Some new inequalities involving the Katugampola fractional integrals for strongly η-convex functions

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Abstract

We introduced several new integral inequalities of the Hermite–Hadamard type for strongly η-convex functions via the Katugampola fractional integrals. Some results in the literature are particular cases of our results.

2010 Mathematics Subject Classification. **26D15**. 26A33, 26A51, 26D10

Keywords. Hermite–Hadamard type inequality, strongly η-convex functions, Hölder's inequality, Katugampola fractional integrals, Riemann–Liouville fractional integrals, Hadamard fractional integrals.

1 Introduction

Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be convex on I if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. The following inequalities which hold for convex functions is known in the literature as the Hermite–Hadamard type inequality.

Theorem 1.1 ([12]). If $f : [a,b] \to \mathbb{R}$ is convex on $[a,b]$ with $a < b$, then

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

Many authors have studied and generalized the Hermite–Hadamard inequality in several ways via different classes of convex functions. For some recent results related to the Hermite–Hadamard inequality, we refer the interested reader to the papers [1, 4–8, 15–17, 22].

In 2016, Gordji et al. [10] introduced the concept of η-convexity as follows:

Definition 1.2 ([10]). A function $f : I \to \mathbb{R}$ is said to be η-convex with respect to the bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if

$$f(tx + (1-t)y) \le f(y) + t\eta(f(x), f(y))$$

for all $x, y \in I$ and $t \in [0,1]$.

Remark 1.3. If we take $\eta(x,y) = x - y$ in Definition 1.2, then we recover the classical definition of convex functions.

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Tbilisi Mathematical Journal 12(1) (2019), pp. 117–130.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 28 November 2018.
Accepted for publication: 20 January 2019.

The following Hermite–Hadamard type inequality holds for η-convex functions.

Theorem 1.4 ([10]). Suppose that $f : I \to \mathbb{R}$ is an η-convex function such that η is bounded from above on $f(I) \times f(I)$. Then for any $a, b \in I$ with $a < b$,

$$2f\left(\frac{a+b}{2}\right) - M_\eta \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(b) + \frac{\eta\Big(f(a), f(b)\Big)}{2},$$

where M_η is an upper bound of η on $f([a,b]) \times f([a,b])$.

In 2017, Awan et al. [2] extended the class of η-convex functions to the class of strongly η-convex functions as follows:

Definition 1.5 ([2]). A function $f : I \to \mathbb{R}$ is said to be strongly η-convex with respect to the bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with modulus $\mu \geq 0$ if

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)) - \mu t(1-t)(x-y)^2$$

for all $x, y \in I$ and $t \in [0,1]$.

Remark 1.6. If $\eta(x, y) = x - y$ in Definition 1.5, then we have the class of strongly convex functions.

The authors in [2] obtained the following refinement of the Hermite–Hadamard inequality for strongly η-convex functions.

Theorem 1.7. Let $f : [a,b] \to \mathbb{R}$ is an η-convex function with modulus $\mu \geq 0$. If η is bounded from above on $f([a,b]) \times f([a,b])$, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} + \frac{\mu}{12}(b-a)^2 &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{f(a)+f(b)}{2} + \frac{\eta\Big(f(a), f(b)\Big) + \eta\Big(f(b), f(a)\Big)}{4} \\ &\quad - \frac{\mu}{6}(b-a)^2 \\ &\leq \frac{f(a)+f(b)}{2} + \frac{M_\eta}{2} - \frac{\mu}{6}(b-a)^2, \end{aligned}$$

where M_η is an upper bound of η on $f([a,b]) \times f([a,b])$.

For some recent results related to the class of η-convex functions, we refer the interested reader to the papers [2, 9–11, 15, 18].

Definition 1.8 ([19]). The left- and right-sided Riemann–Liouville fractional integrals of order $\alpha > 0$ of f are defined by

$$J_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt$$

and

$$J_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt$$

with $a < x < b$ and $\Gamma(\cdot)$ is the gamma function given by

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt, \qquad Re(x) > 0$$

with the property that $\Gamma(x+1) = x\Gamma(x)$.

Definition 1.9 ([21]). The left- and right-sided Hadamard fractional integrals of order $\alpha > 0$ of f are defined by

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt.$$

In what follows, $X_c^p(a,b)$ $(c \in \mathbb{R},\ 1 \le p \le \infty)$ denotes the set of all complex-valued Lebesgue measurable functions f for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p} \qquad (1 \le p < \infty)$$

and for $p = \infty$

$$\|f\|_{X_c^{\infty}} = \operatorname{ess} \sup_{a \le t \le b} |t^c f(t)|.$$

In 2011, Katugampola [13] introduced a new fractional integral operator which generalizes the Riemann–Liouville and Hadamard fractional integrals as follows:

Definition 1.10. Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-sided Katugampola fractional integrals of order $\alpha > 0$ of $f \in X_c^p(a,b)$ are defined by

$${}^{\rho}I_{a+}^{\alpha} f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} f(t)dt$$

and

$${}^{\rho}I_{b-}^{\alpha} f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^{\rho} - x^{\rho})^{1-\alpha}} f(t)dt$$

with $a < x < b$ and $\rho > 0$, if the integrals exist.

Remark 1.11. It is shown in [13] that the Katugampola fractional integral operators are well-defined on $X_c^p(a,b)$.

Theorem 1.12 ([13]). Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$

1. $\lim\limits_{\rho\to 1} {}^\rho I_{a+}^\alpha f(x) = J_{a+}^\alpha f(x)$,
2. $\lim\limits_{\rho\to 0^+} {}^\rho I_{a+}^\alpha f(x) = H_{a+}^\alpha f(x)$.

Similar results also hold for right-sided operators.

For more information about the Katugampola fractional integral and related results, we refer the interested reader to the papers [3, 13, 14].

Recently, Chen and Katugampola [3] proved the following Hermite–Hadamard type inequalities for convex functions via the Katugampola fractional integrals.

Theorem 1.13. Let $\alpha, \rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in X_c^P(a^\rho, b^\rho)$. If f is a convex on $[a^\rho, b^\rho]$, then the following inequalities hold:

$$f\Big(\frac{a^\rho + b^\rho}{2}\Big) \le \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\Big[{}^\rho I_{a+}^\alpha f(b^\rho) + {}^\rho I_{b-}^\alpha f(a^\rho)\Big] \le \frac{f(a^\rho) + f(b^\rho)}{2},$$

where the fractional integrals are considered for the function $f(x^\rho)$ evaluated at a and b, respectively.

Theorem 1.14. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|f'|$ is convex on $[a^\rho, b^\rho]$, then the following inequality holds:

$$\left|\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\Big[{}^\rho I_{a+}^\alpha f(b^\rho) + {}^\rho I_{b-}^\alpha f(a^\rho)\Big]\right| \le \frac{b^\rho - a^\rho}{2(\alpha+1)}\Big[|f'(a^\rho)| + |f'(b^\rho)|\Big].$$

Theorem 1.15. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|f'|$ is convex on $[a^\rho, b^\rho]$, then the following inequality holds:

$$\left|\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\Big[{}^\rho I_{a+}^\alpha f(b^\rho) + {}^\rho I_{b-}^\alpha f(a^\rho)\Big]\right| \le \frac{b^\rho - a^\rho}{2\rho(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)\Big[|f'(a^\rho)| + |f'(b^\rho)|\Big].$$

Motivated by the above results, the goal of this paper is to introduce some new Hermite–Hadamard type inequalities for strongly η-convex functions via the Katugampola fractional integrals. The results in Theorems 1.7, 1.14 and 1.15 are particular cases of some of our results.

2 Main results

Our first result is an extension of Theorem 1.13 to the class of strongly η-convex functions.

Theorem 2.1. Let $\alpha, \rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in X_c^P(a^\rho, b^\rho)$. If f is strongly η-convex with modulus $\mu \ge 0$ on $[a^\rho, b^\rho]$ and η is bounded from above on $f([a^\rho, b^\rho]) \times f([a^\rho, b^\rho])$, then the following inequalities hold:

$$f\Big(\frac{a^\rho + b^\rho}{2}\Big) - \frac{M_\eta}{2} + \frac{\mu(\alpha^2 - \alpha + 2)}{4(\alpha+1)(\alpha+2)}(b^\rho - a^\rho)^2 \le \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\Big[{}^\rho I_{a+}^\alpha f(b^\rho) + {}^\rho I_{b-}^\alpha f(a^\rho)\Big]$$

$$\le \frac{f(a^\rho)+f(b^\rho)}{2} + \frac{\alpha\Big(\eta\Big(f(b^\rho), f(a^\rho)\Big) + \eta\Big(f(a^\rho), f(b^\rho)\Big)\Big)}{2(\alpha+1)} - \frac{\mu\alpha(b^\rho - a^\rho)^2}{(\alpha+1)(\alpha+2)}$$
$$\le \frac{f(a^\rho)+f(b^\rho)}{2} + \frac{\alpha M_\eta}{\alpha+1} - \frac{\mu\alpha(b^\rho - a^\rho)^2}{(\alpha+1)(\alpha+2)}, \tag{1}$$

where M_η is an upper bound of η on $f([a^\rho, b^\rho]) \times f([a^\rho, b^\rho])$.

Proof. We start by considering the following computation which follows directly by using change of variables and the definition of the Katugampola fractional integrals.

$$\int_0^1 t^{\alpha\rho-1} f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt + \int_0^1 t^{\alpha\rho-1} f(t^\rho b^\rho + (1-t^\rho)a^\rho)dt$$
$$= \frac{\rho^{\alpha-1}\Gamma(\alpha)}{(b^\rho - a^\rho)^\alpha}\Big[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\Big]. \tag{2}$$

Since f is strongly η-convex with modulus $\mu \ge 0$ and η is bounded from above on $f\Big([a^\rho, b^\rho]\Big) \times f\Big([a^\rho, b^\rho]\Big)$ by M_η, we have, for any $x, y \in [a^\rho, b^\rho]$

$$f\Big(\frac{x+y}{2}\Big) \le f(x) + \frac{1}{2}\eta\Big(f(y), f(x)\Big) - \frac{\mu}{4}(x-y)^2$$
$$\le f(x) + \frac{M_\eta}{2} - \frac{\mu}{4}(x-y)^2 \tag{3}$$

and

$$f\Big(\frac{x+y}{2}\Big) \le f(y) + \frac{1}{2}\eta\Big(f(x), f(y)\Big) - \frac{\mu}{4}(x-y)^2$$
$$\le f(y) + \frac{M_\eta}{2} - \frac{\mu}{4}(x-y)^2. \tag{4}$$

Adding (3) and (4), and rearranging the terms of the resulting inequality, we have

$$2f\Big(\frac{x+y}{2}\Big) - M_\eta + \frac{\mu}{2}(x-y)^2 \le f(x) + f(y). \tag{5}$$

Now, if we choose $x = t^\rho a^\rho + (1-t^\rho)b^\rho$ and $y = t^\rho b^\rho + (1-t^\rho)a^\rho$, for $t \in [0,1]$ in (5), we have

$$2f\Big(\frac{a^\rho + b^\rho}{2}\Big) - M_\eta + \frac{\mu}{2}(b^\rho - a^\rho)^2(2t^\rho - 1)^2$$
$$\le f(t^\rho a^\rho + (1-t^\rho)b^\rho) + f(t^\rho b^\rho + (1-t^\rho)a^\rho). \tag{6}$$

Multiplying both sides of (6) by $t^{\alpha\rho-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we have

$$\frac{2}{\alpha\rho} f\Big(\frac{a^\rho + b^\rho}{2}\Big) - \frac{M_\eta}{\alpha\rho} + \frac{\mu(\alpha^2 - \alpha + 2)}{2\alpha\rho(\alpha+1)(\alpha+2)}(b^\rho - a^\rho)^2$$

$$\leq \int_0^1 t^{\alpha\rho-1} f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt + \int_0^1 t^{\alpha\rho-1} f(t^\rho b^\rho + (1-t^\rho)a^\rho)dt. \tag{7}$$

Multiplying both sides of (7) by $\frac{\alpha\rho}{2}$ and using (2), we have

$$\begin{aligned} f\Big(\frac{a^\rho+b^\rho}{2}\Big) - \frac{M_\eta}{2} + \frac{\mu(\alpha^2-\alpha+2)}{4(\alpha+1)(\alpha+2)}(b^\rho-a^\rho)^2 \\ \leq \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\Big[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\Big]. \end{aligned}$$

This proves the first inequality of (1). To prove the second inequality, we note that since f is strongly η-convex with modulus $\mu \geq 0$, we have

$$f(t^\rho a^\rho + (1-t^\rho)b^\rho) \leq f(b^\rho) + t^\rho \eta\Big(f(a^\rho), f(b^\rho)\Big) - \mu t^\rho(1-t^\rho)(b^\rho-a^\rho)^2 \tag{8}$$

and

$$f(t^\rho b^\rho + (1-t^\rho)a^\rho) \leq f(a^\rho) + t^\rho \eta\Big(f(b^\rho), f(a^\rho)\Big) - \mu t^\rho(1-t^\rho)(b^\rho-a^\rho)^2 \tag{9}$$

for all $t \in [0,1]$. By adding (8) and (9), we obtain

$$\begin{aligned} &f(t^\rho a^\rho + (1-t^\rho)b^\rho) + f(t^\rho b^\rho + (1-t^\rho)a^\rho) \\ &\quad \leq f(a^\rho) + f(b^\rho) + t^\rho\Big(\eta\Big(f(b^\rho), f(a^\rho)\Big) + \eta\Big(f(a^\rho), f(b^\rho)\Big)\Big) \\ &\quad\quad - 2\mu t^\rho(1-t^\rho)(b^\rho-a^\rho)^2. \end{aligned} \tag{10}$$

Multiplying both sides of (10) by $t^{\alpha\rho-1}$ then integrating the result with respect to t over $[0,1]$ and using (2), we have

$$\begin{aligned} &\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(b^\rho-a^\rho)^\alpha}\Big[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\Big] \\ &\leq \frac{f(a^\rho)+f(b^\rho)}{\alpha\rho} + \frac{\Big(\eta\Big(f(b^\rho), f(a^\rho)\Big) + \eta\Big(f(a^\rho), f(b^\rho)\Big)\Big)}{(\alpha+1)\rho} - \frac{2\mu(b^\rho-a^\rho)^2}{\rho(\alpha+1)(\alpha+2)}. \end{aligned} \tag{11}$$

The second inequality of (1) follows from (11) by multiplying through by $\frac{\alpha\rho}{2}$. The last inequality follows directly from the second inequality. Q.E.D.

Remark 2.2. If we take $\alpha = \rho = 1$ in Theorem 2.1, then we recover Theorem 1.7.

To prove our next results, we need the following lemma obtained by Chen and Katugampola [3].

Lemma 2.3. Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \leq a < b$. Then the following equality holds if the fractional integrals exist:

$$\begin{aligned} &\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\Big[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\Big] \\ &= \frac{b^\rho-a^\rho}{2}\int_0^1 [(1-t^\rho)^\alpha - t^{\rho\alpha}]t^{\rho-1} f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt. \end{aligned}$$

Theorem 2.4. Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|f'|^q$ is a strongly η-convex function on $[a^\rho, b^\rho]$ with modulus $\mu \ge 0$ for $q \ge 1$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right| \\
&\le \frac{b^\rho - a^\rho}{2\rho}\left(\frac{2}{\alpha+1}\Big(1-\frac{1}{2^\alpha}\Big)\right)^{1-\frac{1}{q}}\left(\frac{2}{\alpha+1}\Big(1-\frac{1}{2^\alpha}\Big)|f'(b^\rho)|^q\right. \\
&\qquad + \frac{1}{\alpha+1}\Big(1-\frac{1}{2^\alpha}\Big)\eta\Big(|f'(a^\rho)|^q, |f'(b^\rho)|^q\Big) \\
&\qquad \left. - \frac{\mu(b^\rho - a^\rho)^2(2^{\alpha+2}-\alpha-4)}{2^{\alpha+1}(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. Using Lemma 2.3, the Hölder's inequality and the strong η-convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right| \\
&\le \frac{b^\rho - a^\rho}{2}\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big| t^{\rho-1}\Big|f'(t^\rho a^\rho + (1-t^\rho)b^\rho)\Big| dt \\
&\le \frac{b^\rho - a^\rho}{2}\left(\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big| t^{\rho-1} dt\right)^{1-\frac{1}{q}} \\
&\qquad \times \left(\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big| t^{\rho-1}\Big|f'(t^\rho a^\rho + (1-t^\rho)b^\rho)\Big|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{b^\rho - a^\rho}{2}\left(\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big| t^{\rho-1} dt\right)^{1-\frac{1}{q}}\left(|f'(b^\rho)|^q \int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big| t^{\rho-1} dt\right. \\
&\qquad + \eta(|f'(a^\rho)|^q, |f'(b^\rho)|^q)\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big| t^{\rho-1} t^\rho dt \\
&\qquad \left. - \mu(b^\rho - a^\rho)^2 \int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big| t^{\rho-1} t^\rho (1-t^\rho) dt\right)^{\frac{1}{q}} \\
&= \frac{b^\rho - a^\rho}{2}\left(\frac{2}{\rho(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big)\right)^{1-\frac{1}{q}}\left(\frac{2}{\rho(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big)|f'(b^\rho)|^q\right. \\
&\qquad + \frac{1}{\rho(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big)\eta(|f'(a^\rho)|^q, |f'(b^\rho)|^q) \\
&\qquad \left. - \frac{\mu(b^\rho - a^\rho)^2}{\rho}\left[\frac{2^{\alpha+2}-\alpha-4}{2^{\alpha+1}(\alpha+2)(\alpha+3)}\right]\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big| t^{\rho-1} dt = \frac{1}{\rho}\int_0^1 |(1-u)^\alpha - u^\alpha| du$$

$$= \frac{1}{\rho}\Big[\int_0^{\frac{1}{2}} \Big[(1-u)^\alpha - u^\alpha\Big]du + \int_{\frac{1}{2}}^1 \Big[u^\alpha - (1-u)^\alpha\Big]du\Big]$$
$$= \frac{2}{\rho(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big),$$

$$\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big| t^\rho t^{\rho-1}dt = \frac{1}{\rho}\int_0^1 |(1-u)^\alpha - u^\alpha| u\ du$$
$$= \frac{1}{\rho}\Big[\int_0^{\frac{1}{2}} \Big[(1-u)^\alpha - u^\alpha\Big]u\ du + \int_{\frac{1}{2}}^1 \Big[u^\alpha - (1-u)^\alpha\Big]u\ du\Big]$$
$$= \frac{1}{\rho(\alpha+1)}\Big(1-\frac{1}{2^\alpha}\Big)$$

and

$$\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big| t^\rho t^{\rho-1}(1-t^\rho)dt$$
$$= \frac{1}{\rho}\Big[\int_0^1 \Big|(1-u)^\alpha - u^\alpha\Big| u(1-u)du\Big]$$
$$= \frac{1}{\rho}\Big[\int_0^{1/2} \Big((1-u)^\alpha - u^\alpha\Big)u(1-u)du + \int_{1/2}^1 \Big(u^\alpha - (1-u)^\alpha\Big)u(1-u)du\Big]$$
$$= \frac{1}{\rho}\Big[\frac{2^{\alpha+2}-\alpha-4}{2^{\alpha+1}(\alpha+2)(\alpha+3)}\Big].$$

This completes the proof of the theorem. Q.E.D.

Remark 2.5. If $q = 1, \mu = 0$ and $\eta(x, y) = x - y$ in Theorem 2.4, then we obtain Theorem 1.15.

To prove our next theorem, we need the following lemma which can be found in [20].

Lemma 2.6. For any $\alpha \in (0, 1]$ and $0 \le x < y$, we have

$$\Big|x^\alpha - y^\alpha\Big| \le (y-x)^\alpha.$$

Theorem 2.7. Let $0 < \alpha \le 1$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|f'|^q$ is a strongly η-convex function on $[a^\rho, b^\rho]$ with modulus $\mu \ge 0$ for $q > 1$, then the following inequality holds:

$$\Big|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\Big[{}^\rho I^\alpha_{a+}f(b^\rho) + {}^\rho I^\alpha_{b-}f(a^\rho)\Big]\Big|$$
$$\le \frac{b^\rho-a^\rho}{2}\Big(\mathcal{C}(s,\alpha,\rho)\Big)^{\frac{1}{s}}\Big(|f'(b^\rho)|^q + \frac{1}{\rho+1}\eta\Big(|f'(a^\rho)|^q, |f'(b^\rho)|^q\Big)$$
$$- \frac{\rho\mu(b^\rho-a^\rho)^2}{(\rho+1)(2\rho+1)}\Big)^{\frac{1}{q}},$$

where $\frac{1}{s}+\frac{1}{q}=1$,
$\mathcal{C}(s,\alpha,\rho) := \frac{1}{\rho 2^{\frac{s(\rho-1)+1}{\rho}}}\Big[B\Big(\alpha s+1, \frac{s(\rho-1)+1}{\rho}\Big)+\int_0^1 x^{\alpha s}\Big(1+x\Big)^{\frac{s(\rho-1)+1}{\rho}-1}dx\Big]$ and $B(\cdot,\cdot)$ is the beta function defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad Re(x)>0 \text{ and } Re(y)>0.$$

Proof. Using Lemma 2.3, the Hölder's inequality and the strong η-convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(b^\rho)+{}^\rho I^\alpha_{b-}f(a^\rho)\right]\right|\\
&\le \frac{b^\rho-a^\rho}{2}\int_0^1\Big|(1-t^\rho)^\alpha-t^{\rho\alpha}\Big|t^{\rho-1}\Big|f'(t^\rho a^\rho+(1-t^\rho)b^\rho)\Big|dt\\
&\le \frac{b^\rho-a^\rho}{2}\left(\int_0^1\Big|(1-t^\rho)^\alpha-t^{\rho\alpha}\Big|^s t^{s(\rho-1)}dt\right)^{\frac{1}{s}}\left(\int_0^1\Big|f'(t^\rho a^\rho+(1-t^\rho)b^\rho)\Big|^q dt\right)^{\frac{1}{q}}\\
&\le \frac{b^\rho-a^\rho}{2}\left(\int_0^1\Big|(1-t^\rho)^\alpha-t^{\rho\alpha}\Big|^s t^{s(\rho-1)}dt\right)^{1-\frac{1}{q}}\Bigg(|f'(b^\rho)|^q\int_0^1 1\ dt\\
&\quad+\eta(|f'(a^\rho)|^q,|f'(b^\rho)|^q)\int_0^1 t^\rho dt-\mu(b^\rho-a^\rho)^2\int_0^1 t^\rho(1-t^\rho)dt\Bigg)^{\frac{1}{q}}\\
&= \frac{b^\rho-a^\rho}{2}\Big(\mathcal{C}(s,\alpha,\rho)\Big)^{1-\frac{1}{q}}\Bigg(|f'(b^\rho)|^q+\frac{1}{\rho+1}\eta(|f'(a^\rho)|^q,|f'(b^\rho)|^q)\\
&\qquad\qquad-\frac{\rho\mu(b^\rho-a^\rho)^2}{(\rho+1)(2\rho+1)}\Bigg)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{C}(s,\alpha,\rho) &:= \int_0^1\Big|(1-t^\rho)^\alpha-t^{\rho\alpha}\Big|^s t^{s(\rho-1)}dt\\
&= \frac{1}{\rho}\int_0^1\Big|(1-u)^\alpha-u^\alpha\Big|^s u^{\frac{(s-1)(\rho-1)}{\rho}}du\\
&\le \frac{1}{\rho}\int_0^1\Big|1-2u\Big|^{\alpha s}u^{\frac{(s-1)(\rho-1)}{\rho}}du \qquad \text{(by Lemma 2.6)}\\
&= \frac{1}{\rho}\Big[\int_0^{1/2}\Big(1-2u\Big)^{\alpha s}u^{\frac{(s-1)(\rho-1)}{\rho}}du+\int_{1/2}^1\Big(2u-1\Big)^{\alpha s}u^{\frac{(s-1)(\rho-1)}{\rho}}du\Big]\\
&= \frac{1}{\rho 2^{\frac{s(\rho-1)+1}{\rho}}}\Big[B\Big(\alpha s+1,\frac{s(\rho-1)+1}{\rho}\Big)+\int_0^1 x^{\alpha s}\Big(1+x\Big)^{\frac{s(\rho-1)+1}{\rho}-1}dx\Big].
\end{aligned}$$

This completes the proof of the theorem. Q.E.D.

Theorem 2.8. Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \leq a < b$. If $|f'|^q$ is a strongly η-convex function on $[a^\rho, b^\rho]$ with modulus $\mu \geq 0$ for $q > 1$, then the following inequality holds:

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right|$$
$$\leq \frac{b^\rho - a^\rho}{2\rho}\left(\frac{1}{\alpha s+1}\right)^{\frac{1}{s}}\left(|f'(b^\rho)|^q + \frac{1}{2}\eta\Big(|f'(a^\rho)|^q, |f'(b^\rho)|^q\Big) - \frac{1}{6}\mu(b^\rho-a^\rho)^2\right)^{\frac{1}{q}},$$

where $\dfrac{1}{s} + \dfrac{1}{q} = 1$.

Proof. Using Lemma 2.3, the Hölder's inequality and the strong η-convexity of $|f'|^q$, we obtain

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right|$$
$$\leq \frac{b^\rho - a^\rho}{2}\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big| t^{\rho-1}\Big|f'(t^\rho a^\rho + (1-t^\rho)b^\rho)\Big| dt$$
$$\leq \frac{b^\rho - a^\rho}{2}\left(\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big|^s t^{\rho-1} dt\right)^{\frac{1}{s}}\left(\int_0^1 t^{\rho-1}\Big|f'(t^\rho a^\rho + (1-t^\rho)b^\rho)\Big|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b^\rho - a^\rho}{2}\left(\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big|^s t^{\rho-1} dt\right)^{\frac{1}{s}}\Bigg(|f'(b^\rho)|^q\int_0^1 t^{\rho-1}dt$$
$$+\eta(|f'(a^\rho)|^q, |f'(b^\rho)|^q)\int_0^1 t^{2\rho-1}dt - \mu(b^\rho-a^\rho)^2\int_0^1 t^{2\rho-1}(1-t^\rho)dt\Bigg)^{\frac{1}{q}}$$
$$= \frac{b^\rho - a^\rho}{2}\left(\frac{1}{\rho(\alpha s+1)}\right)^{\frac{1}{s}}\left(\frac{1}{\rho}|f'(b^\rho)|^q + \frac{1}{2\rho}\eta(|f'(a^\rho)|^q, |f'(b^\rho)|^q) - \frac{1}{6\rho}\mu(b^\rho-a^\rho)^2\right)^{\frac{1}{q}}$$
$$= \frac{b^\rho - a^\rho}{2\rho}\left(\frac{1}{\alpha s+1}\right)^{\frac{1}{s}}\left(|f'(b^\rho)|^q + \frac{1}{2}\eta(|f'(a^\rho)|^q, |f'(b^\rho)|^q) - \frac{1}{6}\mu(b^\rho-a^\rho)^2\right)^{\frac{1}{q}},$$

where

$$\int_0^1 \Big|(1-t^\rho)^\alpha - t^{\rho\alpha}\Big|^s t^{\rho-1}dt = \frac{1}{\rho}\int_0^1 \Big|(1-u)^\alpha - u^\alpha\Big|^s du$$
$$\leq \frac{1}{\rho}\int_0^1 \Big|1-2u\Big|^{\alpha s} du \qquad \text{(by Lemma 2.6)}$$
$$= \frac{1}{\rho}\Big[\int_0^{1/2}\Big(1-2u\Big)^{\alpha s}du + \int_{1/2}^1 \Big(2u-1\Big)^{\alpha s}du\Big]$$
$$= \frac{1}{\rho(\alpha s+1)}.$$

This proves the theorem. Q.E.D.

3 More fractional integral inequalities

Lemma 3.1. Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. Then the following equality holds if the fractional integrals exist:

$$\frac{f(a^\rho)+f(b^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]$$
$$= \frac{b^\rho - a^\rho}{\alpha}\int_0^1 t^{\rho(\alpha+1)-1}[f'((1-t^\rho)a^\rho + t^\rho b^\rho) - f'(t^\rho a^\rho + (1-t^\rho)b^\rho)]dt.$$

Proof. The proof follows directly by integration by parts. Q.E.D.

Theorem 3.2. Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|f'|^q$ is strongly η-convex with modulus $\mu \ge 0$ for $q \ge 1$, then the following inequality holds:

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right|$$
$$\le \frac{b^\rho-a^\rho}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{\alpha+1}|f'(a^\rho)|^q + \frac{1}{\alpha+2}\eta\Big(|f'(b^\rho)|^q, |f'(a^\rho)|^q\Big)\right.\right.$$
$$\left. - \frac{\mu(b^\rho-a^\rho)^2}{(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}} + \left(\frac{1}{\alpha+1}|f'(b^\rho)|^q + \frac{1}{\alpha+2}\eta\Big(|f'(a^\rho)|^q, |f'(b^\rho)|^q\Big)\right.$$
$$\left.\left. - \frac{\mu(b^\rho-a^\rho)^2}{(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\right].$$

Proof. Using Lemma 3.1, the Hölder's inequality and the strong η-convexity of $|f'|^q$, we obtain

$$\left|\frac{f(a^\rho)+f(b^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right|$$
$$\le \frac{b^\rho-a^\rho}{\alpha}\int_0^1 t^{\rho(\alpha+1)-1}\Big[\Big|f'((1-t^\rho)a^\rho + t^\rho b^\rho)\Big| + \Big|f'(t^\rho a^\rho + (1-t^\rho)b^\rho)\Big|\Big]dt$$
$$\le \frac{b^\rho-a^\rho}{\alpha}\left(\int_0^1 t^{\rho(\alpha+1)-1}dt\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 t^{\rho(\alpha+1)-1}\Big|f'((1-t^\rho)a^\rho + t^\rho b^\rho)\Big|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1 t^{\rho(\alpha+1)-1}\Big|f'(t^\rho a^\rho + (1-t^\rho)b^\rho)\Big|^q dt\right)^{\frac{1}{q}}\right]$$
$$\le \frac{b^\rho-a^\rho}{\alpha}\left(\int_0^1 t^{\rho(\alpha+1)-1}dt\right)^{1-\frac{1}{q}}\left[\left(|f'(a^\rho)|^q\int_0^1 t^{\rho(\alpha+1)-1}dt\right.\right.$$
$$\left. + \eta\Big(|f'(b^\rho)|^q, |f'(a^\rho)|^q\Big)\int_0^1 t^{\rho(\alpha+2)-1}dt - \mu(b^\rho-a^\rho)^2\int_0^1 t^{\rho(\alpha+2)-1}(1-t^\rho)dt\right)^{\frac{1}{q}}$$
$$+ \left(|f'(b^\rho)|^q\int_0^1 t^{\rho(\alpha+1)-1}dt + \eta\Big(|f'(a^\rho)|^q, |f'(b^\rho)|^q\Big)\int_0^1 t^{\rho(\alpha+2)-1}dt\right.$$

$$- \mu(b^\rho - a^\rho)^2 \int_0^1 t^{\rho(\alpha+2)-1}(1-t^\rho)dt\Big)^{\frac{1}{q}}\Bigg]$$

$$= \frac{b^\rho - a^\rho}{\alpha}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\Bigg[\Bigg(\frac{1}{\rho(\alpha+1)}|f'(a^\rho)|^q + \frac{1}{\rho(\alpha+2)}\eta\Big(|f'(b^\rho)|^q, |f'(a^\rho)|^q\Big) - \frac{\mu(b^\rho-a^\rho)^2}{\rho(\alpha+2)(\alpha+3)}\Bigg)^{\frac{1}{q}}$$

$$+ \left(\frac{1}{\rho(\alpha+1)}|f'(b^\rho)|^q + \frac{1}{\rho(\alpha+2)}\eta\Big(|f'(a^\rho)|^q, |f'(b^\rho)|^q\Big) - \frac{\mu(b^\rho-a^\rho)^2}{\rho(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\Bigg].$$

This completes the proof of the theorem. Q.E.D.

Remark 3.3. If $q = 1, \mu = 0$ and $\eta(x, y) = x - y$ in Theorem 3.2, then we obtain Theorem 1.14.

Theorem 3.4. Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|f'|^q$ is strongly η-convex with modulus $\mu \ge 0$ for $q > 1$, then the following inequality holds:

$$\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(b^\rho) + {}^\rho I^\alpha_{b-}f(a^\rho)\right]\right|$$

$$\le \frac{\rho(b^\rho-a^\rho)}{2}\left(\frac{1}{s\rho(\alpha+1)-s+1}\right)^{\frac{1}{s}}\Bigg[\Bigg(|f'(a^\rho)|^q + \frac{1}{\rho+1}\eta\Big(|f'(b^\rho)|^q, |f'(a^\rho)|^q\Big)$$

$$- \frac{\mu\rho(b^\rho-a^\rho)^2}{(\rho+1)(2\rho+1)}\Bigg)^{\frac{1}{q}} + \left(|f'(b^\rho)|^q + \frac{1}{\rho+1}\eta\Big(|f'(a^\rho)|^q, |f'(b^\rho)|^q\Big) - \frac{\mu\rho(b^\rho-a^\rho)^2}{(\rho+1)(2\rho+1)}\right)^{\frac{1}{q}}\Bigg],$$

where $\dfrac{1}{s} + \dfrac{1}{q} = 1$.

Proof. Using Lemma 3.1, the Hölder's inequality and the strong η-convexity of $|f'|^q$, we obtain

$$\left|\frac{f(a^\rho)+f(b^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(b^\rho) + {}^\rho I^\alpha_{b-}f(a^\rho)\right]\right|$$

$$\le \frac{b^\rho-a^\rho}{\alpha}\int_0^1 t^{\rho(\alpha+1)-1}\left[\Big|f'((1-t^\rho)a^\rho + t^\rho b^\rho)\Big| + \Big|f'(t^\rho a^\rho + (1-t^\rho)b^\rho)\Big|\right]dt$$

$$\le \frac{b^\rho-a^\rho}{\alpha}\left(\int_0^1 t^{s\rho(\alpha+1)-s}dt\right)^{\frac{1}{s}}\Bigg[\left(\int_0^1\Big|f'((1-t^\rho)a^\rho+t^\rho b^\rho)\Big|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1\Big|f'(t^\rho a^\rho+(1-t^\rho)b^\rho)\Big|^q dt\right)^{\frac{1}{q}}\Bigg]$$

$$\le \frac{b^\rho-a^\rho}{\alpha}\left(\int_0^1 t^{s\rho(\alpha+1)-s}dt\right)^{\frac{1}{s}}\Bigg[\Bigg(|f'(a^\rho)|^q\int_0^1 1\,dt$$

$$+ \eta\Big(|f'(b^\rho)|^q, |f'(a^\rho)|^q\Big)\int_0^1 t^\rho dt - \mu(b^\rho-a^\rho)^2\int_0^1 t^\rho(1-t^\rho)dt\Bigg)^{\frac{1}{q}}$$

$$+\left(|f'(b^\rho)|^q\int_0^1 1\ dt+\eta\Big(|f'(a^\rho)|^q,|f'(b^\rho)|^q\Big)\int_0^1 t^\rho dt-\mu(b^\rho-a^\rho)^2\int_0^1 t^\rho(1-t^\rho)dt\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{b^\rho-a^\rho}{\alpha}\left(\frac{1}{s\rho(\alpha+1)-s+1}\right)^{\frac{1}{s}}\Bigg[\Bigg(|f'(a^\rho)|^q+\frac{1}{\rho+1}\eta\Big(|f'(b^\rho)|^q,|f'(a^\rho)|^q\Big)$$

$$-\frac{\mu\rho(b^\rho-a^\rho)^2}{(\rho+1)(2\rho+1)}\Bigg)^{\frac{1}{q}}+\left(|f'(b^\rho)|^q+\frac{1}{\rho+1}\eta\Big(|f'(a^\rho)|^q,|f'(b^\rho)|^q\Big)-\frac{\mu\rho(b^\rho-a^\rho)^2}{(\rho+1)(2\rho+1)}\right)^{\frac{1}{q}}\Bigg].$$

This completes the proof of the theorem. Q.E.D.

4 Conclusion

We have introduced six main results related to the Hermite–Hadamard inequality via the Katugampola fractional integrals for strongly η-convex functions. As noted earlier, some results in the literature are particular cases of our results and several other interesting results can be obtained by considering different bifunctions η and/or the modulus μ as well as different values for the parameters α and ρ.

Acknowledgement

The authors are very grateful to the anonymous referee for his/her valuable comments and suggestions.

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