A characterization of weighted Besov spaces in quantum calculus

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Abstract

In this paper, subspaces of $L^p(\mathbb{R}_{q,+})$ are defined using q-translations $T_{q,x}$ operator and q-differences operator, called q-Besov spaces. We provide characterization of these spaces by using the q-convolution product.

1 Introduction

Much recent research activity has focused on the theory and application of Quantum Calculus. This branch of mathematics continues to find new and useful applications. For example the so-called q-analogs of special functions and hypergeometric series, called q-series have many applications in arithmetic theory, combinatorics, quantum physic, group theory [2], and others areas of science and mathematics. Applications of this mathematics include population biology [5], geometric analysis [6], intelligent robotic control [9], approximation theory [22], and financial engineering [21], among others. Our interest in this paper is to characterize some weighted Besov spaces in Quantum Calculus, called weighted q-Besov spaces. In the classical case there are many ways to define Besov spaces see ([4], [20], [24], [25]).

In this paper we express q-Besov spaces in term of convolution $f *_q \varphi_t$ with different kinds of smooth functions φ. These spaces can be described by means of difference differential operator (see [14], [15], [18], [23]).
Throughout the paper weight $w : \mathbb{R}_{q,+} \longrightarrow \mathbb{R}_+$ will be a q-measurable function, $w > 0$ a.e., and we will give a characterization of weighted q-Besov spaces.
Our objective is to find weights where we can get such a characterization of weighted q-Besov spaces

$$\Lambda^{p,m}_{*,w,q} = B^{p,m}_{*,w,\varphi,q} \quad (\text{ with equivalent seminorms })$$

where $\Lambda^{p,m}_{*,w,q}$ the space of even function $f : \mathbb{R}_q \longrightarrow \mathbb{C}$ such that

$$\| f \|_{\Lambda^{p,m}_{*,w,q}} := \Big\{ \int_0^{+\infty} \frac{\| \nabla_{q,x} f \|_{q,p}^m}{w(x)^m} \frac{d_q x}{x} \Big\}^{\frac{1}{m}} < +\infty, \quad (1 \leq m < \infty)$$

and

$$\| f \|_{\Lambda^{p,\infty}_{*,w,q}} := \inf \Big\{ C > 0; \| \nabla_{q,x} f \|_{q,p} \leq C w(x) \quad a.e. \ x \in \mathbb{R}_{q,+} \Big\}, \quad (m = +\infty)$$

where we have put $\nabla_{q,x} f(y) := T_{q,x} f(y) - f(y)$, and $B^{p,m}_{*,w,\varphi,q}$ the space of the even function $f : \mathbb{R}_q \longrightarrow \mathbb{C}$ belonging to $L^1(\mathbb{R}_{q,+}, \dfrac{d_q x}{(1+x)^2_q})$ satisfying

$$\| f \|_{B^{p,m}_{*,w,\varphi,q}} := \Big\{ \int_0^{+\infty} \frac{\| \varphi_t *_q f \|^m_{q,p} \, d_q t}{w(t)^m} \, \frac{d_q t}{t} \Big\}^{\frac{1}{m}} < +\infty, \quad (1 \le m < +\infty)$$

and

$$\| f \|_{B^{p,\infty}_{*,w,\varphi,q}} := \inf \Big\{ C > 0; \| \varphi_t *_q f \|_{q,p} \le Cw(x) \quad a.e \quad t \in \mathbb{R}_{q,+} \Big\}, \quad (m = +\infty)$$

where $\varphi_t(x) := t^{-1} \varphi(t^{-1}x)$, $t \in \mathbb{R}_{q,+}$ and $x \in \mathbb{R}_+$.

The contents of the paper are as follows. In Section 2, we collect some basic definitions and results about q-harmonic analysis. In Section 3, we give condition about weight and prove the connection between the spaces $\Lambda^{p,m}_{*,w,q}$ and $B^{p,m}_{*,w,\varphi,q}$.

2   Preliminaries

In all the sequel, we assume $q \in (0,1)$ and we adapt the same notations as in [8].

- A q-shifted factorial is defined by

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k); \;\; n = 1, 2, \cdots, \infty,$$

  and more generally

$$(a_1, \cdots, a_r; q)_n := \prod_{k=1}^{r} (a_k; q)_n.$$

- The basic hypergeometric series or q-hypergeometric series are given for r, s integers by

$$_r\varphi_s(a_1, \cdots, a_r; b_1, \cdots, b_s; q, x) := \sum_{n=0}^{\infty} \frac{[(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r} (a_1, \cdots, a_r; q)_n}{(b_1, \cdots, b_s; q)_n} \frac{x^n}{(q;q)_n}.$$

- The q-derivative $D_{q,x} f$ of a function f on an open interval is given by

$$D_{q,x} f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \ne 0$$

  and the q-derivative at zero [17] is defined by

$$D_{q,x} f(0) := \lim_{n \to +\infty} \frac{f(xq^n) - f(0)}{xq^n},$$

  where the limit exists and independent of x.

- We also denote
$$[x]_q := \frac{1 - q^x}{1 - q}, \qquad [n]_q! := \frac{(q; q)_n}{(1 - q)^n}.$$

- The q-shift operators are
$$(\Lambda_{q,x} f)(x) := f(qx), \qquad (\Lambda_{q,x}^{-1} f)(x) := \Lambda_{q^{-1},x} f(x).$$

- We consider the q-difference operator
$$\Delta_{q,x} := \Lambda_{q,x}^{-1} D_{q,x}^2.$$

- The q-analogue of $(a + b)^n$ is a non commutative term $(a + b)_q^n$ given by
$$(a + b)_q^n := \begin{cases} a^n(-\frac{b}{a}; q)_n, & a \neq 0 \\ \\ q^{n(n-1)/2} b^n, & a = 0. \end{cases}$$

It is clear that $(a + b)_q^n$ and $(b + a)_q^n$ are not always the same. For future use, this definition can be generalized in a way similar to its ordinary counterpart by
$$(1 + a)_q^\alpha := \frac{(1 + a)_q^\infty}{(1 + q^\alpha a)_q^\infty}$$

for any number α.

Proposition 2.1. For $n, k \in \mathbb{N}$

(i) $(a - x)_q^n = (-1)^n q^{\frac{n(n-1)}{2}} (x - q^{-n+1} a)_q^n$

(ii) $D_{q,x}^k (x + a)_q^n = [n]_q.[n - 1]_q ... [n - k + 1]_q (x + a)_q^{n-k}$

(iii) $D_{q,x}^k \frac{1}{(1 - x)_q^n} = \frac{[n]_q.[n + 1]_q ... [n + k - 1]_q}{(1 - x)_q^{n+k}}$

Proposition 2.2. For any number α, β

(i) $(1 + x)_q^\alpha.(1 + q^\alpha x)_q^\beta = (1 + x)_q^{\alpha+\beta}$

(ii) $D_{q,x}(1 + x)_q^\alpha = [\alpha]_q(1 + qx)_q^{\alpha-1}$

(iii) For $\alpha > 0$ and $x = a + b$,
$$2 (1 + [a + b])_q^\alpha \leq q^{\alpha(\alpha-1)/2}[(1 + a)_q^\alpha + (1 + b)_q^\alpha]$$

- $\mathbb{R}_q := \{\pm q^k, k \in \mathbb{Z}\}; \quad \mathbb{R}_{q,+} := \{+q^k, k \in \mathbb{Z}\}; \quad \widetilde{\mathbb{R}}_{q,+} := \{+q^k, k \in \mathbb{Z}\} \cup \{0\}.$

- The q-Jackson integral [16] from 0 to a (respectively from a to $+\infty$) is defined by
$$\int_0^a f(x)d_qx := (1 - q)a \sum_{n=0}^{+\infty} f(aq^n)q^n, \quad \int_a^\infty f(x)d_qx := (1 - q)a \sum_{-\infty}^{-1} f(aq^n)q^n.$$

Remark 2.1. Observe that the q-Jackson integral is a Riemann-Stieltjes integral [1] with respect to a step function having infinitely many points of increase at the points q^k, with the jump at the point q^k being q^k. If we call this step function $\Psi_q(t)$ then $d\Psi_q(t) = d_qt$.

Note that for $n \in \mathbb{Z}$ and $a \in \mathbb{R}_{q,+}$, we have

$$\int_0^\infty f(x)d_qx = \int_0^a f(x)d_qx + \int_a^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^\infty f(q^n)q^n,$$

and

$$\int_0^a f\left(\frac{1}{x}\right)d_qx = \int_{qa^{-1}}^\infty f(x)\frac{d_qx}{x^2}, \qquad \int_s^\infty f\left(\frac{1}{x}\right)\frac{d_qx}{x} = \int_0^{qs^{-1}} f(x)\frac{d_qx}{x}. \qquad (2.1)$$

Moreover, if $f \geq 0$ then

$$\int_0^{qs} f(x)d_qx \leq \int_0^s f(x)d_qx, \qquad \int_a^\infty f(x)d_qx \leq \int_{qa}^\infty f(x)d_qx \qquad (2.2)$$

- The following definition of the q-cosine [10] is given by

$$\cos(x;q^2) := {}_1\varphi_1(0,q,q^2;(1-q)^2x^2) = \sum_{n=0}^\infty (-1)^n b_n(x;q^2),$$

where, we have put

$$b_n(x;q^2) := b_n(1;q^2)x^{2n} = q^{n(n-1)}\frac{(1-q)^{2n}}{(q;q)_{2n}}x^{2n}.$$

- The q-cosine Fourier transform $\mathcal{F}_q$ and the q-convolution product are defined for suitable functions f, g as follows

$$\mathcal{F}_q(f)(\lambda) := \frac{(1+q^{-1})^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})}\int_0^\infty f(t)\cos(\lambda t;q^2)d_qt,$$

$$f *_q g(x) := \frac{(1+q^{-1})^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})}\int_0^\infty T_{q,x}f(y)g(y)d_qy.$$

Here $T_{q,x}$, $x \in \mathbb{R}_{q,+}$ are the q-even translation operators defined by

$$T_{q,y}f(x) := \sum_{n=0}^\infty b_n(y;q^2)\Delta_{q,x}^n f(x) \qquad (2.3)$$

Remark that when $q \to 1^-$, the q-translation tends to the classical even translation σ_x given by

$$\sigma_x(f)(y) := \frac{1}{2}[f(x+y) + f(x-y)], \quad y \in [0,+\infty).$$

which has the following properties.

Proposition 2.3. For all $f, g \in L^1(\mathbb{R}_{q,+})$:

(i) $T_{q,x} f(y) = T_{q,y} f(x)$

(ii) $\Delta_{q,x} T_{q,x} f(y) = \Delta_{q,y} T_{q,y} f(x)$

(iii) $\displaystyle\int_0^\infty T_{q,x} f(y) d_q y = \int_0^\infty f(y) d_q y$

(iv) $\displaystyle\int_0^\infty T_{q,x} f(y) g(y) d_q y = \int_0^\infty f(y) T_{q,x} g(y) d_q y$

(v) $T_{q,x} \cos(ty; q^2) = \cos(tx; q^2) \cos(ty; q^2), \quad x, y, t \in \mathbb{R}_{q,+}.$

In [7] the q-cosine Fourier transform satisfies the following:

Theorem 2.1. For $f \in L^1(\mathbb{R}_{q,+})$, $\mathcal{F}_q(f) \in \mathcal{C}_{*,q,0}(\mathbb{R}_q)$ and
$$\| \mathcal{F}_q(f) \|_{\mathcal{C}_{*,q,0}} \le \frac{1}{(q(1-q))^{\frac{1}{2}} (q;q)_\infty} \|f\|_{q,1}.$$

Theorem 2.2. (Inversion formula)
(i) Let $f \in L^1(\mathbb{R}_{q,+})$ such that $\mathcal{F}_q(f) \in L^1(\mathbb{R}_{q,+})$, then for all $x \in \mathbb{R}_{q,+}$, we have

$$f(x) = \frac{(1 + q^{-1})^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})} \int_0^\infty \mathcal{F}_q(f)(y) \cos(xy; q^2) d_q y. \tag{2.4}$$

(ii) $\mathcal{F}_q$ is an isomorphism of $\mathcal{S}_{*,q}(\mathbb{R}_q)$ and $\mathcal{F}_q{}^2 = Id.$

They proved that $\mathcal{F}_q$ can be extended to $L^2(\mathbb{R}_{q,+})$ and we have

Theorem 2.3. (q-Plancherel theorem type)
$\mathcal{F}_q$ is an isomorphism of $L^2(\mathbb{R}_{q,+})$, we have $\|\mathcal{F}_q(f)\|_{q,2} = \|f\|_{q,2}$, for $f \in L^2(\mathbb{R}_{q,+})$ and $\mathcal{F}_q{}^{-1} = \mathcal{F}_q.$

In [13], the authors proved that

Proposition 2.4. For $f, g \in L^1(\mathbb{R}_{q,+})$ we have

(i) $\mathcal{F}_q(f *_q g) = \mathcal{F}_q(f) \mathcal{F}_q(g)$

(ii) $\displaystyle\int_0^\infty \mathcal{F}_q(f)(\xi) g(\xi) d_q \xi = \int_0^\infty f(\xi) \mathcal{F}_q(g)(\xi) d_q \xi$

(iii) $\mathcal{F}_q(T_{q,x} f)(\xi) = \cos(\xi x; q^2) \mathcal{F}_q(f)(\xi))$

(iv) For $f \in L^p(\mathbb{R}_{q,+})$, $g \in L^1(\mathbb{R}_{q,+})$ then $f *_q g \in L^p(\mathbb{R}_{q,+})$ and $\|f *_q g\|_{q,p} \le \|f\|_{q,p} \|g\|_{q,1}.$

Specially, we choose $q \in [0, q_0]$ where q_0 is the first zero of the function [11]: $q \mapsto {}_1\varphi_1(0, q, q; q)$ under the condition $\frac{\log(1-q)}{\log q} \in \mathbb{Z}.$
Let us now introduce some q-functional spaces which one will need in this work.

▶ $\mathcal{C}_{*,q,0}(\mathbb{R}_q)$ the space of even functions f defined on $\mathbb{R}_q$ continuous at 0, and satisfying

$$\lim_{x \to \infty} f(x) = 0 \quad \text{and} \quad \| f \|_{\mathcal{C}_{*,q,0}} = \sup_{x \in \mathbb{R}_q} |f(x)| < +\infty.$$

▶ $\mathcal{C}_{*,q}^m(\mathbb{R}_q)$ the space of even functions m times q-differentiable on $\mathbb{R}_q$, continuous at 0. We equip this space with the topology of the uniform convergence of the functions and their q-derivatives.

▶ $\mathcal{S}_{*,q}(\mathbb{R}_q)$ the q-analogue of Schwartz space formed by the functions $f \in \mathcal{C}_{*,q,0}^\infty(\mathbb{R}_q)$ such that

$$\forall k, n \in \mathbb{N}, \ \ N_{q,n,k}(f) = \sup_{x \in \mathbb{R}_q} (1 + x^2)^n \mid D_{q,x}^k f(x) \mid < +\infty.$$

▶ $L^p(\mathbb{R}_{q,+})$, $p \in [1, +\infty]$, the space of functions f such that $\| f \|_{p,q} < +\infty$, where

$$\| f \|_{p,q} = (\int_0^\infty |f(x)|^p d_q x)^{\frac{1}{p}} < +\infty, \ \ \text{for } p < \infty,$$

and

$$\| f \|_{q,\infty} = ess \sup_{x \in \mathbb{R}_{q,+}} \mid f(x) \mid < +\infty.$$

▶ $\mathcal{S}_{*,q,0}$ the space of even functions $f \in \mathcal{S}_{*,q}$ such that $\int_0^{+\infty} f(x) d_q x = 0$.

▶ $\mathcal{A}_{*,q}$ the space of even function $\varphi \in \mathcal{S}_{*,q,0}$ such that $\int_0^{+\infty} (\mathcal{F}_q \varphi(t\xi))^2 \dfrac{d_q t}{t} = 1$ for $\xi \in \mathbb{R}_{q,+}$.

▶ $\mathcal{A}_{*,1,q}$, the space of even function $\varphi \in \mathcal{A}_{*,q}, supp\varphi \subseteq [0,1]$, such that
$$\int_0^{+\infty} x\varphi(x) d_q x = 0.$$

To establish the results of the paper, let us first the following notions.

▶ A weight w is said to satisfy q-Dini condition if there exists $C > 0$ such that

$$\int_0^s \frac{w(t)}{t} d_q t < C w(s); \quad a.e. \quad s \in \mathbb{R}_{q,+}.$$

▶ A weight w is said to be a $(b_{1,q})$-weight if there exists $C > 0$ such that

$$\int_{qs}^{+\infty} \frac{w(t)}{t^2} d_q t < C \frac{w(s)}{s}; \quad a.e. \quad s \in \mathbb{R}_{q,+}.$$

We also put $\mathcal{W}_{0,1;q}$ the space of $(b_{1,q})$-weight satisfy q-Dini condition.

Definition 2.1. Let $\varepsilon \geq 0$, $\delta \geq 0$ and w be a weight.
w is said to be a $(d_{\varepsilon;q})$-weight if there exists $C > 0$ such that

$$\int_0^s t^\varepsilon w(t) \frac{d_q t}{t} \leq C s^\varepsilon w(s); \quad \text{a.e. } s > 0$$

w is said to be $(b_{\delta;q})$ weight if there exist $C > 0$

$$\int_{qs}^{+\infty} \frac{w(t)}{t^\delta} \frac{d_q t}{t} \leq C \frac{w(s)}{s^\delta}; \quad \text{a.e. } s > 0.$$

We write $\mathcal{W}_{\varepsilon,\delta;q} = (d_{\varepsilon;q}) \cap (b_{\delta;q})$.

Proposition 2.5. Let $\varepsilon \geq 0$, $\delta \geq 0$ and w be a weight, we have

(i) If $w \in (d_{\varepsilon;q})$ then $w \in (d_{\varepsilon';q})$ for any $\varepsilon' > \varepsilon$

(ii) If $w \in (b_{\delta;q})$, then $w \in (b_{\delta';q})$ for any $\delta' > \delta$

(iii) If $\overline{w}(t) = w(t^{-1})$; then $w \in (b_{\varepsilon;q})$, if and only if, $\overline{w} \in (d_{\varepsilon;q})$

(iv) If $w \in \mathcal{W}_{\varepsilon,\delta;q}$; then $w(t) \geq C \min(t^{-\varepsilon}, t^\delta)$; $C > 0$.

Proof. The assertions (i) and (ii) follows after minor computations. Let prove (iii),
($\Rightarrow$) Let $w \in (b_{\varepsilon;q})$, from (2.1) we have

$$
\begin{aligned}
\int_0^s t^\varepsilon \overline{w}(t) \frac{d_q t}{t} &= \int_0^s t^\varepsilon w(\tfrac{1}{t}) \frac{d_q t}{t} = \int_{qs^{-1}}^\infty \frac{w(u)}{u} \frac{d_q u}{u} \\
&\leq C \frac{w(s^{-1})}{s^{-1}} \\
&\leq C s^\varepsilon \overline{w}(s)
\end{aligned}
$$

($\Leftarrow$) Let now $\overline{w} \in (d_{\varepsilon;q})$, again from (2.1)

$$
\begin{aligned}
\int_{qs}^\infty \frac{w(t)}{t^\delta} \frac{d_q t}{t} &= \int_0^{s^{-1}} \frac{w(\tfrac{1}{x})}{\tfrac{1}{x^\delta}} \frac{d_q x}{x} = \int_0^{s^{-1}} x^\delta \overline{w} x \frac{d_q x}{x} \\
&\leq C s^{-\delta} \overline{w}(s^{-1}) \\
&\leq C \frac{w(s)}{s^\delta}
\end{aligned}
$$

then $w \in (b_{\varepsilon;q})$.
To prove (iv) we use the fact that w is a wight then there exist $s_1, s_2 > 0, C_1, C_2$ such that

$$
\begin{aligned}
w(s) &\geq cs^\delta . \int_s^\infty \frac{w(t)}{t^\delta} \frac{d_q t}{t} = cs^\delta . A(s) \geq cs^\delta . A(s_1) \geq C_1 s^\delta \\
w(s) &\geq cs^{-\varepsilon} . \int_0^\varepsilon w(t) t^{-\varepsilon} \frac{d_q t}{t} = cs^{-\varepsilon} . B(s) \geq cs^{-\varepsilon} . B(s_2) \geq C_2 s^{-\varepsilon}
\end{aligned}
$$

then $w(s) \geq C \min(s^{-\varepsilon}, s^{\delta})$, where $C = \max(C_1, C_2)$. $\square$

Lemma 2.1. Every function $f \in \mathcal{S}_{*,q}$, can be written as

$$f(x) = \int_0^{+\infty} f *_q \varphi_t *_q \varphi_t(x) \frac{d_q t}{t}; \quad \text{for all} \quad \varphi \in \mathcal{A}_{*,q}.$$

Proof. Let $g(x) = \int_0^{+\infty} f *_q \varphi_t *_q \varphi_t(x) \frac{d_q t}{t}$, then

$$
\begin{aligned}
\mathcal{F}_q g(\lambda) &= \int_0^{+\infty} \int_0^{+\infty} f *_q \varphi_t *_q \varphi_t(x) t^{-1} \cos(\lambda x; q^2) d_q x d_q t \\
&= \int_0^{+\infty} \mathcal{F}_q(f *_q \varphi_t *_q \varphi_t)(\lambda) t^{-1} d_q t \\
&= \int_0^{+\infty} \mathcal{F}_q(f)(\lambda)(\mathcal{F}_q(\varphi_t)(\lambda))^2 t^{-1} d_q t \\
&= \mathcal{F}_q(f)(\lambda) \int_0^{+\infty} (\mathcal{F}_q(\varphi)(t\lambda))^2 \frac{d_q t}{t} = \mathcal{F}_q(f)(\lambda).
\end{aligned}
$$

From the fact that $\varphi \in \mathcal{A}_{*,q}$ and the use of the inversion formula we obtain the result. $\square$

Proposition 2.6. Let $\varphi \in \mathcal{A}_{*,q}$ and $\psi \in \mathcal{S}_{*,q}$, then for all $\xi \in \mathbb{R}_{q,+}$

$$\mathcal{F}_q \psi(\xi) = \int_0^{+\infty} \mathcal{F}_q(\varphi_t *_q \varphi_t *_q \psi)(\xi) \frac{d_q t}{t}.$$

Indeed, since $\varphi \in \mathcal{A}_{*,q}$, so

$$
\begin{aligned}
1 &= \int_0^{+\infty} (\mathcal{F}_q \varphi(t\xi))^2 \frac{d_q t}{t} \\
&= \int_0^{+\infty} \mathcal{F}_q(\varphi_t *_q \varphi_t)(\xi) \frac{d_q t}{t},
\end{aligned}
$$

then from Lemma 2.1, and the first relation in Proposition 2.4, we can see easily that

$$
\begin{aligned}
\mathcal{F}_q \psi(\xi) &= \int_0^{+\infty} \mathcal{F}_q(\varphi_t *_q \varphi_t)(\xi) \mathcal{F}_q \psi(\xi) \frac{d_q t}{t} \\
&= \int_0^{+\infty} \mathcal{F}_q(\varphi_t *_q \varphi_t *_q \psi)(\xi) \frac{d_q t}{t}.
\end{aligned}
$$

Remark 2.2. The last proposition shows that

$$\psi_{\varepsilon, \delta; q} = \int_\varepsilon^\delta \varphi_t *_q \varphi_t *_q \psi \frac{d_q t}{t} \text{ converges to } \psi \text{ in } \mathcal{S}_{*,q} \text{ as } \varepsilon \to 0 \text{ and } \delta \to \infty.$$

On the same way we recall the q-Calderón's formula studied and given in [19]

Theorem 2.4. Let $f \in L^1(\mathbb{R}_{q,+}, \frac{d_q x}{(x+1)_q^2})$ and $\varphi \in \mathcal{A}_{*,q}$. For $0 < \varepsilon < \delta$ define

$$f_{\varepsilon,\delta;q}(x) = \int_\varepsilon^\delta (\varphi_t *_q \varphi_t *_q f)(x) \frac{d_q t}{t}.$$

Then $f_{\varepsilon,\delta;q}$ converge to f in $\mathcal{S}'_{q,*,0}$ as $\varepsilon \to 0$ and $\delta \to \infty$.

To finish this preliminary section let us state the chief tool in our investigation, that is the Schur lemma [3] that will be useful for our purposes.

Lemma 2.2. (Schur lemma [3]) Let $1 < p \leq \infty$ and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two σ-finite measure spaces and let $K : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}_+$ be a measurable function and write $T_K(f)$ for

$$T_K(f)(w_2) = \int_{\Omega_1} K(w_1, w_2) f(w_1) d\mu_1(w_1).$$

If there exist $C > 0$ and measurable function $h_i : \Omega_i \longrightarrow \mathbb{R}_+ (i = 1, 2)$ such that

$$\int_{\Omega_1} K(w_1, w_2) h_1^{p'}(w_1) d\mu_1(w_1) \quad \leq \quad Ch_2^{p'}(w_2); \quad \mu_2 - a.e, \tag{2.5}$$

$$\int_{\Omega_2} K(w_1, w_2) h_2^{p}(w_2) d\mu_2(w_2) \quad \leq \quad Ch_1^{p}(w_1); \quad \mu_1 - a.e. \tag{2.6}$$

Then T_K defines a bounded operator from $L^p(\Omega_1, \mu_1)$ into $L^p(\Omega_2, \mu_2)$.

3   Characterization of q-Besov spaces

We begin first by establish some general technical results that can be used to relate properties about q-difference $\nabla_{q,x} f$ and q-convolutions $\varphi_t *_q f$.

Lemma 3.1. For all $y \in [0, 1]$ and $\varphi \in \mathcal{A}_{*,q}$., we have

$$\|\nabla_{q,y}\varphi\|_{1,q} \leq y \int_0^{+\infty} |D_{q,y}(T_{q,x}\varphi)(0)| d_q x.$$

Proof. Let $y \in [0, 1]$ and $\varphi \in \mathcal{A}_{*,q}$, from Proposition 2.3 we have

$$
\begin{aligned}
\|\nabla_{q,y}\varphi\|_{1,q} &= \int_0^\infty | T_{q,y}\varphi(x) - \varphi(x) | d_q x \\
&= \int_0^\infty y . \left| \frac{T_{q,y}\varphi(x) - T_{q,0}\varphi(x)}{y} \right| d_q x \\
&= y \int_0^\infty \left| \frac{T_{q,x}\varphi(y) - T_{q,x}\varphi(0)}{y} \right| d_q x \\
&\leq y \int_0^\infty | D_{q,y}(T_{q,x}\varphi)(0) | d_q x
\end{aligned}
$$

which leads to the result. □

Proposition 3.1. Let $p \in [1, \infty]$, $\zeta \geq 0$ and $\varphi \in \mathcal{A}_{*,q}$. Then there exist $C_q > 0$ such that if $f \in L^1(\mathbb{R}_{q,+}, \frac{d_q x}{(x+1)_q^2})$, then we have that

$$\| \varphi_t *_q f \|_{p,q} \leq C_q \int_0^\infty \min((\frac{x}{t}), (\frac{t}{x})^\zeta) \| \nabla_{q,x} f \|_{p,q} \frac{d_q x}{x} \tag{3.1}$$

and

$$\| \nabla_{q,x} f \|_{p,q} \leq C_q \int_0^\infty \min((\frac{x}{t}), 1) \| \varphi_t *_q f \|_{p,q} \frac{d_q t}{t}. \tag{3.2}$$

Proof. Since $\int_0^{+\infty} \varphi(x) d_q x = 0$, it follows that

$$\varphi_t *_q f(y) = \int_0^{+\infty} \varphi_t(x) \nabla_{q,x} f(y) d_q x,$$

from q-Minkowski's inequality one get

$$
\begin{aligned}
\|\varphi_t *_q f\|_{p,q} &= \left(\int_0^{+\infty} | \int_0^{+\infty} \varphi_t(x) \nabla_{q,x} f(y) d_q x |^p d_q y \right)^{\frac{1}{p}} \\
&\leq \int_0^{+\infty} |\varphi_t(x)| \left(\int_0^{+\infty} |\nabla_{q,x} f(y)|^p d_q y \right)^{\frac{1}{p}} d_q x \\
&\leq \int_0^{+\infty} \frac{x}{t} |\varphi(\frac{x}{t})| \|\nabla_{q,x} f\|_{p,q} \frac{d_q x}{x} \\
&\leq \int_0^1 \frac{x}{t} |\varphi(\frac{x}{t})| \|\nabla_{q,x} f\|_{p,q} \frac{d_q x}{x} + \int_1^{+\infty} \frac{x}{t} |\varphi(\frac{x}{t})| \|\nabla_{q,x} f\|_{p,q} \frac{d_q x}{x}.
\end{aligned}
$$

Hence, the relation (3.1) follows from the trivial estimates, for $y = x/t$

$$
\begin{aligned}
y^{1+\zeta} |\varphi(y)| &\leq C_q & \text{if} \quad y \in [1, \infty), \\
|\varphi(y)| &\leq C_q & \text{if} \quad y \in [0, 1].
\end{aligned}
$$

To prove (3.2) we recall that for $0 < \varepsilon < \delta$,

$$\nabla_{q,x} f_{\varepsilon,\delta}(y) = \int_\varepsilon^\delta (\nabla_{q,x} \varphi_t) *_q \varphi_t *_q f(y) \frac{d_q t}{t} \tag{3.3}$$

Hence q-Minkowski's inequality and q-Young's inequality give

$$\|\nabla_{q,x} f_{\varepsilon,\delta}\|_{p,q} \leq \int_\varepsilon^\delta \|\nabla_{q,x} \varphi_t\|_{1,q} \|\varphi_t *_q f\|_{p,q} \frac{d_q t}{t}$$

Since $\|\nabla_{q,y}\varphi\|_{1,q} \leq 2\|\varphi\|_{1,q}, \ y \geq 1$, from Proposition 2.3, for $a = 1/t$ and Lemma 3.1 one has

$$
\begin{aligned}
\|\nabla_{q,x}\varphi_t\|_{1,q} &= \int_0^\infty \mid \nabla_{q,x}\varphi_t(y) \mid d_qy = \int_0^\infty \mid T_{q,x}\varphi_t(y) - \varphi_t(y) \mid d_qy \\
&= \int_0^\infty \frac{1}{t} \mid T_{q,x}\left[\varphi(\frac{y}{t})\right] - \varphi(\frac{y}{t}) \mid d_qy \\
&= \int_0^\infty \frac{1}{t} \mid T_{q,\frac{x}{t}}\left(\varphi(\frac{y}{t})\right) - \varphi(\frac{y}{t}) \mid d_qy \\
&= \int_0^\infty \mid T_{q,\frac{x}{t}}\left(\varphi(u)\right) - \varphi(u) \mid d_qu \\
&= \|\nabla_{q,x/t}\varphi\|_{1,q} \\
&\leq C_q \min(1, \frac{x}{t}).
\end{aligned}
$$

Therefore, using the previous estimate relation (3.3), we get

$$
\begin{aligned}
\|\nabla_{q,x}f_{\varepsilon,\delta}\|_{p,q} &\leq \int_\varepsilon^\delta \|\nabla_{q,x}\varphi_t\|_{1,q}\|\varphi_t *_q f\|_{p,q}\frac{d_qt}{t} \\
&\leq C_q \int_\varepsilon^\delta \min(1, \frac{x}{t})\|\varphi_t *_q f\|_{p,q}\frac{d_qt}{t}
\end{aligned}
$$

Now using the q-Calderon's formula in Theorem 2.4 we obtain $\nabla_{q,x}f_{\varepsilon,\delta} \xrightarrow[\substack{\varepsilon\to 0 \\ \delta\to\infty}]{} \nabla_{q,x}f$ in L^p, and then

$$
\begin{aligned}
\|\nabla_{q,x}f\|_{p,q} &\leq \int_0^\infty \|\nabla_{q,x}\varphi_t\|_{1,q}\|\varphi_t *_q f\|_{p,q}\frac{d_qt}{t} \\
&\leq C_q \int_0^\infty \min(1, \frac{x}{t})\|\varphi_t *_q f\|_{p,q}\frac{d_qt}{t}.
\end{aligned}
$$

$\square$

Although for the purposes of this paper only a particular case of the following proposition will be used, we shall state a general version of it that we find interesting in its own right.

Proposition 3.2. Given $\varepsilon, \delta \in [0, \infty]$, $p \in (1, \infty)$ and w a weight, let us consider

$$
\Theta_{\varepsilon,\delta}(s,t) = \frac{w(s)}{w(t)} \min((\frac{s}{t})^\varepsilon, (\frac{t}{s})^\delta).
$$

If $w(s) = \lambda^{\frac{1}{p'}}(s)\mu^{-\frac{1}{p}}(s^{-1})$ for some pair of weights $\lambda, \ \mu \in \mathcal{W}_{\varepsilon,\delta;q}$, then there exist $C_q > 0$ and $g : \mathbb{R}_{q,+} \longrightarrow \mathbb{R}_+$ q-measurable such that

$$
\int_0^\infty \Theta_{\varepsilon,\delta}(s,t)g^{p'}(s)\frac{d_qs}{s} \leq C_q g^{p'}(t), \tag{3.4}
$$

and

$$
\int_0^\infty \Theta_{\varepsilon,\delta}(s,t)g^p(t)\frac{d_qt}{t} \leq C_q g^p(s). \tag{3.5}
$$

Proof. Let us take $g(t) = \lambda^{1/pp'}(t)\mu^{1/pp'}(t^{-1})$. Then $g^{p'}(s) = \lambda(s)/w(s)$ and $g^p(t) = w(t)\mu(t^{-1})$.

Therefore from (2.2),

$$
\begin{aligned}
\int_0^{+\infty} \Theta_{\varepsilon,\delta}(s,t)g^{p'}(s)\frac{d_q s}{s} &= \frac{1}{w(t)}\int_0^{+\infty} \lambda(s)\min\left(\left(\frac{s}{t}\right)^\varepsilon, \left(\frac{t}{s}\right)^\delta\right)\frac{d_q s}{s} \\
&= \frac{1}{t^\varepsilon w(t)}\int_0^t s^\varepsilon \lambda(s)\frac{d_q s}{s} + \frac{t^\delta}{w(t)}\int_t^{+\infty} \frac{\lambda(s)}{s^\delta}\frac{d_q s}{s} \\
&\leq \frac{1}{t^\varepsilon w(t)}\int_0^t s^\varepsilon \lambda(s)\frac{d_q s}{s} + \frac{t^\delta}{w(t)}\int_{qt}^{+\infty} \frac{\lambda(s)}{s^\delta}\frac{d_q s}{s} \\
&\leq C_q \frac{\lambda(t)}{w(t)} = C_q g^{p'}(t).
\end{aligned}
$$

On the other hand, same from (2.2),

$$
\begin{aligned}
\int_0^{+\infty} \Theta_{\varepsilon,\delta}(s,t)g^{p}(t)\frac{d_q t}{t} &= w(s)\int_0^{+\infty} \mu(t^{-1})\min\left(\left(\frac{s}{t}\right)^\varepsilon, \left(\frac{t}{s}\right)^\delta\right)\frac{d_q t}{t} \\
&= \frac{w(s)}{s^\delta}\int_0^s t^\delta \mu(t^{-1})\frac{d_q t}{t} + s^\varepsilon w(s)\int_s^{+\infty} \frac{\mu(t^{-1})}{t^\varepsilon}\frac{d_q t}{t} \\
&= \frac{w(s)}{s^\delta}\int_{s^{-1}}^{\infty} \frac{\mu(t)}{t^\delta}\frac{d_q t}{t} + s^\varepsilon w(s)\int_0^{s^{-1}} t^\varepsilon \mu(t)\frac{d_q t}{t} \\
&\leq \frac{w(s)}{s^\delta}\int_{qs^{-1}}^{\infty} \frac{\mu(t)}{t^\delta}\frac{d_q t}{t} + s^\varepsilon w(s)\int_0^{s^{-1}} t^\varepsilon \mu(t)\frac{d_q t}{t} \\
&\leq C_q \mu(s^{-1})w(s) = C_q g^p(s).
\end{aligned}
$$

$\square$

We need the following Lemma that we will use after in several of the remaining proofs.

Lemma 3.2. For all $(x,y) \in (q^{\mathbb{Z}}, q^{\mathbb{N}})$, $y \neq q$ we have

$$
T_{q,y}\left(\frac{1}{(x+1)_q^2}\right) \leq \frac{1}{1-q}\cdot\frac{1}{(x+1)_q^2}.
$$

Proof. Let $x, y \in \mathbb{R}_q$, from (2.3) a simple recurrence on $n \in \mathbb{N}$ leads

$$
\begin{aligned}
T_{q,y}\left(\frac{1}{(x+1)_q^2}\right) &= \sum_{n=0}^{\infty} b_n(y;q^2)\Delta_{q,x}^n\left(\frac{1}{(x+1)_q^2}\right) \\
&= \sum_{n=0}^{\infty} q^{n(n-1)}\frac{(1-q)^{2n}}{(q;q)_{2n}}[-2]_q[-3]_q...[-1-2n]_q\frac{y^{2n}}{(q^nx+1)_q^{2n+2}} \\
&= \sum_{n=0}^{\infty} q^{n(n-1)}q^{2n^2+n}[2n+1]_q\frac{y^{2n}}{(q^nx+1)_q^{2n+2}} \\
&= \sum_{n=0}^{\infty} q^{n(n-1)}q^{2n^2+n}[2n+1]_q q^{-n(2n+2)}\frac{y^{2n}}{(x+q^{-n})_q^{2n+1}}
\end{aligned}
$$

from the fact that for $x \in \mathbb{R}_q$

$$
\begin{aligned}
\frac{1}{(x+q^{-n})_q^{2n+1}} &= \frac{1}{(x+q^{-n})(x+q^{-n+1})...(x+q^{-1})}\cdot\frac{1}{(x+1)_q^2}\cdot\frac{1}{(x+q^2)(x+q^3)...(x+q^{n+1})} \\
&\leq \frac{1}{(x+1)_q^2}\, q^{\frac{n(n+1)}{2}}q^{\frac{-n(n+3)}{2}} \\
&\leq \frac{q^{-n}}{(x+1)_q^2}
\end{aligned}
$$

we can deduce that for $y = q^k, k \neq 1$

$$
\begin{aligned}
T_{q,y}\left(\frac{1}{(x+1)_q^2}\right) &= \sum_{n=0}^{\infty} q^{n(n-1)}q^{-n}[2n+1]_q\frac{y^{2n}}{(x+q^{-n})_q^{2n+1}} \\
&\leq \frac{1}{(1-q)}\sum_{n=0}^{\infty} q^{n(n-1)}q^{-n}\frac{y^{2n}}{(x+1)_q^2} \\
&\leq \frac{1}{(1-q)}\cdot\frac{1}{(x+1)_q^2}\cdot\sum_{n=0}^{\infty} q^{n(n-1)}(q^{-2}y^2)^n \\
&\leq \frac{1}{(1-q)}\cdot\frac{1}{(x+1)_q^2}\cdot\frac{1}{1-q^{-2}y^2} \\
&\leq \frac{1}{(1-q)}\cdot\frac{1}{(x+1)_q^2}.
\end{aligned}
$$

the result follow. $\qquad\square$

Proposition 3.3. Let $p \in [1,\infty]$ and let f be a q-measurable function.

If $\| \nabla_{q,x}f \|_{p,q}\in L^1(\mathbb{R}_{q,+}, \frac{d_qx}{(x+1)_q^2})$ then $f \in L^1(\mathbb{R}_{q,+}, \frac{d_qx}{(x+1)_q^2})$.

Proof. Let $\Psi \in L^{p'}(\mathbb{R}_{q,+}, d_q x)$ such that $\Psi > 0$ a.e. Then q-Hölder's inequality and q-Fubini's theorem give

$$\int_0^{+\infty} \Big[\int_0^{+\infty} \frac{|T_{q,x}f(y) - f(y)|}{(x+1)_q^2} d_q x \Big] \Psi(y) d_q y < \infty.$$

Therefore,

$$\int_0^{+\infty} \frac{|T_{q,x}f(y) - f(y)|}{(x+1)_q^2} d_q x < \infty \quad \text{for} \quad a.e. \quad y \in \mathbb{R}_{q,+}.$$

Now we have to prove that $f \in L^1(\mathbb{R}_{q,+}, \frac{d_q x}{(x+1)_q^2})$. From Proposition 2.3 and Lemma 3.2

$$
\begin{aligned}
\int_0^{+\infty} \frac{|f(x)|}{(x+1)_q^2} d_q x \quad &\leq \quad \int_0^{+\infty} \frac{|T_{q,y}f(x) - f(x)|}{(x+1)_q^2} d_q x + \int_0^{+\infty} \frac{|T_{q,y}f(x)|}{(x+1)_q^2} d_q x \\
&\leq \quad \int_0^{+\infty} \frac{|T_{q,y}f(x) - f(x)|}{(x+1)_q^2} d_q x + \int_0^{+\infty} f(x) \, | \, T_{q,y}\left(\frac{1}{(x+1)_q^2}\right) d_q x \\
&\leq \quad \int_0^{+\infty} \frac{|T_{q,y}f(x) - f(x)|}{(x+1)_q^2} d_q x + \frac{1}{1-q} \int_0^{+\infty} \frac{|f(x)|}{(x+1)_q^2} d_q x \\
&\leq \quad \int_0^{+\infty} \frac{|T_{q,x}f(y) - f(y)|}{(x+1)_q^2} d_q x + \int_0^{+\infty} \frac{|f(y)|}{(x+1)_q^2} d_q x + \frac{2-q}{1-q} \int_0^{+\infty} \frac{|f(x)|}{(x+1)_q^2} d_q x.
\end{aligned}
$$

Since $x \longmapsto \frac{1}{(x+1)_q^2} \in L^1(\mathbb{R}_{q,+}, d_q x)$, then using (3.6), we obtain

$$
\begin{aligned}
\frac{1}{q-1} \int_0^{+\infty} \frac{|f(x)|}{(x+1)_q^2} d_q x \quad &\leq \quad \int_0^{+\infty} \frac{|T_{q,y}f(x) - f(x)|}{(x+1)_q^2} \, | \, d_q x + \int_0^{+\infty} \frac{|f(y)|}{(x+1)_q^2} d_q x \\
&< \quad \infty \quad \text{for} \quad a.e. \quad y \in \mathbb{R}_{q,+}
\end{aligned}
$$

which leads to the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

▶ Now we start the main result of this paper with the case $m = \infty$ which easily follows from Proposition 3.1.

Theorem 3.1. Let $p \in [1, \infty]$, $\varphi \in \mathcal{A}_{*,q}$ and $w \in \mathcal{W}_{0,1;q}$. Then

$$\Lambda_{*,w,q}^{p,\infty} = B_{*,w,\varphi,q}^{p,\infty} \qquad \text{with equivalent seminorms.}$$

Proof. Let $f \in \Lambda_{*,w,q}^{p,\infty}$, then one has

$$
\begin{aligned}
\int_0^{+\infty} \frac{\|\nabla_{q,x}f\|_{p,q}}{(x+1)_q^2} d_q x \quad &\leq \quad C_q \int_0^{\infty} \frac{w(x)}{(x+1)_q^2} d_q x \\
&\leq \quad C_q \Big[\int_0^1 w(x) \frac{d_q x}{x} + \int_1^{\infty} w(x) \frac{d_q x}{x^2} \Big] \\
&< \quad \infty,
\end{aligned}
$$

what combined with Proposition 3.3 gives $\int_0^\infty \frac{|f(x)|}{(x+1)_q^2} d_q x < \infty$.

Let us prove that $\|\varphi_t *_q f\|_{p,q} \leq C_q w(t)$. From (2.2) and (3.1) for $\zeta = 1$ one has

$$
\begin{aligned}
\|\varphi_t *_q f\|_{p,q} &\leq C_q \Big[\frac{1}{t} \int_0^t \|\nabla_{q,x} f\|_{p,q} d_q x + t \int_t^\infty \|\nabla_{q,x} f\|_{p,q} \frac{d_q x}{x^2} \Big] \\
&\leq C_q \Big[\int_0^t \frac{x}{t} w(x) \frac{d_q x}{x} + t \int_t^\infty w(x) \frac{d_q x}{x^2} \Big] \\
&\leq C_q \Big[\int_0^t \frac{x}{t} w(x) \frac{d_q x}{x} + t \int_{qt}^\infty w(x) \frac{d_q x}{x^2} \Big] \\
&\leq C_q w(t).
\end{aligned}
$$

Conversely if $f \in B_{*,w,\varphi,q}^{p,\infty}$, then from (2.2) and (3.2) one has

$$
\begin{aligned}
\|\nabla_{q,x} f\|_{p,q} &\leq C_q \Big[\int_0^x \|\varphi_t *_q f\|_{p,q} \frac{d_q t}{t} + x \int_x^\infty \|\varphi_t *_q f\|_{p,q} \frac{d_q t}{t^2} \Big] \\
&\leq C_q \Big[\int_0^x \frac{w(t)}{t} d_q t + x \int_x^\infty \frac{w(t)}{t^2} d_q t \Big] \\
&\leq C_q \Big[\int_0^x \frac{w(t)}{t} d_q t + x \int_{qx}^\infty \frac{w(t)}{t^2} d_q t \Big] \\
&\leq C_q w(x).
\end{aligned}
$$

$\square$

▶ We prove now the main Theorem in the case $m = 1$.

Theorem 3.2. Let $p \in [1,\infty]$, $\varphi \in \mathcal{A}_{*,q}$ and $w \in \mathcal{W}_{0,1;q}$ such that $\mu(t) = w^{-1}(t^{-1})$. Then

$$
\Lambda_{*,w,q}^{p,1} = B_{*,w,\varphi,q}^{p,1} \qquad \text{with equivalent seminorms.}
$$

Proof. Assume $f \in \Lambda_{*,w,q}^{p,1}$. Let us first prove that

$$
\int_0^\infty \frac{|f(x)|}{(x+1)_q^2} d_q x < \infty.
$$

From Proposition 2.5, we have

$$
\frac{1}{xw(x)} \geq C_q \frac{1}{x} \min(1, \frac{1}{x}) \geq C_q \frac{1}{x} \min(x, \frac{1}{x}) \geq \frac{C_q}{(x+1)_q^2}.
$$

Hence

$$
\int_0^\infty \frac{\|\nabla_{q,x} f(x)\|_{p,q}}{(x+1)_q^2} d_q x \leq C_q \int_0^\infty \frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)} \frac{d_q x}{x} < \infty
$$

and we apply Proposition 3.3 again.

We will prove that $\|f\|_{B^{p,1}_{*,w,\varphi,q}} \leq C_q \|f\|_{\Lambda^{p,1}_{*,w,q}}$.

Using (2.2) and (3.1) with $\zeta = 1$ we have

$$
\begin{aligned}
\int_0^\infty \frac{\|\varphi_t *_q f\|_{p,q}}{w(t)} d_q t &\leq C_q \int_0^\infty \Big[\int_0^\infty \min\Big(\frac{x}{t}, \frac{t}{x}\Big) \frac{\|\nabla_{q,x} f\|_{p,q}}{w(t)} \frac{d_q x}{x} \Big] \frac{d_q t}{t} \\
&= C_q \int_0^\infty \|\nabla_{q,x} f\|_{p,q} \Big[\int_0^\infty \min\Big(\frac{x}{t}, \frac{t}{x}\Big) \mu(t^{-1}) \frac{d_q t}{t} \Big] \frac{d_q x}{x} \\
&= C_q \int_0^\infty \|\nabla_{q,x} f\|_{p,q} \Big[\int_0^x \frac{t\mu(t^{-1})}{x} \frac{d_q t}{t} + \int_x^\infty \frac{x\mu(t^{-1})}{t} \frac{d_q t}{t} \Big] \frac{d_q x}{x} \\
&\leq C_q \int_0^\infty \|\nabla_{q,x} f\|_{p,q} \Big[\frac{1}{x} \int_{qx^{-1}}^\infty \mu(t) \frac{d_q t}{t^2} + \int_0^{qx^{-1}} \mu(t) \frac{d_q t}{t} \Big] \frac{d_q x}{x} \\
&\leq C_q \int_0^\infty \|\nabla_{q,x} f\|_{p,q} \Big[\frac{1}{x} \int_{qx^{-1}}^\infty \mu(t) \frac{d_q t}{t^2} + \int_0^{x^{-1}} \mu(t) \frac{d_q t}{t} \Big] \frac{d_q x}{x} \\
&\leq C_q \int_0^\infty \|\nabla_{q,x} f\|_{p,q} \mu(x^{-1}) \frac{d_q x}{x} \\
&\leq C_q \int_0^\infty \frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)} \frac{d_q x}{x}.
\end{aligned}
$$

Let $f \in B^{p,1}_{*,w,\varphi,q}$, from (2.2), (3.2) and q-Fubini's theorem we get

$$
\begin{aligned}
\int_0^\infty \frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)} \frac{d_q x}{x} &\leq C_q \int_0^\infty \|\varphi_t *_q f\|_{p,q} \Big[\int_0^\infty \mu(x^{-1}) \min\Big(1, \frac{x}{t}\Big) \frac{d_q x}{x} \Big] \frac{d_q t}{t} \\
&= C_q \int_0^\infty \|\varphi_t *_q f\|_{p,q} \Big[\int_0^\infty \mu(s) \min\Big(1, \frac{1}{st}\Big) \frac{d_q s}{s} \Big] \frac{d_q t}{t} \\
&= C_q \int_0^\infty \|\varphi_t *_q f\|_{p,q} \Big[\int_0^{t^{-1}} \frac{\mu(s)}{s} d_q s + \int_{t^{-1}}^\infty \frac{\mu(s)}{s^2} d_q s \Big] \frac{d_q t}{t} \\
&\leq C_q \int_0^\infty \|\varphi_t *_q f\|_{p,q} \Big[\int_0^{t^{-1}} \frac{\mu(s)}{s} d_q s + \int_{qt^{-1}}^\infty \frac{\mu(s)}{s^2} d_q s \Big] \frac{d_q t}{t} \\
&\leq C_q \int_0^\infty \|\varphi_t *_q f\|_{p,q} \mu(t^{-1}) \frac{d_q t}{t} \\
&= C_q \int_0^\infty \frac{\|\varphi_t *_q f\|_{p,q}}{w(t)} \frac{d_q t}{t}.
\end{aligned}
$$

$\square$

Theorem 3.3. Let $p \in [1, \infty]$, $m \in (1, \infty)$, and w be a weight such that $w(t) = \lambda^{\frac{1}{m'}}(t) \mu^{-\frac{1}{m}}(t^{-1})$ for some pair of weights $\lambda, \mu \in \mathcal{W}_{0,1;q}$. Then for $\varphi \in \mathcal{A}_{*,q}$,

$$
\Lambda^{p,m}_{*,w,q} = B^{p,m}_{*,w,\varphi,q} \qquad \text{with equivalent seminorms.}
$$

Proof. Assume that $f \in \Lambda^{p,m}_{*,w,q}$. We will first show that

$$\int_0^\infty \frac{|f(x)|}{(x+1)_q^2} d_q x < \infty.$$

We denote

$$\Phi(x) = \frac{x}{(x+1)_q^2} w(x)$$

under the assumptions $\lambda, \mu \in \mathcal{W}_{0,1;q}$ one has $\Phi \in L^{m'}(\mathbb{R}_{q,+}, \frac{d_q x}{x})$. Indeed,

$$\int_0^\infty \Phi^{m'}(t) \frac{d_q t}{t} \leq \int_0^\infty \lambda(t) \mu^{-\frac{m'}{m}}(t^{-1}) \frac{t^{m'}}{(t+1)_q^{2m'}} \frac{d_q t}{t}.$$

Using Proposition 2.5 we have $\mu(s) \geq C_q \min(1, s)$. Therefore

$$\begin{aligned}
\int_0^\infty \Phi^{m'}(t) \frac{d_q t}{t} &\leq \int_0^\infty \lambda(t) \max(1, t^{m'-1}) \frac{t^{m'}}{(t+1)_q^{2m'}} \frac{d_q t}{t} \\
&\leq C_q \left[\int_0^1 \lambda(t) \frac{d_q t}{t} + \int_1^\infty \frac{\lambda(t)}{t} \frac{d_q t}{t} \right] \\
&< \infty.
\end{aligned}$$

Then using q-Hölder's inequality one has

$$\int_0^\infty \frac{\|\nabla_{q,x} f\|_{p,q}}{(x+1)_q^2} d_q x = \int_0^\infty \frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)} \Phi(x) \frac{d_q x}{x} < \infty$$

and we apply Proposition 3.3.

Now we prove that

$$\|f\|_{B^{p,m}_{*,w,\varphi,q}} \leq C_q \|f\|_{\Lambda^{p,m}_{*,w,q}}.$$

From (3.1) in Proposition 3.1 and taking $\rho = 1$, it follows that

$$\frac{\|\varphi_t *_q f\|_{p,q}}{w(t)} \leq \int_0^\infty K(x,t) \frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)} \frac{d_q x}{x}$$

where

$$K(x,t) = \frac{w(x)}{w(t)} \min(1, \frac{t}{x}).$$

If we take

$$(\Omega_1, \mu_1) = (\mathbb{R}_{q,+}, \frac{d_q x}{x})$$

and

$$(\Omega_2, \mu_2) = (\mathbb{R}_{q,+}, \frac{d_q x}{x}).$$

Since $K(x,t) = \Theta_{0,1}(x,t)$, we can apply Proposition 3.2 with $\varepsilon = 0$ and $\delta = 1$ to get a q-measurable function g satisfying (3.4) and (3.5).

Now write $h_1(x) = g(x)$ and $h_2(t) = g(t)$. Obviously, using (3.4) and (3.5) give (2.5) and (2.6) in Lemma 2.2, what shows T_K is bounded from $L^m(\mathbb{R}_{q,+}, \frac{d_q x}{x})$ into $L^m(\mathbb{R}_{q,+}, \frac{d_q t}{t})$. Therefore,

$$
\begin{aligned}
\|f\|_{B^{p,m}_{*,w,\varphi,q}} &\leq C_q \|T_K(\frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)})\|_{L^m(\mathbb{R}_{q,+}, \frac{d_q t}{t})} \\
&\leq C_q \|\frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)}\|_{L^m(\mathbb{R}_{q,+}, \frac{d_q x}{x})} \\
&\leq C_q \|f\|_{\Lambda^{p,m}_{*,w,q}}.
\end{aligned}
$$

Conversely, let $f \in \Lambda^{p,m}_{*,w,q}$. From (3.2) in Proposition 3.1 we obtain

$$\frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)} \leq C_q \int_0^\infty \Theta(t,x) \frac{\|\varphi_t *_q f\|_{p,q}}{w(t)} \frac{d_q t}{t}$$

where

$$\Theta(t,x) = \frac{w(t)}{w(x)} \min(1, \frac{x}{t}).$$

Now take

$$(\Omega_1, \mu_1) = (\mathbb{R}_{q,+}, \frac{d_q x}{x})$$

and

$$(\Omega_2, \mu_2) = (\mathbb{R}_{q,+}, \frac{d_q x}{x}).$$

Combine now again Proposition 3.2 and Lemma 2.2 to get the boundedness of T_K from $L^m(\mathbb{R}_{q,+}, \frac{d_q x}{x})$ into $L^m(\mathbb{R}_{q,+}, \frac{d_q t}{t})$. Therefore,

$$
\begin{aligned}
\|f\|_{\Lambda^{p,m}_{*,w,q}} &\leq C_q \|T_K(\frac{\|\varphi_t *_q f\|_{p,q}}{w(t)})\|_{L^m(\mathbb{R}_{q,+}, \frac{d_q x}{x})} \\
&\leq C_q \|\frac{\|\varphi_t *_q f\|_{p,q}}{w(t)}\|_{L^m(\mathbb{R}_{q,+}, \frac{d_q t}{t})} \\
&\leq C_q \|f\|_{B^{p,m}_{*,w,\varphi,q}}.
\end{aligned}
$$

$\square$

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