A discrete orthogonal polynomials approach for coupled systems of nonlinear fractional order integro-differential equations

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Abstract

This paper develops a numerical approach for solving coupled systems of nonlinear fractional order integro-differential equations(NFIDE). Shifted discrete Chebyshev polynomials (SDCPs) have been introduced and their attributes have been checked. Fractional operational matrices for the orthogonal polynomials are also acquired. A numerical algorithm supported by the discrete orthogonal polynomials and operational matrices are used to approximate solution of coupled systems of NFIDE. The operational matrices of fractional integration and product are applied for approximate the unknown functions directly. These approximations were put in the coupled systems of NFIDE. A comparison has been made between the absolute error of approximate solutions of SDCPs method with previous published. The gained numerical conclusions disclose that utilizing discrete Chebyshev polynomials are more efficient in comparison to the other methods.

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1 Introduction

Newly, fractional calculus, the theory of differentiation and integration to non-integer order, was recognized to describe literally many fundamental problems in science and engineering [1, 2]. Especially, fractional calculus is applied for modelling problems in viscoelasticity [4], economics [5] nonlinear oscillation of earthquake [6], fluid-dynamic traffic [7], continuum and statistical mechanics [8], signal processing [9].

Fractional integral-differential equations(FIDEs) plays a significant role in modelling of various problems arise in scientific fields. Owing to the fractional order exponents in differential operators, analytical solutions of FIDEs are usually difficult to acquire. Consequently, many numerical methods have been developed to solve FIDEs. For example, fractional differential transform [12, 13], Adomian decomposition method [14], Taylor expansion [15], collocation method [16–18], Legendre Tau method [19], wavelet method [21] and radial basis functions [22] have been proposed to approximate solution of FIDEs.

Recently, notable attention was paid to the approximate solution of FIDEs by means of orthogonal polynomials. For example, numerical schemes based on Legendre [23], Chebyshev [24,25], second kind of Chebyshev [26, 27] and Jacobi [28–30] polynomials were proposed. By using the orthogonal properties of the polynomials and spectral methods, reduce the fractional equations to a system of

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algebraic equations that are much easier to solve. According to the defined inner product in the solution space, orthogonal polynomials are usually classified into two main classes: continuous and discrete. Continuous orthogonal polynomials, such as those of Legendre, Chebyshev, Hermite, and Laguerre, require the evaluation of an integral in the inner product, whereas discrete orthogonal polynomials require a discrete scalar product. Continous orthogonal polynomials have been more frequently used to approximate the solution of functional equations than discrete versions. However, there are some advantages of using discrete orthogonal polynomials. The Fourier coefficients can be calculated with a sumation and the computational cost is significantly reduced [31, 48]. Moreover, there is close connection between stochastic processes and discrete orthogonal polynomials which motivated their use forsolving stochastic differential equations [31, 33].

In the current article we discuss on coupled system of NFIDE with initial conditions. The study on this type of FIDEs has gotten less attention. For instance [34, 35] considered and proposed the numerical solution for coupled system of NFIDE. Consider coupled system of NFIDE:

$$\begin{cases} D^{\nu}u(t) = F_1(t, u(t), v(t)) + \int_0^t F_2(\eta, u(\eta), v(\eta))d\eta, & 0 < \nu \leq 1, \\ \\ D^{\mu}v(t) = G_1(t, u(t), v(t)) + \int_0^t G_2(\eta, u(\eta), v(\eta))d\eta, & 0 < \mu \leq 1, \end{cases} \tag{1}$$

subjected to the initial conditions

$$\begin{cases} u(0) = u_0 \\ v(0) = v_0 \end{cases}, \qquad t, \eta \in [0, 1]. \tag{2}$$

where $u(t)$ and $v(t)$ are the unknown functions, $D^{\nu}u(t)$ and $D^{\mu}v(t)$ denote the fractional derivative of $u(t)$ and $v(t)$ in the Liouville-Caputo sense, respectively.

The SDCPs along with their fractional operational matrices are employed to find the approximate solution of coupled system of NFIDE (1) subject to the initial conditions(2). First, the discrete Chebyshev polynomials and their properties are introduced. In this method, the fractional derivative of the unknown functions are expanded with polynomial basis and unknown coefficients. The operational matrix of fractional integration is used to transform the under consideration coupled systems of NFIDEs into an algabric system. A comparison has been made between the absolute error of approximate solution of this proposed method and published previous methods.

The structure of the paper is as follows. Section 2, presents several definitions and notation of fractional calculus. Section 3 is devoted to the formulation of the SDCPs and their properties. Section 4 deals with the numerical solution of NFIDEs. Section 5, discusses illustrative examples that demonstrate the superiority of the SDCPs. Finally, Section 6 summarises the main concluding remarks.

2 Preliminary remarks

Frequently used definitions for fractional integral and derivative are the Liouville-Caputo and Riemann-Liouville, formulated as follows [1, 2].

Definition 2.1. The fractional integration of order $\nu \geq 0$ in the Riemann-Liouville sense is defined as

$$(I^{\nu} f)(t) = \begin{cases} f(t), & \nu = 0, \\ \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t \frac{f(s)ds}{(t-s)^{1-\nu}}, & \nu > 0. \end{cases}$$

Some properties of the fractional operator I^{ν} can be listed as follow:

$$\begin{aligned} &I^{\nu_1}\left(I^{\nu_2} f(t)\right) = I^{\nu_2}\left(I^{\nu_1} f(t)\right), \quad \nu_1, \nu_2 \geq 0, \\ &I^{\nu_1}\left(I^{\nu_2} f(t)\right) = I^{\nu_1+\nu_2} f(t), \quad \nu_1, \nu_2 \geq 0, \\ &I^{\nu} t^r = \tfrac{\Gamma(r+1)}{\Gamma(r+\nu+1)} t^{\nu+r}, \;\; \nu \geq 0, \;\; r > -1. \end{aligned} \tag{3}$$

Definition 2.2. Let ν be a real positive number. The Liouville-Caputo fractional derivative operator of order ν, i.e. D^{ν}, is defined as follow:

$$D^{\nu} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\nu)} \displaystyle\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\nu-n+1}} ds, \;\; t > 0, & 0 \leq n-1 < \nu < n, \\ \dfrac{d^n f(t)}{dt^n}, & \nu = n \in \mathbb{N}. \end{cases}$$

where n is an integer, $t > 0$.

Also, the following properties of the Liouville-Caputo fractional operator D^{ν} are verified:

$$D^{\nu} I^{\nu} f(t) = f(t).$$

$$I^{\nu} D^{\nu} f(t) = f(t) - \sum_{j=0}^{n-1} f^{(j)}(0^+) \frac{t^j}{j!}, \qquad t > 0.$$

$$D^{\nu} t^r = \begin{cases} 0, & for\ r \in \mathbb{N} \cup \{0\}\ and\ r < \nu, \\ \dfrac{\Gamma(r+1)}{\Gamma(r-\nu+1)} t^{r-\nu}, & otherwise. \end{cases}$$

3 Discrete Chebyshev polynomials

The discrete Chebyshev polynomials belong to a rich family of the orthogonal polynomials introduced by P.L. Chebyshev. In this section, we restrict our attention to the fundemental definition, formula and properties. For more details and applications of these polynomials one can refer to Refs. [44, 45].

Definition 3.1. Let N be a positive integer number. The discrete Chebyshev polynomials $\mathcal{T}_{n,N}(x)$ are defined as follows:

$$\mathcal{T}_{n,N}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+k}{n} \binom{N-k}{n-k} \binom{x}{k}, \quad n = 0, 1, ..., N. \tag{4}$$

Orthogonality:

The set of discrete Chebyshev polynomials $\{\mathcal{T}_{n,N}, n = 0, 1, ..., N\}$ are orthogonal on the interval $[0, N]$ with respect to the following discrete norm:

$$\langle f, g\rangle = \sum_{r=0}^{N} f(r)g(r). \tag{5}$$

Moreover, the orthogonality condition for these polynomials is as follow:

$$\langle \mathcal{T}_{m,N}, \mathcal{T}_{n,N}\rangle = \sum_{r=0}^{N} \mathcal{T}_{m,N}(r)\mathcal{T}_{n,N}(r) = \mu_n \delta_{mn},$$

where δ_{mn} is the Kronecker delta and μ_n can be defined as:

$$\mu_n = \frac{(N + n + 1)!}{(2n + 1)(N - n)!(n!)^2}. \tag{6}$$

Recurrence relation:

The set of discrete Chebyshev polynomials $\{\mathcal{T}_{n,N},\ n = 0, 1, ..., N\}$ can be determined with the aid of the following recurrence formula:

$$\alpha_n \mathcal{T}_{n+1,N}(x) = \frac{(N - k)(\alpha_n + \beta_n - x)}{k + 1}\mathcal{T}_{n,N}(x) - \frac{(N - k + 1)(N - k)\beta_n}{k(k + 1)}\mathcal{T}_{n-1,N}(x), \quad n = 0, 1, ..., N,$$

where $\mathcal{T}_{0,N}(x) = 1,\ \mathcal{T}_{1,N}(x) = N - 2x$ and

$$\alpha_n = \frac{(N - n)(n + 1)^2}{(2n + 1)(2n + 2)}, \qquad \beta_n = \frac{N^2(N + n + 1)}{2n(2n + 1)}.$$

Explicit formula:

By using the relation (4), the discrete Chebyshev polynomial $\mathcal{T}_{n,N}$ can be written as:

$$\mathcal{T}_{n,N}(x) = \sum_{k=0}^{n} \frac{(-1)^k \binom{n+k}{n}\binom{N-k}{n-k}}{k!} x(x - 1)...(x - k + 1)$$
$$= \sum_{k=0}^{n} \frac{(-1)^k \binom{n+k}{n}\binom{N-k}{n-k}}{k!} \sum_{i=0}^{k} s(k, i)x^i, \quad n = 0, 1, ...N, \tag{7}$$

where $s(k, i)$ represents the Stirling numbers of the first kind [46, 47]. Therefore, the analytical form of $\mathcal{T}_{n,N}(x)$ can be derived as:

$$\mathcal{T}_{n,N}(x) = \sum_{k=0}^{n} a_{k,n} x^k, \quad n = 0, 1, ...N, \tag{8}$$

where

$$a_{k,n} = \sum_{r=k}^{n} \frac{(-1)^r \binom{n+r}{n}\binom{N-r}{n-r} s(r, i)}{r!}. \tag{9}$$

Shifted discrete Chebyshev polynomials:

In order to use the discrete Chebyshev polynomials on the interval $[0,1]$ we define the shifted discrete Chebyshev polynomials (SDCPs) by introducing change of variable $x = Nt$. Let nth shifted discrete Chebyshev polynomial, i.e. $\mathcal{T}_{n,N}(Nt)$, to be denoted by $\mathfrak{T}_{n,N}(t)$. Then, the set of SDChPs $\{\mathfrak{T}_{n,N}, n = 0,1,...,N\}$ are orthogonal on the interval $[0,1]$ with respect to the following discrete norm:

$$\langle f,g\rangle_* = \sum_{r=0}^{N} f\left(\frac{r}{N}\right) g\left(\frac{r}{N}\right). \tag{10}$$

Also, the orthogonality condition for SDChPs can be defined such as:

$$\langle \mathfrak{T}_{m,N}(t), \mathfrak{T}_{n,N}(t)\rangle_* = \sum_{r=0}^{N} \mathfrak{T}_{m,N}\left(\frac{r}{N}\right) \mathfrak{T}_{n,N}\left(\frac{r}{N}\right) = \mu_n \delta_{mn}, \tag{11}$$

where μ_n is defined in relation (6).

Function approximation:

Any function $f(t)$, defined over the interval $[0,1]$, may be expanded by the SDChPs as follows:

$$f(t) \simeq \sum_{i=0}^{N} c_i \mathfrak{T}_{i,N}(t) = C^T \Psi(t), \tag{12}$$

where C and $\Psi(x)$ are $(N+1)$ vectores given by

$$C^T = [c_0, c_1, ..., c_N]^T, \tag{13}$$

$$\Psi(t) = [\mathfrak{T}_{0,N}(t), \mathfrak{T}_{1,N}(t), ..., \mathfrak{T}_{N,N}(t)]^T, \tag{14}$$

and the coefficients c_i can be derived by means of the expression:

$$c_i = \frac{\langle f(t), \mathfrak{T}_{i,N}(t)\rangle_*}{\mu_i}, \quad i = 0,1,...N. \tag{15}$$

Convergence analysis:

The discrete Chebyshev polynomials $\mathcal{T}_{n,N}(x)$ are special case of the well-known Hahn polynomials $Q_n(x;\alpha,\beta,N)$ with $\alpha = \beta = 0$ [48]. The following Theorem provides conditions that ensure the series expansion of a function by the discrete Chebyshev polynomials converges.

Theorem 3.2. The series expansion $\sum_{k=0}^{n} \frac{\langle f,\mathcal{T}_{k,N}\rangle}{\langle \mathcal{T}_{k,N},\mathcal{T}_{k,N}\rangle}\mathcal{T}_{k,N}(x)$ of a function f by the discrete Chebyshev polynomials converges pointwise, if the series expansionof the function f by the Jacobi polynomials converges pointwise and if $\frac{n^4}{N} \to 0$ as $n, N \to \infty$.

Proof. The proof follows directly from Theorems 1.1, 2.1 and 2.2 in Ref. [48]. Q.E.D.

3.1 Operational matrices:

The explicit formulations for operational matrices of fractional integration, in the Riemann-Liouville sense, for the SDCPs are obtained in the follow-up. Moreover, the product operational matrix of the SDChPs vector are also derived.

Lemma 3.3. For a positive number r the inner product of the $\mathfrak{T}_{n,N}(t)$ and t^r, denoted by $\beta(n,r)$, can be derived as:

$$\beta\left(n,r\right)=\frac{1}{N^r}\sum_{j=0}^{N}\sum_{k=0}^{n}a_{j,n}j^{k+r}, \tag{16}$$

where $a_{j,n}$ is defined in relation (9).

Proof. By using definition of the discrete inner product $\langle .,. \rangle_*$ in (11), we get:

$$\beta\left(n,r\right)=\langle t^r,\mathfrak{T}_{n,N}(t)\rangle_*=\sum_{j=0}^{N}\mathfrak{T}_{n,N}\left(\frac{j}{N}\right)\frac{j^r}{N^r}=\frac{1}{N^r}\sum_{j=0}^{N}\mathcal{T}_{n,N}\left(j\right)j^r.$$

The analytical form of the discrete Chebyshev polynomial $\mathcal{T}_{n,N}$ in relation (8) leads to (16). Q.E.D.

Theorem 3.4. Let $\Psi(t)$ be the SDCPs vector as defined in (14). Then, its fractional integral of order ν in the Riemann-Liouville sense is given by

$$I^{\nu}\Psi(t)=\mathcal{P}^{(\nu)}\Psi(t), \tag{17}$$

where $\mathcal{P}^{(\nu)}$ is called operational matrix of fractional integration and

$$\mathcal{P}^{\nu}_{i,j}=\sum_{k=0}^{i}\frac{a_{k,N}N^k\beta(k+\nu,j)\Gamma(k+1)}{\mu_j\Gamma(k+\nu+1)},\qquad i,j=1,2,...,N+1. \tag{18}$$

Proof. Consider the ith element of the SDCPs vector $\Psi(t)$, that is $\mathfrak{T}_{i-1,N}(t)$. The fractional integration of order ν for $\mathfrak{T}_{i-1,N}(t)$ with using of relation (3), may be written as

$$I^{\nu}\mathfrak{T}_{i-1,N}(t)=I^{\nu}\sum_{k=0}^{i-1}N^k a_{k,i-1}t^k=\sum_{k=0}^{i-1}N^k a_{k,i-1}\left(I^{\nu}t^k\right)=\sum_{k=0}^{i-1}\frac{\Gamma(k+1)N^k a_{k,i-1}}{\Gamma(\nu+k+1)}t^{k+\nu}. \tag{19}$$

Expanding the term $t^{k+\nu}$ we get

$$t^{k+\nu}\simeq\sum_{j=0}^{N}b_{k,j}\mathfrak{T}_{j,N}(t), \tag{20}$$

and $b_{k,j}$ can be derived as

$$b_{k,j}=\frac{\left\langle x^{k+\nu},\mathfrak{T}_{j,N}(t)\right\rangle_*}{\langle\mathfrak{T}_{j,N}(t),\mathfrak{T}_{j,N}(t)\rangle_*}=\frac{\beta(k+\nu,j)}{\mu_j}. \tag{21}$$

Substituting Eqs. (21) and (20) in (19), we have:

$$I^{\nu}\mathfrak{T}_{i-1,N}(t)=\sum_{j=0}^{N}\left(\sum_{k=0}^{i-1}\frac{\beta(k+\nu,j)\Gamma(k+1)N^{k}a_{k,i-1}}{\mu_j\Gamma(\nu+k+1)}\right)\mathfrak{T}_{j,N}(t). \tag{22}$$

Accordingly, the desired result (17) is derived. Q.E.D.

Theorem 3.5. Let $\Psi(t)$ be the SDCPs vector defined in (14) and V be an arbitrary $(N+1)$ vector. Then, we can write

$$\Psi(t)\Psi^{T}(t)V=\tilde{V}\Psi(t), \tag{23}$$

where $\tilde{V}$ is the $(N+1)\times(N+1)$ product operational matrix and

$$\tilde{V}_{i+1,j+1}=\frac{1}{\mu_j}\sum_{k=0}^{N}V_k\left\langle\mathfrak{T}_{k,N}(t)\mathfrak{T}_{i,N}(t),\mathfrak{T}_{j,N}(t)\right\rangle_*,\quad i,j=0,1,...,N.$$

Proof. The product of two SDChPs vectors $\Psi(t)$ and $\Psi^{T}(t)$ is a $(N+1)\times(N+1)$ matrix as follow

$$\Psi(t)\Psi^{T}(t)=\begin{bmatrix}\mathfrak{T}_{0,N}(t)\mathfrak{T}_{0,N}(t) & \mathfrak{T}_{0,N}(t)\mathfrak{T}_{1,N}(t) & \dots & \mathfrak{T}_{0,N}(t)\mathfrak{T}_{N,N}(t)\\ \mathfrak{T}_{1,N}(t)\mathfrak{T}_{0,N}(t) & \mathfrak{T}_{1,N}(t)\mathfrak{T}_{1,N}(t) & \dots & \mathfrak{T}_{1,N}(t)\mathfrak{T}_{N,N}(t)\\ \vdots & \vdots & \ddots & \vdots\\ \mathfrak{T}_{N,N}(t)\mathfrak{T}_{0,N}(t) & \mathfrak{T}_{N,N}(t)\mathfrak{T}_{1,N}(t) & \dots & \mathfrak{T}_{N,N}(t)\mathfrak{T}_{N,N}(t)\end{bmatrix}.$$

As a result, relation (23) can be rewritten as:

$$\sum_{k=0}^{N}\mathfrak{T}_{k,N}(t)\mathfrak{T}_{i,N}(t)V_{k+1}=\sum_{k=0}^{N}\mathfrak{T}_{j,N}(t)\tilde{V}_{i+1,k+1},\quad i=0,1,...,N.$$

Multiplying both sides of the above relation by $\mathfrak{T}_{j,N}(t)$ and using the defined inner product in (10), we get:

$$\sum_{k=0}^{N}\left\langle\mathfrak{T}_{k,N}(t)\mathfrak{T}_{i,N}(t),\mathfrak{T}_{j,N}(t)\right\rangle_* V_{k+1}=\left\langle\mathfrak{T}_{j,N}(t),\mathfrak{T}_{j,N}(t)\right\rangle_*\tilde{V}_{i+1,j+1},\quad i,j=0,1,...,N.$$

Therefore, the matrix $\tilde{V}$ is given by

$$\tilde{V}_{i+1,j+1}=\frac{\sum\limits_{k=0}^{N}V_k\left\langle\mathfrak{T}_{k,N}(t)\mathfrak{T}_{i,N}(t),\mathfrak{T}_{j,N}(t)\right\rangle_*}{\left\langle\mathfrak{T}_{j,N}(t),\mathfrak{T}_{j,N}(t)\right\rangle_*}=\frac{1}{\mu_j}\sum_{k=0}^{N}V_k\left\langle\mathfrak{T}_{k,N}(t)\mathfrak{T}_{i,N}(t),\mathfrak{T}_{j,N}(t)\right\rangle_*.$$

Q.E.D.

4 The numerical method

Consider the coupled systems of NFIDE(1) subject to the initial conditions (2). First, we approximate $D^{\nu}(t)$ and $D^{\mu}(t)$ in terms of the SDCPs as follows First, we approximate all functions involved in the problem as follows:

$$D^{\nu}u(t) \simeq U^T\Psi(t), \tag{24}$$

$$D^{\mu}v(t) \simeq V^T\Psi(t), \tag{25}$$

where $\Psi(t)$ is the SDCPs vector as defined in relations (14). Moreover, U and V are unknown vectors to be determined. With the aid of fractional Riemann-Liouville operators I^{ν} and I^{μ} we can write:

$$u(t) \simeq I^{\nu}U^T\Psi(t) + u_0 = U^T\mathcal{P}^{(\nu)}\Psi(t) + d^T\Psi(t), \tag{26}$$

and

$$v(t) \simeq I^{\mu}V^T\Psi(t) + v_0 = V^T\mathcal{P}^{(\mu)}\Psi(t) + q^T\Psi(t), \tag{27}$$

where d and q denote the coefficient vectors of the constant values u_0 and v_0 respectively and $\mathcal{P}^{(\nu)}$ and $\mathcal{P}^{(\mu)}$ are the fractional operational matrix of SDChPs vector derived in (17).
Substituting (24)-(27) in Eq. (1) we get:

$$\begin{cases} D^{\nu}u(t) = F_1(t, U^T\mathcal{P}^{(\nu)}\Psi(t) + d^T\Psi(t), V^T\mathcal{P}^{(\mu)}\Psi(t) + q^T\Psi(t)) + \int_0^t F_2(\eta, U^T\mathcal{P}^{(\nu)}\Psi(\eta) + d^T\Psi(\eta), \\ V^T\mathcal{P}^{(\mu)}\Psi(\eta) + q^T\Psi(\eta))d\eta, \\ D^{\mu}v(t) = G_1(t, U^T\mathcal{P}^{(\nu)}\Psi(t) + d^T\Psi(t), V^T\mathcal{P}^{(\mu)}\Psi(t) + q^T\Psi(t)) + \int_0^t G_2(\eta, U^T\mathcal{P}^{(\nu)}\Psi(\eta) + d^T\Psi(\eta), \\ V^T\mathcal{P}^{(\mu)}\Psi(\eta) + q^T\Psi(\eta))d\eta, \end{cases}$$

Residual functions $R_1(t)$ and $R_2(t)$ for the NFIDEs(1) can be derived as

$$\begin{cases} R_1(t) = U^T\Psi(t) - F_1(t, U^T\mathcal{P}^{(\nu)}\Psi(t) + d^T\Psi(t), V^T\mathcal{P}^{(\mu)}\Psi(t) + q^T\Psi(t)) - \int_0^t F_2(\eta, U^T\mathcal{P}^{(\nu)}\Psi(\eta) \\ +d^T\Psi(\eta), V^T\mathcal{P}^{(\mu)}\Psi(\eta) + q^T\Psi(\eta))d\eta, \\ R_2(t) = V^T\Psi(t) - G_1(t, U^T\mathcal{P}^{(\nu)}\Psi(t) + d^T\Psi(t), V^T\mathcal{P}^{(\mu)}\Psi(t) + q^T\Psi(t)) - \int_0^t G_2(\eta, U^T\mathcal{P}^{(\nu)}\Psi(\eta) \\ +d^T\Psi(\eta), V^T\mathcal{P}^{(\mu)}\Psi(\eta) + q^T\Psi(\eta))d\eta, \end{cases} \tag{28}$$

In order to approximate solution of the NFIDE (1), we collocate the residual system (28) at the $2N+2$ zeros of shifted Chebyshev polynomials in the interval $[0,1]$, as follow:

$$R(r_j) = 0, \quad j = 1, 2..., 2N+2. \tag{29}$$

Eqs. (29) generate a nonlinear system of $2N+2$ algebraic equations unknown elements of the unknown vectors U and V. This nonlinear system can be solved for unknown coefficient vectors U and V by means of the Newton-Raphson method. Consequently, unknown functions $u(t)$ and $v(t)$ can be obtained by substituting the vectors U and V in Eqs. (26) and (27).

5 Computational results and comparisons

In order to verify the efficiency of the SDCPs method, we consider three examples in the follow-up. The root mean squared error (RMSE) is used to examine accuraccy of the numerical solution. Let $\tilde{g}(t)$ be an approximation for the function $g(t)$. The RMSE for this approximate function $\tilde{g}(t)$ can be estimated as:

$$\|g-\tilde{g}\|_2 = \sqrt{\frac{\sum_{i=1}^{m} |g(t_i)-\tilde{g}(t_i)|^2}{m}}.$$

The algorithms are performed using Maple 17 with 30 digits precision.

Example 5.1. Let us consider the coupled system of NFIDE

$$\begin{cases} D^{\nu}u(t) = \frac{1}{3}u(t)v(t) + \frac{1}{2}v^2(t) + 2v(t) - \int_0^t [u(\eta)+v(\eta)]d\eta, & 0<\nu\leq 1, \\ D^{\mu}v(t) = \frac{1}{3}u(t)v(t) - u(t) + 1 - \int_0^t [u(\eta)-2v(\eta)]d\eta, & 0<\mu\leq 1. \end{cases}$$

subjected to FVIDE

$$\begin{cases} u(0)=0, \\ v(0)=0, \end{cases}, \qquad t,\eta\in[0,1].$$

The exact solutions of this coupled system of NFIDE are $u(t)=t^2$ and $v(t)=t$ for $\nu,\mu=1$. The SDCP method is applied to solve this problem for various values of N and ν,μ. Fig. 1 shows the approximate solutions for $N=10$ and different choices of $\nu=\mu$. The approximate solutions for $N=10$ and different choices of μ and constant value of $\nu=0.8$ are shown in Fig. 2. The charts show that it was misleading this way when $\nu,\mu=1$, the numerical solution is very close to the exact one. The required CPU time (in second) for different values of ν,μ and N are listed in Table1. To confirm accuracy of the proposed methods, for $\nu,\mu=1$ and various values of N, Table2 provides the RMSE of the approximate solutions. For visual results obtained by presented method when $N=6$, and BWM, HWM and LWM methods [35] when $k=6, M=2$, where $\mu=\nu=1$, the reader is referrede to Table 3. From this results in these Tables, it is easy to conclude that the SDCPs method is an efficient for solving this coupled system of NFIDE and that the RMSE of the approximate solutions decreases signicantly as the number of basis functions increases.

Table 1: Comparison of the required CPU time for N and different values of ν,μ (Example 5.1).

	The elapsed CPU time of SDCPs method(in second)				
	$\nu=\mu=0.75$	$\nu=\mu=0.85$	$\nu=\mu=0.95$	$\nu=\mu=0.99$	$\nu=\mu=1.0$
$N=6$	2.047	1.719	1.765	1.719	1.141
$N=10$	3.172	2.734	2.953	2.735	1.703
$N=16$	6.250	5.969	6.125	6.594	3.702

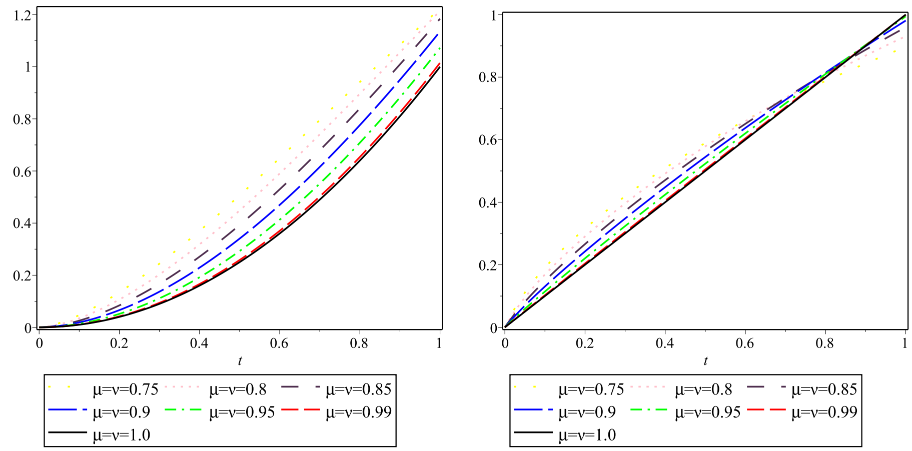

Figure 1: The approximate solutions for $N = 10$ and different values of ν, μ and (Example 5.1).

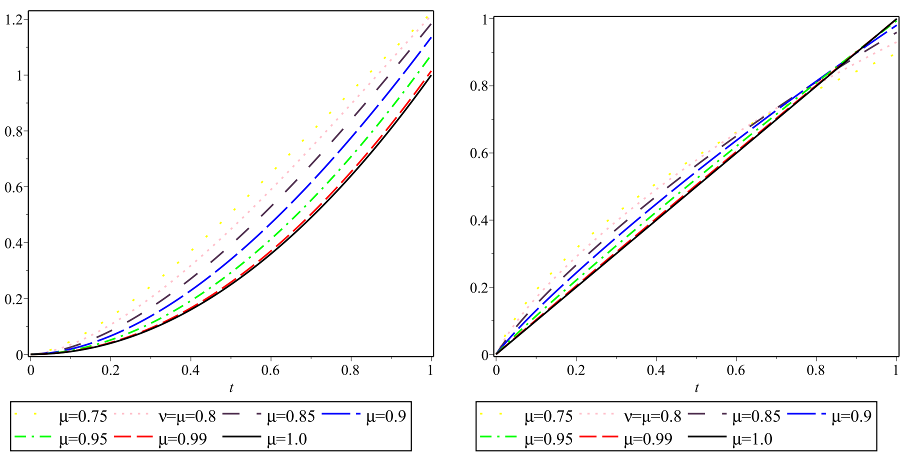

Figure 2: The approximate solutions for $N = 10$ and different values of ν and (Example 5.1).

Table 2: The RMSE of the approximate solutions (Example 5.1).

	$N=4$	$N=6$	$N=8$	$N=10$	$N=12$	$N=14$	$N=16$
$\|u-\tilde{u}\|_2$	1.84×10^{-30}	1.05×10^{-30}	1.85×10^{-30}	1.02×10^{-30}	1.94×10^{-30}	2.67×10^{-30}	2.05×10^{-30}
$\|v-\tilde{v}\|_2$	2.89×10^{-30}	3.17×10^{-30}	8.05×10^{-30}	1.88×10^{-30}	2.58×10^{-31}	2.74×10^{-31}	6.15×10^{-31}

Example 5.2. Let us consider the coupled system of NFIDE

$$\begin{cases} D^{\nu}u(t) = u^2(t) - \frac{3}{2}u(t) - v(t) + 1 + \int_0^t [u(\eta) + v(\eta)]d\eta, & 0 < \nu \leq 1, \\ D^{\mu}v(t) = u^2(t) - 2u(t) + +v(t) + 2 + \int_0^t u(\eta)d\eta, & 0 < \mu \leq 1. \end{cases}$$

Table 3: Comparison of the obtained absolute error for different values of $\nu = \mu = 1$ and $N = 6$ (Example 5.1).

	e_u				e_v			
	HWM [35]	LWM [35]	BWM [35]	SDCPs	HWM [35]	LWM [35]	BWM [35]	SDCPs
$t = 0.1$	$3.60E-04$	$3.75E-04$	$1.22E-04$	$4.91E-31$	$0.92E-04$	$1.07E-06$	$3.04E-09$	$5.30E-31$
$t = 0.2$	$3.23E-04$	$4.07E-04$	$3.24E-05$	$1.05E-30$	$0.46E-04$	$5.33E-06$	$4.31E-06$	$1.04E-30$
$t = 0.3$	$2.72E-04$	$9.16E-04$	$1.07E-04$	$1.29E-30$	$3.75E-04$	$1.35E-05$	$9.06E-06$	$1.59E-30$
$t = 0.4$	$1.40E-04$	$2.82E-05$	$1.03E-05$	$1.98E-30$	$8.17E-04$	$1.02E-05$	$1.41E-05$	$2.12E-30$
$t = 0.5$	$3.58E-04$	$7.47E-05$	$1.85E-05$	$2.59E-30$	$3.29E-04$	$3.66E-05$	$1.95E-05$	$2.58E-30$
$t = 0.6$	$6.02E-04$	$8.97E-06$	$2.48E-06$	$3.10E-30$	$8.17E-04$	$2.48E-05$	$3.06E-05$	$3.03E-30$
$t = 0.7$	$1.32E-04$	$1.01E-05$	$7.08E-06$	$3.38E-30$	$1.34E-04$	$1.45E-04$	$3.62E-05$	$3.39E-30$
$t = 0.8$	$0.42E-04$	$1.03E-05$	$6.01E-06$	$3.04E-30$	$0.95E-04$	$1.52E-04$	$4.17E-05$	$3.63E-30$
$t = 0.9$	$0.71E-04$	$8.44E-06$	$2.94E-06$	$3.46E-30$	$4.86E-04$	$3.50E-05$	$4.70E-05$	$4.01E-30$
$t = 1.0$	$5.48E-04$	$3.01E-05$	$1.97E-04$	$4.57E-30$	$3.32E-04$	$1.12E-04$	$5.20E-05$	$4.40E-30$

subjected to FVIDE

$$\begin{cases} u(0) = 1, \\ v(0) = 0, \end{cases} \qquad t, \eta \in [0, 1].$$

The exact solutions of this NFIDE are $u(t) = e^{\frac{t}{2}}$ and $v(t) = te^t$ when $\nu, \mu = 1$. The SDCP method is applied to solve this problem for various values of N and ν, μ. Fig. 3 shows the approximate solutions for $N = 10$ and different choices of $\nu = \mu$. The approximate solutions for $N = 10$ and different choices of μ and constant value of $\nu = 0.8$ are shown in Fig. 4. The charts show that it was misleading this way when $\nu, \mu = 1$, the numerical solution is very close to the exact one. The required CPU time (in second) for different values of ν, μ and N are listed in Table4. To confirm accuracy of the proposed methods, for $\nu, \mu = 1$ and various values of N, Table5 provides the RMSE of the approximate solutions. For visual results obtained by presented method when $N = 6$ and BWM method [35] when $k = 6, M = 4$, where $\mu = \nu = 1$, the reader is referrede to Table 6. From this results in these Tables, it is easy to conclude that the SDCPs method is an efficient for solving this coupled system of NFIDE and that the RMSE of the approximate solutions decreases signicantly as the number of basis functions increases.

Example 5.3. Let us consider the coupled system of NFIDE

$$\begin{cases} D^\nu u(t) = u^2(t) + v^2(t) - \int_0^t u(\eta) d\eta, & 0 < \nu \leq 1, \\ D^\mu v(t) = \frac{-1}{2} v^2(t) - u(t) + \frac{1}{2} + \int_0^t u(\eta) v(\eta) d\eta, & 0 < \mu \leq 1. \end{cases}$$

subjected to FVIDE

$$\begin{cases} u(0) = 0, \\ v(0) = 0, \end{cases} \qquad t, \eta \in [0, 1].$$

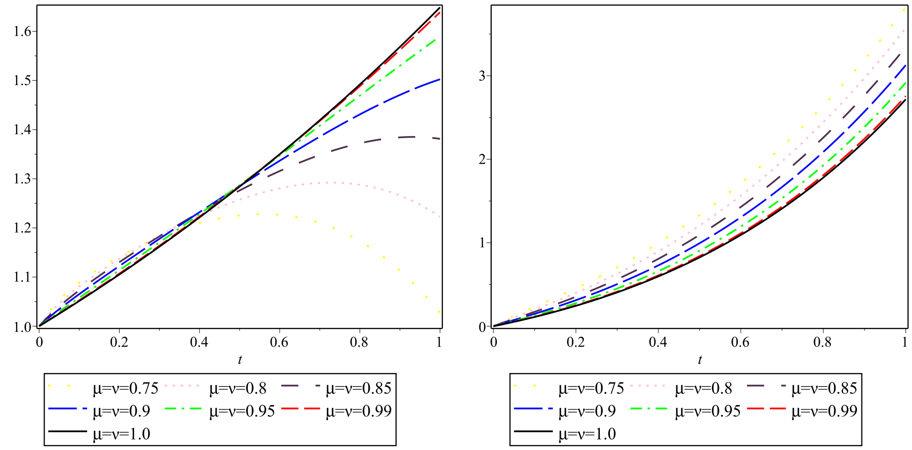

Figure 3: The obtained approximate solutions for $N = 10$ and different values of ν, μ (Example 5.2).

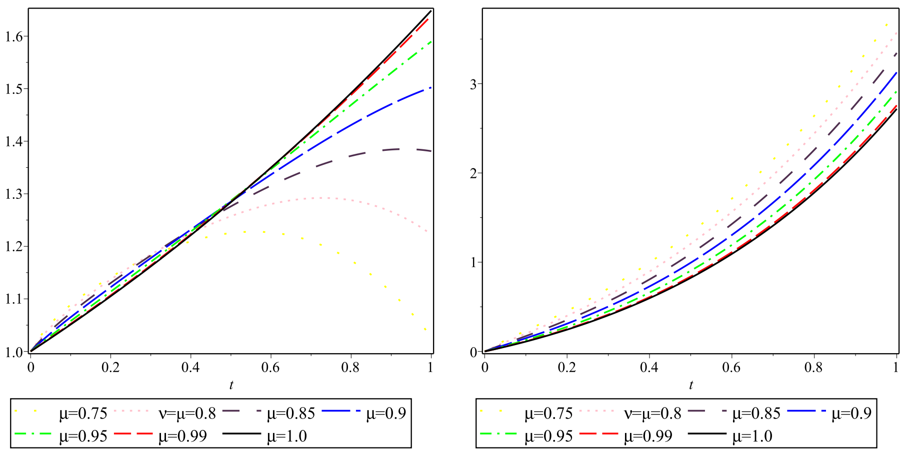

Figure 4: The obtained approximate solutions for $N = 10$ and different values of ν (Example 5.2).

Table 4: Comparison of the required CPU time for N and different values of ν, μ (Example 5.2).

	The elapsed CPU time of SDCPs method(in second)				
	$\nu=\mu=0.75$	$\nu=\mu=0.85$	$\nu=\mu=0.95$	$\nu=\mu=0.99$	$\nu=\mu=1.0$
$N=6$	1.829	1.750	1.703	1.672	1.281
$N=10$	3.391	2.688	2.469	2.610	2.016
$N=16$	5.703	5.313	5.250	5.390	3.110

Table 5: The RMSE of the approximate solutions (Example 5.2).

	$N=4$	$N=6$	$N=8$	$N=10$	$N=12$	$N=14$	$N=16$
$\|u-\tilde{u}\|_2$	5.35×10^{-05}	1.06×10^{-07}	1.38×10^{-10}	1.27×10^{-13}	8.57×10^{-17}	4.46×10^{-20}	1.85×10^{-23}
$\|v-\tilde{v}\|_2$	1.54×10^{-04}	2.54×10^{-07}	1.93×10^{-10}	7.58×10^{-14}	6.79×10^{-17}	3.62×10^{-20}	1.25×10^{-23}

Table 6: Comparison of the obtained absolute error for different values of $\nu=\mu$ and N (Example 5.2).

	e_u		e_v	
	BWM [35]	SDCPs	BWM [35]	SDCPs
$t=0.1$	$3.32E-05$	$5.05E-8$	$3.04E-04$	$6.50E-7$
$t=0.2$	$3.60E-05$	$9.14E-8$	$3.75E-04$	$6.51E-8$
$t=0.3$	$3.90E-05$	$9.61E-8$	$4.59E-04$	$1.40E-8$
$t=0.4$	$4.22E-05$	$8.46E-8$	$5.58E-04$	$1.92E-7$
$t=0.5$	$1.51E-05$	$8.43E-8$	$2.50E-04$	$9.66E-8$
$t=0.6$	$4.94E-05$	$1.08E-7$	$8.12E-04$	$2.13E-8$
$t=0.7$	$5.34E-05$	$1.46E-7$	$9.72E-04$	$1.63E-7$
$t=0.8$	$5.77E-05$	$1.73E-7$	$1.15E-03$	$2.34E-7$
$t=0.9$	$6.22E-05$	$1.67E-7$	$1.37E-03$	$5.32E-7$
$t=1.0$	$6.71E-05$	$1.47E-7$	$1.62E-03$	$5.30E-8$

For $\nu,\mu=1$, the exact solutions of this problem are $u(t)=\sin(t)$ and $v(t)=\cos(t)$. The SDCP method is applied to solve this problem for various values of N and ν,μ. Fig. 5 shows the approximate solutions for $N=10$ and different choices of $\nu=\mu$. The approximate solutions for $N=10$ and different choices of μ and constant value of $\nu=0.8$ are shown in Fig. 6. The charts show that it was misleading this way when $\nu,\mu=1$, the numerical solution is very close to the exact one. The required CPU time (in second) for different values of ν,μ and N are listed in Table7. To confirm accuracy of the proposed methods, for $\nu,\mu=1$ and various values of N, Table8 provides the RMSE of the approximate solutions. For visual results obtained by presented method when $N=6$ and BWM method [35] with $k=6, M=1$, where $\mu=\nu=1$, the reader is referrede to Table 9. From this results in these Tables, it is easy to conclude that the SDCPs method is an efficient for solving this coupled system of NFIDE and that the RMSE of the approximate solutions decreases signicantly as the number of basis functions increases.

6 Concluding remarks

A comparative study has been carried out to illustrate the efficiency and supriority of the discrete Chebyshev polynomials in solving coupled systems of nonlinear fractional order integro-differential equations. First, the discrete Chebyshev polynomials and their properties are introduced. Hence, the fractional derivative of the unknown functions in the consideration problem is approximated by orthogonal polynomials with unknown coefficients. The operational matrix of fractional integration is used to transform the under consideration coupled systems of NFIDEs into an algabric system.

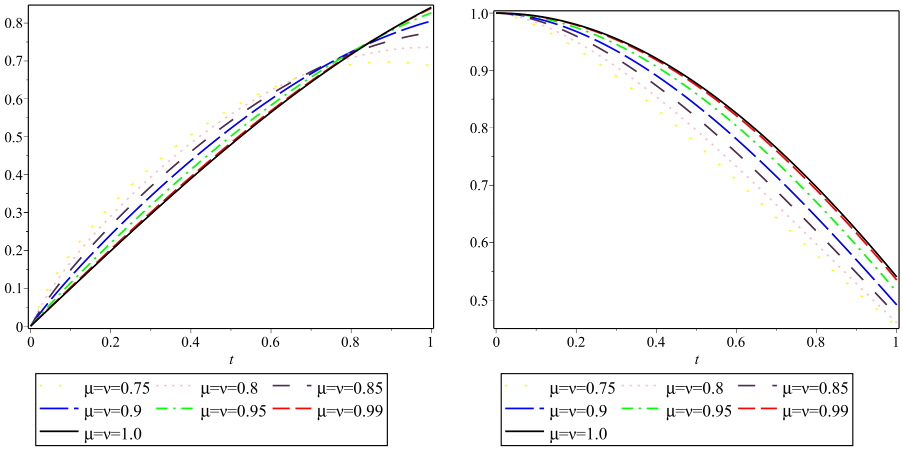

Figure 5: The obtained approximate solutions for $N = 10$ and different values of μ (Example 5.3).

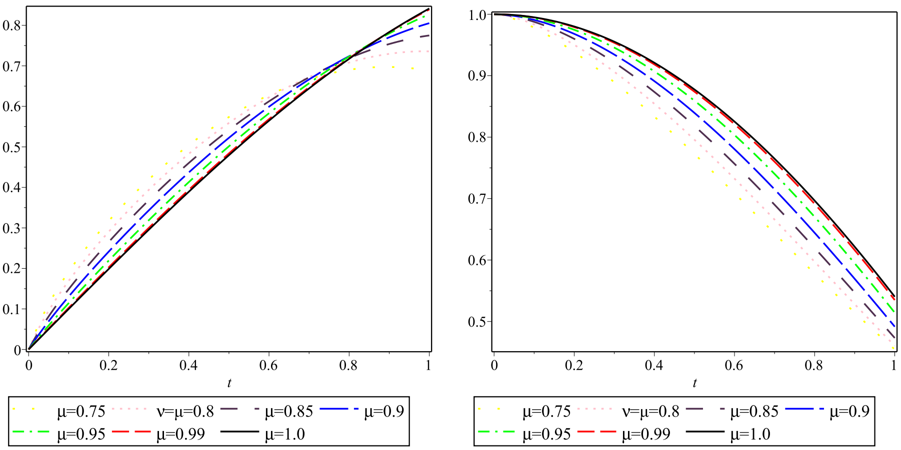

Figure 6: The obtained approximate solutions for $N = 10$ and different values of ν, μ (Example 5.3).

Table 7: Comparison of the required CPU time for N and different values of $\nu = \mu$ (Example 5.3).

	The elapsed CPU time of SDCPs method(in second)				
	$\nu = \mu = 0.75$	$\nu = \mu = 0.85$	$\nu = \mu = 0.95$	$\nu = \mu = 0.99$	$\nu = \mu = 1.0$
$N = 6$	2.156	2.094	2.109	2.094	1.4468
$N = 10$	5.235	4.797	4.953	4.844	2.735
$N = 16$	25.469	25.516	25.359	25.422	10.688

Table 8: The RMSE of the approximate solutions (Example 5.3).

	$N=4$	$N=6$	$N=8$	$N=10$	$N=12$	$N=14$	$N=16$
$\|u-\tilde{u}\|_2$	1.74×10^{-05}	2.07×10^{-08}	1.41×10^{-11}	7.33×10^{-15}	4.43×10^{-18}	2.01×10^{-21}	7.07×10^{-25}
$\|v-\tilde{v}\|_2$	8.75×10^{-06}	1.05×10^{-08}	6.50×10^{-12}	2.28×10^{-15}	1.40×10^{-18}	6.28×10^{-22}	2.06×10^{-25}

Table 9: Comparison of the obtained absolute error for different values of $\nu=\mu=1$ and N (Example 5.3).

	e_u		e_v	
	BWM [35]	SDCPs	BWM [35]	SDCPs
$t=0.03125$	$4.51E-05$	$4.90E-08$	$4.77E-04$	$2.82E-08$
$t=0.15625$	$2.20E-04$	$1.51E-09$	$4.19E-04$	$5.33E-09$
$t=0.28125$	$3.68E-04$	$1.65E-08$	$3.38E-04$	$4.69E-09$
$t=0.40625$	$5.41E-04$	$1.12E-09$	$2.33E-04$	$4.93E-09$
$t=0.53125$	$6.87E-04$	$1.45E-08$	$1.05E-04$	$1.74E-09$
$t=0.65625$	$8.24E-04$	$1.22E-08$	$4.59E-05$	$3.10E-09$
$t=0.78125$	$9.55E-04$	$2.18E-09$	$2.18E-04$	$4.79E-09$
$t=0.90625$	$1.14E-03$	$5.91E-08$	$4.11E-04$	$2.82E-08$

Finally, a comparison has been made between the obtained results by the proposed SDCPs method and previous published methods.

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