Mappings preserving unit distance on Heisenberg group

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Abstract

Let H^m be a Heisenberg group provided with a norm ρ. A mapping $f : H^m \to H^m$ is called preserving the distance n if for all x, y of H^m with $\rho(x^{-1}y) = n$ then $\rho(f(x)^{-1}f(y)) = n$. We obtain some results for the Aleksandrov problem in the Heisenberg group.

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1 Introduction

Posed around 1970, the Aleksandrov problem investigates an isometry by the preservation of some properties of distance [1]. Several studies have been established to this subject on different normed spaces. I quote in this connection, the studies made by H. Y. Chu, C. G. Park, W. G. Park in 2004 on linear 2-normed spaces [8], J. M. Rassias, S. Xiang, M. J. Rassias in 2007 on the Aleksandrov and triangle isometry Ulam stability problem [16], X.Y. Chen, M.M. Song in 2010 on linear n-normed spaces [7] and D. Wang, Y. Liu, M. Song in 2012 on non-Archimedean normed spaces [23]. For more details the reader may also study [[2]-[6], [9]-[13], [15, 17],[18]-[22]].
The purpose of our contribution, is an idea introduced by J. M. Rassias that consists to apply it here to study the Aleksandrov problem in a Heinsenberg group.
Following this Introduction, some preliminary notations are set in the second Section, as well as our main new results are investigated in the third Section, respectively.

2 Preliminary

In this section we fix notations and special vocabulary that will be used later in the document.
Let m be a fixed nonzero integer number. The m^{th} Heisenberg group H^m is of course a near isomorphism $\mathbb{C}^m \times \mathbb{R}$ endowed with the following group law

$$(z,t)(z',t') = (z + z', t + t' + Im\,\langle z, z'\rangle),\ (z,t),(z',t') \in \mathbb{C}^m \times \mathbb{R},$$

where $z = (z_i)_{1\le i\le m},\ z' = (z'_i)_{1\le i\le m}$ and $\langle z, z'\rangle = \sum_{i=1}^{m} z_i \overline{z'}_i$, with identity element $(0,0)$ and an inverse given by $(z,t)^{-1} = (-z,-t)$. The dilation δ_s, for $s > 0$, acts on the Heisenberg group as $\delta_s(z,t) = (sz, s^2t)$ and is its automorphism. The homogeneous norm

$$\rho(z,t) = (|z|^4 + t^2)^{\frac{1}{4}}$$

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defines the Heisenberg metric d_ρ via the formula

$$d_\rho(x, y) = \rho(x^{-1}y), x, y \in H^m.$$

Observe that the Heisenberg metric is really a metric and not just a quasi-metric since,

$$\rho(xy) \leq \rho(x) + \rho(y)$$

for all $x, y \in H^m$ (see [7, 8] for instance). It is also known that the Heisenberg metric d_ρ and the Carnot Caratheodory metric d are equivalent; that is, there exists a constant $c > 1$ such that $c^{-1}d(x, y) \leq d_\rho(x, y) \leq cd(x, y)$ for all $x, y \in H^m$.

3 Main result

Let us etablish in this section the main results of this paper. We note that, throughout this section H^m designates a Heisenberg group with its norm ρ.

Definition 3.1. A mapping f of H^m on itself is called an isometry if

$$\rho(f(x)^{-1}f(y)) = \rho(x^{-1}y)$$

for all $x, y \in H^m$.

If a mapping f of H^m on itself is an isometry then the inverse mapping is an isometry of H^m onto H^m.

Definition 3.2. A mapping f of H^m on itself, satisfies the strong distance one preserving property (SDOPP) if and only if for all $x, y \in H^m$ with $\rho(x^{-1}y) = 1$ it follows that $\rho(f(x)^{-1}f(y)) = 1$.

Definition 3.3. A mapping $f : H^m \to H^m$ satisfies the strong distance n preserving property (SDnPP) if only if for all $x, y \in H^m$ with $\rho(x^{-1}y) = n$ it follows that $\rho(f(x)^{-1}f(y)) = n$.

Definition 3.4. Let H^m be a Heisenberg group. We call a mapping $f : H^m \to H^m$ Lipschitz mapping if there is a $K > 0$ such that

$$\rho(f(x)^{-1}f(y)) \leq K\rho(x^{-1}y)$$

for any $x, y \in H^m$.

Definition 3.5. We call a mapping $f : H^m \to H^m$ locally Lipschitz mapping if there is a $K > 0$ such that

$$\rho(f(x)^{-1}f(y)) \leq K\rho(x^{-1}y),$$

whenever $\rho(x^{-1}y) \leq 1$.

We consider in this paper only the Lipschitz constant $K \leq 1$.

In this paper we shall study, mappings satisfying the weaker assumption that they preserve distance n in both directions, instead of isometries. We shall see that such mappings are not far from being isometries. Let us prove the following Lemma.

Lemma 3.6. Let H^m the Heisenberg group. Suppose that $f : H^m \to H^m$ is a surjective mapping satisfying (SDOPP). Then f is bijective.

Proof. We shall show that f is injective. Suppose that there exist x and y in H^m with $x \neq y$ such that $f(x) = f(y)$. Since $\rho(x^{-1}y) \neq 0$, we can set

$$z = x\delta_{\frac{1}{\rho(y^{-1}x)}}(y^{-1}x),$$

thus,

$$\rho(x^{-1}z) = \rho(\delta_{\frac{1}{\rho(y^{-1}x)}}(y^{-1}x)) = 1.$$

Since f verify f preserves the unit distance, that $\rho(f(x)^{-1}f(z)) = 1$. Now we will prove that $\rho(y^{-1}z) \neq 1$. Suppose on the contrary, that $\rho(y^{-1}z) = 1$. We have $y^{-1}z = y^{-1}x\delta_{\frac{1}{\rho(y^{-1}x)}}(y^{-1}x)$, by identification we can put $y^{-1}x = (x_1, t_1)$ and denote $\alpha = \frac{1}{\rho(y^{-1}x)}$. This implies that

$$\rho(y^{-1}z) = \rho((1+\alpha)x_1, (1+\alpha^2)t_1) = ((1+\alpha)^4|x_1|^4 + (1+\alpha^2)^2t_1^2)^{\frac{1}{4}}.$$

Since $\rho(y^{-1}z) = 1$ and $\alpha^4|x_1|^4 + \alpha^4t_1^2 = 1$, so

$$(1 + 4\alpha^3 + 4\alpha^3 + 2\alpha^2)|x_1|^4 + (1 + 2\alpha^2)t_1^2 = 0$$

and so $x_1 = 0$ and $t_1 = 0$, then $x = y$, which is a contradiction. Therefore $\rho(y^{-1}z) \neq 1$. Now, we get $1 = \rho(f(x)^{-1}f(z)) = \rho(f(y)^{-1}f(z))$. Since f preserves the unit distance, that $\rho(y^{-1}z) = 1$ which is a contradiction. Therefore f is a injective and surjective mapping then f is bijective.

Q.E.D.

The following theorem gives the n-distance preserving mapping in both directions

Theorem 3.7. Let H^m the Heisenberg group. Suppose that $f : H^m \to H^m$ is a surjective mapping satisfying (SDOPP) such that

$$\rho(x^{-1}y) < 1 \text{ } if \text{ } an \text{ } only \text{ } if \text{ } \rho(f(x)^{-1}f(y)) < 1. \tag{3.1}$$

Then f preserves the area n for each $n \in \mathbb{N}$.

Proof. By Lemma 3.6 f is a injective and since f is surjective mapping then f is bijective. Both f and f^{-1} preserves the unit distance and verify 3.1. Now we will prove that f preserves distance n in both directions for any positive integer n. In the sequel we shall need the following notations:

$$\begin{aligned} B(x; r) &= \{z : \rho(x^{-1}z) \leq r\}; \\ B_0(x; r) &= \{z : \rho(x^{-1}z) < r\}; \end{aligned}$$

Let x be an arbitrary vector in H^m and n any positive integer, $n > 1$. Assume that $z \in B(x, n)$. We can find a sequence $x = x_0, \ldots, z = x_n$, such that

$$x_i = x_{i-1}\delta_{\frac{1}{\rho(x^{-1}z)}}(z^{-1}x)), \; i = 1, \ldots, n.$$

Then,

$$\rho(x_{i-1}^{-1}x_i) = \rho(\delta_{\frac{1}{\rho(x^{-1}z)}}(z^{-1}x)) = 1, \; i = 1, \ldots, n.$$

Since f preserves the unit distance, that

$$\rho(f(x)^{-1}f(z)) \leq n$$

Consequently, we have

$$f(B(x,n)) \subset B(f(x),n), n > 1.$$

The same result can be obtained for f^{-1}. Hence,

$$f(B(x,n)) = B(f(x),n),\ n > 1.$$

Since f and f^{-1} verify 3.1 then

$$f(B_0(x,1)) = B_0(f(x),1).$$

hold for all $x \in H^m$. Now we will prove that

$$f(B_0(x,n)) = B_0(f(x),n).$$

for all $x \in X$, and $n \in \mathbb{N}$. Let $z \in B_0(x,n)$ and consider a sequence $x = x_0, x_1, ..., x_n = z$ such that

$$x_i = x_{i-1}\delta_{\frac{1}{n}}(z^{-1}x),\ i = 1,\dots,n.$$

Then

$$\rho(x_{i-1}^{-1}x_i) = \rho(\delta_{\frac{1}{n}}(z^{-1}x)) = 1,\ i = 1,\dots,n.$$

Since $f(B_0(x,1)) = B_0(f(x),1)$, that

$$\rho(f(x)^{-1}f(z)) < n$$

Consequently, we have

$$f(B_0(x,n)) \subset B_0(f(x),n), n \in \mathbb{N}.$$

The same result can be obtained for f^{-1}. Hence,

$$f(B_0(x,n)) = B_0(f(x),n), n \in \mathbb{N}.$$

Consequently

$$f(B(x,n)) \setminus f(B_0(x,n)) = B(f(x),n) \setminus B_0(f(x),n), n \in \mathbb{N}.$$

So f preserve the area n for each $n \in \mathbb{N}$. Q.E.D.

We will study the problem of mappings which preserve unit distance is an isometry.

Lemma 3.8. If a mapping f is locally Lipschitz, then f is a Lipschitz mapping.

Proof. We may assume that $\rho(x^{-1}y) \geq 1$, then there is $n_0 \in \mathbb{N}$ such that $\rho(x^{-1}y) \leq n_0$. Let $x = x_0 = (y_0, t_0), x_1, ..., x_{n_0} = y$, such that

$$x_i = x_{i-1}\delta_{\frac{1}{n_0}}(y^{-1}x),\ i = 1,\dots,n_0.$$

Then

$$\rho(x_{i-1}^{-1}x_i) = \rho(\delta_{\frac{1}{n_0}}(y^{-1}x)) \leq 1,\ i = 1, \ldots, n.$$

Since f is locally Lipschitz, so

$$\rho(f(x_{i-1})^{-1}f(x_i)) \leq K\rho(x_{i-1}^{-1}x_i),$$

consequently

$$\rho(f(x)^{-1}f(y)) \leq K\rho(x^{-1}y).$$

Q.E.D.

Theorem 3.9. Let $f : H^m \to H^m$ be a locally Lipschitz mapping with the Lipschitz constant $K \leq 1$. Assume that f is a surjective mapping satisfying (SDOPP). Then f is a isometry.

Proof. By Lemma 3.6 and 3.8, f Lipschitz mapping with the Lipschitz constant $K \leq 1$ and f is bejective.For $x, y \in X$, there are two cases depending upon whether $\rho(f(x)^{-1}f(y)) = 0$ or not.

In the first case $\rho(f(x)^{-1}f(y)) = 0$ equivalent to $f(x) = f(y)$. Since f is injective so $x = y$ and so $\rho(x^{-1}y) = 0$.

In the remaining case $\rho(f(x)^{-1}f(y)) > 0$, there is an $n_0 \in \mathbb{N}$ such that$\rho(f(x)^{-1}f(y)) < n_0$. Assume that $\rho(f(x)^{-1}f(y)) < \rho(x^{-1}y)$. Set $x = x_0, x_1, ..., x_{n_0} = y$, such that

$$x_i = x_{i-1}\delta_{\frac{1}{n_0}}(f(y)^{-1}f(x)),\ i = 1, \ldots, n_0.$$

Then we obtain that

$$\rho(x_{i-1}^{-1}x_i) = \frac{\rho(f(y)^{-1}f(x))}{n_0},\ i = 1, \ldots, n_0.$$

Thus

$$\begin{aligned}
\rho(x^{-1}y) &= \rho(\prod_{i=1}^{n_0} x_{i-1}^{-1}x_i) \\
&\leq \sum_{i=1}^{n_0} \rho(x_{i-1}^{-1}x_i) \\
&= \sum_{i=1}^{n_0} \frac{\rho(f(y)^{-1}f(x))}{n_0} \\
&= \rho(f(x)^{-1}f(y)).
\end{aligned}$$

Which is a contradiction, hence f is a isometry. Q.E.D.

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