written as Cauchy integral

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta,\ \ z \in G.$$

Since S_Γ is bounded in the space $X(\Gamma, \omega)$, the $S_\Gamma(f)(z)$ singular integral exists. Then, for the function f the following identity

$$f(z) = (S_\Gamma f)(z) + f(z)/2$$

holds a. e. on Γ. Taking into account the last relation, we have

$$f(z) = \frac{1}{2\pi}\int_0^\pi K_n(h) f(z) dh + \frac{1}{\pi}\int_0^\pi K_n(h)[(S_\Gamma f)(z)]dh. \tag{9}$$

Consideration of (4) and (9) gives us

$$\begin{aligned}\|f - P_n\|_{X(\Gamma,\omega)} \le \Bigg\| &\frac{1}{2\pi}\int_0^\pi K_n(h) f(z) dh + \frac{1}{\pi}\int_0^\pi K_n(h)[(S_\Gamma f)(z)]dh P_n(z) \\ &- \frac{1}{2\pi}\int_0^\pi K_n(h)\left[S_\Gamma(F_h)(z)\right]dh- \\ &\frac{1}{4\pi}\int_0^\pi K_n(h)\left[f(z_h) + f(z_{-h})\right]dh\Bigg\|_{X(\Gamma,\omega)}.\end{aligned} \tag{10}$$

Using (10), Minkowski's inequality and the boundedness of singular operator S_Γ we get

$$\begin{aligned}\|f - P_n\|_{X(\Gamma,\omega)} &\le c_7 \left\|\int_0^\pi K_n(h)\left[S_\Gamma(F_h - 2f)(z)\right]dh\right\|_{X(\Gamma,\omega)} \\ +c_8\left\|\int_0^\pi K_n(h)\left[(F_h - 2f)(z)\right]dh\right\|_{X(\Gamma,\omega)} &\le c_9\int_0^\pi K_n(h)dh\left\|S_\Gamma(F_h - 2f)(z)\right\|_{X(\Gamma,\omega)} \\ +c_{10}\int_0^\pi K_n(h)dh\left\|(F_h - 2f)(z)\right\|_{X(\Gamma,\omega)} &\le c_{11}\int_0^\pi K_n(h)\left\|(F_h - 2f)(z)\right\| dh_{X(\Gamma,\omega)} \\ +c_{12}\int_0^\pi K_n(h)dh\left\|(F_h - 2f)(z)\right\|_{X(\Gamma,\omega)} &\le c_{13}\int_0^\pi K_n(h)\Omega^{(2)}_{\Gamma,\ X,\ \omega}(h,f)dh \\ &\le c_{14}\Omega^{(2)}_{\Gamma,\ X,\ \omega}(h,f)\int_0^\pi K_n(h)(nh+1)dh. \le c_{15}\omega(1/n).\end{aligned} \tag{11}$$

According to (5), (7) and (11), we obtain the inequality (1) of Theorem 1.

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