Crossed Semimodules of Categories and Schreier 2-Categories

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Abstract

The main idea of this paper is to introduce the notion of a Schreier 2-category and of a crossed semimodule over categories and to prove the categorical equivalence between their categories.

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1 Introduction

A *group-groupoid* (alternatively named $\mathcal{G}$-groupoid [7] or 2-group [1, 2]) is an internal category (which is automatically an internal groupoid) in the category of groups. A *crossed module* of groups as defined by Whitehead is a pair of groups M, N with an action $\bullet\colon N \times M \to M$ and a morphism $\partial\colon M \to N$ of groups such that $\partial(n \bullet m) = n \cdot \partial(m) \cdot n^{-1}$ and $\partial(m) \bullet m' = m \cdot m' \cdot m^{-1}$ [16, 17]. The equivalence of the categories of crossed modules and group-groupoids is well known, see [7]. This equivalence is described in [1] by considering a group-groupoid as a 2-category with one object. The notion of 2-category was first introduced by Bénabou in 1967 [3]. A *2-category* consists of objects, 1-morphisms as in a classical category and 2-morphisms between 1-morphisms as follows

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd} .$$

which can be composed vertically and horizontally. Further details are recalled later in the paper. A *2-groupoid* is a 2-category whose all 1-morphisms are invertible as in a classical groupoid and all 2-morphisms are invertible vertically and horizontally [1, 11].

For the groupoid version of crossed modules, basic references are Brown-Higgins [4, 5] and Brown-Icen [6]. One can find in [9] that the categories of 2-groupoids and of crossed modules over groupoids are equivalent. For the topological aspect of this equivalence, see [8].

Let MON be the category of monoids. A *Schreier internal category* in MON is an internal category in MON which satisfies the Schreier condition [12]. A *crossed semimodule of monoids* is a pair of monoids M, N with an action $\bullet\colon N \times M \to M$ of monoids and a morphism $\partial\colon M \to N$ of monoids such that $\partial(n \bullet m) \cdot n = n \cdot \partial(m)$ and $\big(\partial(m) \bullet m'\big) \cdot m = m \cdot m'$ [12]. The categorical equivalence between Schreier internal categories in MON and crossed semimodules is proved in [12]. This

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equivalence is the generalization of the equivalence between the category of group-groupoids and of crossed modules of groups. On the other hand in [15], a similar equivalence is proved for topological monoids by viewing a Schreier internal category in the category of topological monoids as a 2-category with a single object.

The main purpose of this paper is to develope the construction of Schreier 2-categories and of crossed semimodules of categories and to prove that the categories of crossed semimodules of categories and of Schreier 2-categories are equivalent. The results which are obtained in this paper are the generalization of the results in [12], [13] and [9].

2 Preliminaries

Let $\mathcal{C} = (C_0, C_1, s, t, \varepsilon, m)$ be a finitely complete category. An *internal category* $\mathcal{D} = (D_0, D_1, s, t, \varepsilon, m)$ in $\mathcal{C}$ consists of a set of objects $D_0 \in C_0$ and a set of morphisms $D_1 \in C_0$ with morphisms $s, t\colon D_1 \to D_0$, $\varepsilon\colon D_0 \to D_1$ in $\mathcal{C}$ called the source, the target and the identity maps, respectively, such that $s\varepsilon = t\varepsilon = 1_{D_0}$ and a morphism $m\colon D_1 \times_{D_0} D_1 \to D_1$ of $\mathcal{C}$ called the composition map (usually expressed as $m(f, g) = g \circ f$) where $D_1 \times_{D_0} D_1$ is the pullback of s and t such that $h \circ (g \circ f) = (h \circ g) \circ f$ and $\varepsilon s(f) \circ f = f = f \circ \varepsilon s(f)$ [12, 10]. An *internal groupoid* in $\mathcal{C}$ is an internal category with a morphism $\eta\colon D_1 \to D_1$, $\eta(f) = \overline{f}$ of $\mathcal{C}$ called inverse such that $\overline{f} \circ f = 1_{s(f)}$, $f \circ \overline{f} = 1_{t(f)}$.

Let $\mathcal{M} = (M_0, M_1, s, t, \varepsilon, m)$ be an internal category in MON. If for any $f \in M_1$ there exists a unique $\tilde{f} \in \mathrm{Ker}s$ such that $f = \tilde{f} \cdot \varepsilon s(f)$, then $\mathcal{M}$ is called a *Schreier internal category* in MON and this condition is called *the Schreier condition* [12]. A *Schreier internal groupoid* in MON is a Schreier internal category in MON whose all morphisms are invertible.

A crossed semimodule $K = (M, N, \partial, \bullet)$ of monoids consists of monoids M, N with a homomorphism $\partial\colon M \to N$ and an action $\bullet\colon N \times M \to M$ of monoids such that $\partial(n \bullet m) \cdot n = n \cdot \partial(m)$ and $(\partial(m) \bullet m') \cdot m = m \cdot m'$.

The following theorem and corollary are given in [12]:

Theorem 2.1. The category of Schreier internal categories in MON and of crossed semimodules are equivalent.

Restricting this equivalence we have

Corollary 2.2. The category of schreier internal groupoids in MON is equivalent to the category of crossed semimodules where M is a group.

Note that Corollary 2.2 is obtained as a special case of the theorem of [13].

Group-groupoids can be thought of as internal categories in the category of groups which is denoted by GP [14]. A crossed module $K = (M, N, \partial, \bullet)$ of groups consists of groups M, N with a homomorphism $\partial\colon M \to N$ of groups and an action $\bullet\colon N \times M \to M$ of groups which satisfy $\partial(n \bullet m) = n \cdot \partial(m) \cdot n^{-1}$ and $\partial(m) \bullet m' = m \cdot m' \cdot m^{-1}$ [16, 17]. Restricting of Corollary 2.2, the following theorem is obtained as given by Brown and Spencer in [7]:

Theorem 2.3. The category of internal categories in the category of groups is equivalent to the category of crossed modules.

We present the following definition as in [1]:

A *2-category* $\mathcal{C} = (C_0, C_1, C_2)$ consists of a set of objects C_0, a set of 1-morphisms C_1 and a set of 2-morphisms C_2 as follows

$$\begin{tikzcd} x \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$$

with

- the source and the target maps
  $s\colon C_1 \to C_0,\ s(f) = x,\ s_h\colon C_2 \to C_0,\ s_h(\alpha) = x,\ s_v\colon C_2 \to C_1,\ s_v(\alpha) = f,$
  $t\colon C_1 \to C_0,\ t(f) = y,\ t_h\colon C_2 \to C_0,\ t_h(\alpha) = y,\ t_v\colon C_2 \to C_1,\ t_v(\alpha) = g,$
- the composition of 1-morphisms as in an ordinary category,
- the associative horizontal composition of 2-morphisms $\circ_h\colon C_2 \times_{C_0} C_2 \to C_2$ where $C_2 \times_{C_0} C_2 = \{(\alpha, \delta) \in C_2 \times C_2 | s_h(\delta) = t_h(\alpha)\}$ as

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, bend left, "f_1"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\delta"] & z \end{tikzcd} \quad = \quad \begin{tikzcd} x \arrow[r, bend left, "f_1 \circ f"] \arrow[r, bend right, "g_1 \circ g"'] \arrow[r, phantom, "\Downarrow\delta\circ_h\alpha"] & z \end{tikzcd},$$

- the associative vertical composition of 2-morphisms $\circ_v\colon C_2 \times_{C_1} C_2 \to C_2$ where $C_2 \times_{C_1} C_2 = \{(\alpha, \beta) \in C_2 \times C_2 | s_v(\beta) = t_v(\alpha)\}$ as

$$\begin{tikzcd} x \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=2ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2ex, "\Downarrow\beta"] & y \end{tikzcd} \quad = \quad \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta\circ_v\alpha"] & y \end{tikzcd}$$

- the identity maps $\varepsilon\colon C_0 \to C_1, \varepsilon(x) = 1_x$, $\varepsilon_h\colon C_0 \to C_2$, $\varepsilon_h(x) = 1_{1_x}$ such that $\alpha \circ_h 1_{1_x} = \alpha = 1_{1_y} \circ_h \alpha$ and $\varepsilon_v\colon C_1 \to C_2$, $\varepsilon_v(f) = 1_f$ such that $\alpha \circ_v 1_f = \alpha = 1_g \circ_v \alpha$ whenever the compositions are defined.
- the interchange rule
$$(\theta \circ_v \delta) \circ_h (\beta \circ_v \alpha) = (\theta \circ_h \beta) \circ_v (\delta \circ_h \alpha)$$
  whenever the compositions are defined.

Hence the construction of a 2-category $\mathcal{C} = (C_0, C_1, C_2)$ contains three compatible category structures $\mathcal{C}_1 = (C_0, C_1, s, t, \varepsilon, \circ)$, $\mathcal{C}_2 = (C_0, C_2, s_h, t_h, \varepsilon_h, \circ_h)$ and $\mathcal{C}_3 = (C_1, C_2, s_v, t_v, \varepsilon_v, \circ_v)$ such that the following diagram commutes.

$$\begin{tikzcd}
C_2 \arrow[rr, shift left, "s_h"] \arrow[rr, shift right, "t_h"'] \arrow[dr, shift left, "t_v"] \arrow[dr, shift right, "s_v"'] & & C_0 \arrow[ll, bend right, "\varepsilon_h"'] \arrow[dl, bend left, "\varepsilon"] \\
& C_1 \arrow[ul, bend left, "\varepsilon_v"] \arrow[ur, shift left, "t"] \arrow[ur, shift right, "s"'] &
\end{tikzcd}$$

A *2-functor* is a map $F \colon \mathcal{C} \to \mathcal{C}'$ between 2-categories $\mathcal{C}$ and $\mathcal{C}'$ sending each object of $\mathcal{C}$ to an object of $\mathcal{C}'$, each 1-morphism of $\mathcal{C}$ to 1-morphism of $\mathcal{C}'$ and 2-morphism of $\mathcal{C}$ to 2-morphism of $\mathcal{C}'$ as follows

$$\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, phantom, "\mapsto"] & F(x) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Downarrow F(\alpha)"] & F(y)
\end{tikzcd}$$

such that $F(f_1 \circ f) = F(f_1) \circ F(f)$, $F(\delta \circ_h \alpha) = F(\delta) \circ_h F(\alpha)$, $F(\beta \circ_v \alpha) = F(\beta) \circ_v F(\alpha)$, $F(1_{1_x}) = 1_{F(1_x)} = 1_{1_{F(x)}}$, $F(1_f) = 1_{F(f)}$. Thus, 2-categories and 2-functors form a category which is denoted by 2CAT [11].

A 2-groupoid is a 2-category whose all 1-morphisms and 2-morphisms are invertible as follows

$$\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, bend left, "\bar{f}"] \arrow[r, bend right, "\bar{g}"'] \arrow[r, phantom, "\Downarrow\bar{\alpha}^h"] & x \arrow[r, phantom, "="] & x \arrow[r, bend left, "1_x"] \arrow[r, bend right, "1_x"'] \arrow[r, phantom, "\Downarrow 1_{1_x}"] & x, & x \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\bar{\alpha}^v"] & y \arrow[r, phantom, "="] & x \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & y .
\end{tikzcd}$$

Let $\mathcal{G}, \mathcal{G}'$ be 2-groupoids. A *morphism of 2-groupoids* is a 2-functor $F \colon \mathcal{G} \to \mathcal{G}'$ which preserves the 2-groupoid structures. Thus 2-groupoids form a category which is denoted by 2GPD [11].

We recall an action of groupoids and crossed modules over groupoids as given in [6, 9]. Let $\mathcal{G} = (X, G)$ and $\mathcal{H} = (X, H)$ be groupoids over the same object set and let $\mathcal{H}$ be totally disconnected. An action of $\mathcal{G}$ on $\mathcal{H}$ is a partially defined map

$$\bullet \colon G \times H \to H, \ \ (g, h) \mapsto g \bullet h$$

such that the following conditions holds
AG1. $g \bullet h$ is defined $iff \;\; t(h) = s(g)$, and $t(g \bullet h) = t(g)$,
AG2. $(g_2 \circ g_1) \bullet h = g_2 \bullet (g_1 \bullet h)$,
AG3. $g \bullet (h_2 \circ h_1) = (g \bullet h_2) \circ (g \bullet h_1)$, for $h_1, h_2 \in H(x, x)$ and $g \in G(x, y)$,
AG4. $1_x \bullet h = h$, for $h \in H(x, x)$.

Let $\mathcal{G} = (X, G)$ and $\mathcal{H} = (X, H)$ be groupoids and let $\mathcal{H}$ be totally disconnected. A crossed module $K = (\mathcal{H}, \mathcal{G}, \partial, \bullet)$ of groupoids consists of a morphism $\partial = (1, \partial)\colon \mathcal{H} \to \mathcal{G}$ of groupoids which is the identity on objects together with an action $\bullet\colon G \times H \to H$ of groupoids which satisfy
CMG1. $\partial(g \bullet h) = g \circ \partial(h) \circ \overline{g}$
CMG2. $\partial(h) \bullet h_1 = h \circ h_1 \circ \overline{h}$, for $h, h_1 \in H(x, x)$ and $g \in G(x, y)$.

Let $K = (\mathcal{H}, \mathcal{G}, \partial, \bullet)$ and $K' = (\mathcal{H}', \mathcal{G}', \partial', \bullet')$ be crossed modules of groupoids. Recall from [6, 9] that a *morphism of crossed modules of groupoids* is a triple $\lambda = (\lambda_0, \lambda_1, \lambda_2)\colon K \to K'$ such that $(\lambda_0, \lambda_1)\colon \mathcal{H} \to \mathcal{H}'$ and $(\lambda_0, \lambda_2)\colon \mathcal{G} \to \mathcal{G}'$ are morphisms of groupoids which satisfy $\lambda_2 \partial = \partial' \lambda_1$ and $\lambda_1(g \bullet h) = \lambda_2(g) \bullet' \lambda_1(h)$.

3 Crossed Semimodules of Categories and Schreier 2-categories

To define a crossed semimodule over categories we first define the notion of action of categories, similarly to the notion of action of groupoids.

Definition 3.1. *Let $\mathcal{C} = (X, C)$ and $\mathcal{D} = (X, D)$ be categories over the same object set and let $\mathcal{D}$ be totally disconnected. An action of $\mathcal{C}$ on $\mathcal{D}$ is a partially defined map $\bullet\colon C \times D \to D,\ \ (c, d) \mapsto c \bullet d$ such that the following conditions holds*
AC1. $c \bullet d$ is defined if and only if $t(d) = s(c)$, and $t(c \bullet d) = t(c)$,
AC2. $(c_2 \circ c_1) \bullet d = c_2 \bullet (c_1 \bullet d)$,
AC3. $c \bullet (d_2 \circ d_1) = (c \bullet d_2) \circ (c \bullet d_1)$, for $d_1, d_2 \in D(x, x)$ and $c \in C(x, y)$,
AC4. $1_x \bullet d = d$, for $d \in D(x, x)$.

The following definition is due to Porter [13]:

Definition 3.2. *Let $\mathcal{C} = (X, C)$ and $\mathcal{D} = (X, D)$ be categories over the same object set and let $\mathcal{D}$ be totally disconnected. A crossed semimodule of categories $K = (\mathcal{D}, \mathcal{C}, \partial, \bullet)$ consists of a functor $\partial\colon \mathcal{D} \to \mathcal{C}$ of categories which is the identity on objects together with an action $\bullet\colon C \times D \to D$ of categories which satisfy*
CSC1. $\partial(c \bullet d) \circ c = c \circ \partial(d)$
CSC2. $\big(\partial(d) \bullet d_1\big) \circ d = d \circ d_1$ for $d, d_1 \in D(x, x)$ and $c \in C(x, y)$.

Definition 3.3. *Let $K = (\mathcal{D}, \mathcal{C}, \partial, \bullet), K' = (\mathcal{D}', \mathcal{C}', \partial', \bullet')$ be crossed semimodules over categories. A morphism of crossed semimodules of categories is a mapping $\lambda = (\lambda_2, \lambda_1, \lambda_0)\colon K \to K'$ such that $(\lambda_0, \lambda_1)\colon \mathcal{D} \to \mathcal{D}'$ and $(\lambda_0, \lambda_2)\colon \mathcal{C} \to \mathcal{C}'$ are functors which satisfy $\lambda_2 \partial = \partial' \lambda_1$ and $\lambda_1(c \bullet d) = \lambda_2(c) \bullet' \lambda_1(d)$. Hence crossed semimodules of categories and their morphisms form a category which we denoted by* CSC.

The following definition is due to Porter [13]. Although this condition is given under the name Condition (B) in [13], we will call this condition "Schreier" inspired by Patchkoria's paper [12] since every Schreier internal category in MON can be viewed as a Schreier 2-category with a single object. We will mention this special kind of Schreier 2-categories in section 4.

Definition 3.4. *A Schreier 2-category $\mathcal{C} = (C_0, C_1, C_2)$ is a 2-category which satisfies the Schreier condition: for any $\alpha \in C_2$ there exists a unique $\widetilde{\alpha}$ whose source is the identity 1-morphism such that*

$$\alpha = \widetilde{\alpha} \circ_h \varepsilon_v s_v(\alpha).$$

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & y \arrow[r, bend left, "1_y"] \arrow[r, bend right, "\widetilde{g}"'] \arrow[r, phantom, "\Downarrow \widetilde{\alpha}"] & y \end{tikzcd}$$

Example 3.5. *Consider any monoid $(M, +)$ and any group $(G, \cdot)$. We assume that triples (m, g_1, n) and (m, g_2, n) are 1-morphisms from m to $m + n$ and a 4-tuple (m, g_1, g_2, n) is a 2-morphism from (m, g_1, n) to (m, g_2, n) as follows*

$$\begin{tikzcd} m \arrow[r, bend left, "{(m,g_1,n)}"] \arrow[r, bend right, "{(m,g_2,n)}"'] \arrow[r, phantom, "\Downarrow(m{,}g_1{,}g_2{,}n)"] & m+n \end{tikzcd}$$

for $m, n \in M$ and $g_1, g_2 \in G$. Hence we can construct a Schreier 2-category when compositions are defined as follows

- $(m + n, h_1, p) \circ (m, g_1, n) = (m, h_1 \cdot g_1, n + p)$
- $(m + n, h_1, h_2, p) \circ_h (m, g_1, g_2, n) = (m, h_1 \cdot g_1, h_2 \cdot g_2, n + p)$
- $(m, g_2, g_3, n) \circ_v (m, g_1, g_2, n) = (m, g_1, g_3, n)$.

Since $(m, g_1, g_2, n) = (m + n, e_G, g_2 \cdot g_1^{-1}, e_M) \circ_h (m, g_1, g_1, n)$, the Schreier condition is satisfied for all 2-morphisms.

Proposition 3.6. In a Schreier 2-category, the vertical composition of 2-morphisms can be written in terms of the horizontal composition as follows

$$\beta \circ_v \alpha = \widetilde{\beta} \circ_h \widetilde{\alpha} \circ_h \varepsilon_v s_v(\alpha)$$

whenever compositions are defined.

Proof: Let $\begin{tikzcd} x \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & y \end{tikzcd}$. Due to Schreier condition, we write

$$\beta = \widetilde{\beta} \circ_h \varepsilon_v s_v(\beta) = \widetilde{\beta} \circ_h \varepsilon_v t_v(\alpha) = \widetilde{\beta} \circ_h \varepsilon_v t_v(\widetilde{\alpha} \circ_h \varepsilon_v s_v(\alpha)) = \widetilde{\beta} \circ_h \varepsilon_v t_v(\widetilde{\alpha}) \circ_h \varepsilon_v s_v(\alpha).$$

Then

$$\begin{aligned} \beta \circ_v \alpha &= \Big(\widetilde{\beta} \circ_h \varepsilon_v t_v(\widetilde{\alpha}) \circ_h \varepsilon_v s_v(\alpha)\Big) \circ_v \Big(\widetilde{\alpha} \circ_h \varepsilon_v s_v(\alpha)\Big) \\ &= \Big(\big(\widetilde{\beta} \circ_h \varepsilon_v t_v(\widetilde{\alpha})\big) \circ_v \widetilde{\alpha}\Big) \circ_h \Big(\varepsilon_v s_v(\alpha) \circ_v \varepsilon_v s_v(\alpha)\Big) \\ &= \Big(\big(\widetilde{\beta} \circ_h \varepsilon_v t_v(\widetilde{\alpha})\big) \circ_v (\varepsilon_h t_h(\widetilde{\alpha}) \circ_h \widetilde{\alpha})\Big) \circ_h \varepsilon_v s_v(\alpha) \\ &= \Big(\big(\widetilde{\beta} \circ_v \varepsilon_h t_h(\widetilde{\alpha})\big) \circ_h (\varepsilon_v t_v(\widetilde{\alpha}) \circ_v \widetilde{\alpha})\Big) \circ_h \varepsilon_v s_v(\alpha) \\ &= \widetilde{\beta} \circ_h \widetilde{\alpha} \circ_h \varepsilon_v s_v(\alpha) \end{aligned}$$

Definition 3.7. *Let $\mathcal{C} = (C_0, C_1, C_2)$ and $\mathcal{C}' = (C'_0, C'_1, C'_2)$ be Schreier 2-categories. A morphism of Schreier 2-categories is a 2-functor $F = (F_0, F_1, F_2)\colon \mathcal{C} \to \mathcal{C}'$. Therefore Schreier 2-categories form a category which we denote by* S2CAT.

Theorem 3.8. The category of Schreier 2-categories is equivalent to the category of crossed semimodules of categories.

Proof: Given any Schreier 2-category $\mathcal{C} = (C_0, C_1, C_2)$, we know that $\mathcal{N} = (C_0, C_1, s, t, \varepsilon, \circ)$ is a category. Let $M(x) = \{\alpha \in C_2 | s_v(\alpha) = \varepsilon(x)\}$, for $x \in C_0$. Then $\mathcal{M} = (C_0, M, s_h, t_h, \varepsilon_h, \circ_h)$ is a category where $M = \{M(x)\}_{x \in C_0}$. Now we can define a functor $\gamma\colon$ S2CAT $\to$ CSC as equivalence of categories such that $\gamma(\mathcal{C}) = (\mathcal{M}, \mathcal{N}, \partial, \bullet)$ when

$$\partial\colon \mathcal{M} \to \mathcal{N}, \ \ \partial(\alpha) = t_v(\alpha)$$

and

$$\bullet\colon C_1 \times M \to M$$

such that

$$(f \bullet \alpha) \circ_h \varepsilon_v(f) = \varepsilon_v(f) \circ_h \alpha$$

$$\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & y \arrow[r, bend left, "1_y"] \arrow[r, bend right, "\partial(f\bullet\alpha)"'] \arrow[r, phantom, "\Downarrow f\bullet\alpha"] & y \ := \ x \arrow[r, bend left, "1_x"] \arrow[r, bend right, "\partial(\alpha)"'] \arrow[r, phantom, "\Downarrow \alpha"] & x \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & y
\end{tikzcd}$$

We will verify that $\bullet$ is an action of $\mathcal{N}$ on $\mathcal{M}$.
AC1. $f \bullet \alpha$ is defined *iff* $t_h(\alpha) = s(f)$, and $t_h(f \bullet \alpha) = t(f)$,
AC2. Since

$$\begin{aligned}
\Big((f_1 \circ f) \bullet \alpha\Big) \circ_h 1_{f_1 \circ f} = 1_{f_1 \circ f} \circ_h \alpha = 1_{f_1} \circ_h 1_f \circ_h \alpha &= 1_{f_1} \circ_h (f \bullet \alpha) \circ_h 1_f \\
&= (f_1 \bullet (f \bullet \alpha)) \circ_h 1_{f_1} \circ_h 1_f \\
&= (f_1 \bullet (f \bullet \alpha)) \circ_h 1_{f_1 \circ f},
\end{aligned}$$

under the Schreier condition

$$(f_1 \circ f) \bullet \alpha = f_1 \bullet (f \bullet \alpha)$$

whenever $f_1 \circ f$ is defined.
AC3. Since

$$\Big(f \bullet (\beta \circ_h \alpha)\Big) \circ_h 1_f = 1_f \circ_h \beta \circ_h \alpha = (f \bullet \beta) \circ_h 1_f \circ_h \alpha \ \ = \ \ (f \bullet \beta) \circ_h (f \bullet \alpha) \circ_h 1_f$$

under the Schreier condition

$$f \bullet (\beta \circ_h \alpha) = (f \bullet \beta) \circ_h (f \bullet \alpha)$$

whenever $s_v(\beta) = s_v(\alpha)$.
AC4. Since $(1_x \bullet \alpha) \circ_h 1_{1_x} = 1_{1_x} \circ_h \alpha = \alpha \circ_h 1_{1_x}$, we obtain $1_x \bullet \alpha = \alpha$.

It is obvious that $\partial(f \bullet \alpha) \circ f = f \circ \partial(\alpha)$ and $\Big(\partial(\alpha) \bullet \alpha_1\Big) \circ_h \alpha = \alpha \circ_h \alpha_1$, for $f \in C_1(x, y)$ and $\alpha, \alpha_1 \in M(x)$. Thus $\gamma(\mathcal{C})$ is a crossed semimodule of categories.

Let $F = (F_0, F_1, F_2)$ be a morphism of Schreier 2-categories. Then $\gamma(F) = (F_2\big|_M, F_1, F_0)$ is a morphism of crossed semimodules over categories.

Now, let us define a functor $\theta\colon \mathrm{C{\small SC}} \to \mathrm{S2C{\small AT}}$ which is a weak inverse for λ. Given a crossed semimodule $K = (\mathcal{M}, \mathcal{N}, \partial, \bullet)$ over categories $\mathcal{M} = (X, M)$ and $\mathcal{N} = (X, N)$, then a Schreier 2-category $\theta(K) = (X, N, N \ltimes M)$ can be constructed as in the following way where the set of 2-morphisms is $N \ltimes M = \{(n, m) | n \in N, m \in M, s(m) = t(m) = t(n)\}$. We suppose that if

$$x \xrightarrow{n} y \quad , \quad y \xrightarrow{m} y \ ,$$

then (n, m) is a 2-morphisms as follows

$$\begin{tikzcd} x \arrow[r, bend left, "n"{name=U}] \arrow[r, bend right, "\partial(m)\circ n"'{name=D}] & y \arrow[from=U, to=D, phantom, "\Downarrow(n{,}m)"] \end{tikzcd}$$

where the source and the target maps are defined by $s_v(n, m) = n, t_v(n, m) = \partial(m) \circ n$, respectively, the identity map is defined by $\varepsilon_v(n) = (n, \varepsilon t(n))$ and the vertical composition of 2-morphisms is defined by

$$\Big((\partial(m) \circ n), m'\Big) \circ_v (n, m) = (n, m' \circ m)$$

when $y \xrightarrow{m'} y$. For the horizontal composition, the source and the target maps are defined by $s_h(n, m) = s(n), t_h(n, m) = t(n)$, respectively, the identity map is defined by $\varepsilon_h(x) = (\varepsilon(x), \varepsilon(x))$ where the horizontal composition of 2-morphisms is defined by

$$(n_1, m_1) \circ_h (n, m) = (n_1 \circ n, m_1 \circ (n_1 \bullet m))$$

when compositions are defined. It is easy to check that the vertical composition and the horizontal composition satisfy the usual interchange rule. Since

$$(n, m) = (\varepsilon t(n), m) \circ_h (n, \varepsilon t(n)),$$

all 2-morphisms satisfy the Schreier condition. Now we will verify that the horizontal and vertical compositions are associative and satisfy the interchange rule.

$$\begin{aligned}
(n_2, m_2) \circ_h \Big((n_1, m_1) \circ_h (n, m)\Big) &= (n_2, m_2) \circ_h (n_1 \circ n, m_1 \circ (n_1 \bullet m)) \\
&= \big(n_2 \circ (n_1 \circ n), m_2 \circ (n_2 \bullet (m_1 \circ (n_1 \bullet m)))\big) \\
&= \big((n_2 \circ n_1) \circ n, m_2 \circ (n_2 \bullet m_1) \circ (n_2 \bullet (n_1 \bullet m))\big) \\
&= \Big(n_2 \circ n_1, m_2 \circ (n_2 \bullet m_1)\Big) \circ_h (n, m) \\
&= \Big((n_2, m_2) \circ_h (n_1, m_1)\Big) \circ_h (n, m),
\end{aligned}$$

$$
\begin{aligned}
& \Big(\partial(m' \circ m) \circ n, m''\Big) \circ_v \left[\Big((\partial(m) \circ n), m'\Big) \circ_v (n, m)\right] \\
= \quad & \Big(\partial(m' \circ m) \circ n, m''\Big) \circ_v (n, m' \circ m) \\
= \quad & (n, m'' \circ m' \circ m) \\
= \quad & (\partial(m) \circ n, m'' \circ m') \circ_v (n, m) \\
= \quad & \left[\Big(\partial(m' \circ m) \circ n, m''\Big) \circ_v \Big((\partial(m) \circ n), m'\Big)\right] \circ_v (n, m)
\end{aligned}
$$

and

$$
\begin{aligned}
& \left[\Big((\partial(m_1) \circ n_1), m_1'\Big) \circ_h \Big(\partial(m) \circ n, m'\Big)\right] \circ_v \Big[(n_1, m_1) \circ_h (n, m)\Big] \\
= \quad & \Big[\partial(m_1) \circ n_1 \circ \partial(m) \circ n, m_1' \circ \Big((\partial(m_1) \circ n_1) \bullet m'\Big)\Big] \circ_v \Big[n_1 \circ n, m_1 \circ (n_1 \bullet m)\Big] \\
= \quad & \Big(n_1 \circ n, m_1' \circ \Big(\partial(m_1) \bullet (n_1 \bullet m')\Big) \circ m_1 \circ (n_1 \bullet m)\Big) \\
= \quad & \Big(n_1 \circ n, m_1' \circ m_1 \circ (n_1 \bullet m') \circ (n_1 \bullet m)\Big) \\
= \quad & \Big(n_1 \circ n, m_1' \circ m_1 \circ (n_1 \bullet (m' \circ m))\Big) \\
= \quad & (n_1, m_1' \circ m_1) \circ_h (n, m' \circ m) \\
= \quad & \left[\Big((\partial(m_1) \circ n_1), m_1'\Big) \circ_v (n_1, m_1)\right] \circ_h \left[\Big((\partial(m) \circ n), m'\Big) \circ_v (n, m)\right]
\end{aligned}
$$

whenever all compositions are defined.

Let $\lambda = (\lambda_2, \lambda_1, \lambda_0)$ be a morphism of crossed semimodules of categories. Then $\theta(\lambda) = (\lambda_0, \lambda_2, \lambda_2 \times \lambda_1)$ is morphism of Schreier 2-categories.

To define a natural equivalence $S\colon \theta\gamma \to \mathbf{1}_{\mathrm{S2CAT}}$, a mapping $S_{\mathcal{C}}\colon \theta\gamma(\mathcal{C}) \to \mathcal{C}$ is defined to be identity on objects and on 1-morphisms, on 2-morphisms is defined by $\alpha \mapsto (s_v(\alpha), \widetilde{\alpha})$. Since

$$
\begin{aligned}
S_{\mathcal{C}}(\beta \circ_h \alpha) &= S_{\mathcal{C}}(\widetilde{\beta} \circ_h \varepsilon_v s_v(\beta) \circ_h \widetilde{\alpha} \circ_h \varepsilon_v s_v(\alpha)) \\
&= S_{\mathcal{C}}(\widetilde{\beta} \circ_h (s_v(\beta) \bullet \widetilde{\alpha}) \circ_h \varepsilon_v s_v(\beta) \circ_h \varepsilon_v s_v(\alpha)) \\
&= (s_v(\beta \circ_h \alpha), \widetilde{\beta} \circ_h (s_v(\beta) \bullet \widetilde{\alpha})) \\
&= (s_v(\beta), \widetilde{\beta}) \circ_h (s_v(\alpha), \widetilde{\alpha}) \\
&= S_{\mathcal{C}}(\beta) \circ_h S_{\mathcal{C}}(\alpha)
\end{aligned}
$$

and

$$
\begin{aligned}
S_{\mathcal{C}}(\delta \circ_v \alpha) = S_{\mathcal{C}}(\widetilde{\delta} \circ_h \widetilde{\alpha} \circ_h \varepsilon_v s_v(\alpha)) = (s_v(\alpha), \widetilde{\delta} \circ_h \widetilde{\alpha}) \quad &= \quad (s_v(\delta), \widetilde{\delta}) \circ_v (s_v(\alpha), \widetilde{\alpha}) \\
&= \quad S_{\mathcal{C}}(\delta) \circ_v S_{\mathcal{C}}(\alpha),
\end{aligned}
$$

$S_{\mathcal{C}}$ preserves the compositions, whenever all above compositions are defined.

For a crossed semimodule $K = (\mathcal{M}, \mathcal{N}, \partial, \bullet)$ over categories, a natural equivalence $T\colon \mathbf{1}_{\mathrm{CSC}} \to \gamma\theta$ is given by a map $T_K\colon K \to \gamma\theta(K)$ which is defined to be identity on objects, while on N and on M is defined by $m \mapsto (\varepsilon s(m), m)$. Clearly T_K is an isomorphism of crossed semimodules over categories.

If a Schreier 2-category satifies the following condition, the theorem of [13] is obtained: for each $\alpha \in M$, the horizontal and the vertical inverses are related by

$$\overline{\alpha}^v = \varepsilon_v s_v(\alpha) \circ_h \overline{\alpha}^h$$

Note that this condition is called Condition (A) in [13].

The following corollary is the theorem of [9].

Corollary 3.9. The category of 2-groupoids is equivalent to the category of crossed modules of groupoids.

Proof: The vertical inverse of (n, m) is defined by $\overline{(n,m)}^v = (\partial(m) \circ n, \overline{m})$ and the horizontal inverse of (n, m) is defined by $\overline{(n,m)}^h = (\overline{n}, \overline{n} \bullet \overline{m})$.

4 Schreier internal categories in MON as Schreier 2-categories

We have mentioned that a Schreier internal category in MON is an internal category in MON which satisfies the Schreier condition, see [12]. However, a different approach is to think of a Schreier internal category in MON as a Schreier 2-category with a single object using similar method of [1]. To understand this, we can think a monoid M as a category with a single object $\star$. Hence, elements of M can be thought as morphisms and composition of morphisms is defined by multiplication of M:

$$\star \xrightarrow{x} \star \xrightarrow{x'} \star \qquad = \qquad \star \xrightarrow{x' \cdot x} \star$$

Then, we can think a Schreier internal category $\mathcal{M}$ in MON as a 2-category with a single object $\star$. Here 2-morphisms labeled by morphisms of $\mathcal{M}$, and the horizontal composition defined by the multiplication of M:

$$\begin{tikzcd} \star \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow f"] & \star \arrow[r, bend left, "x'"] \arrow[r, bend right, "y'"'] \arrow[r, phantom, "\Downarrow f'"] & \star \end{tikzcd} = \begin{tikzcd} \star \arrow[r, bend left, "x' \cdot x"] \arrow[r, bend right, "y' \cdot y"'] \arrow[r, phantom, "\Downarrow f' \cdot f"] & \star \end{tikzcd}$$

Thus, the Schreier condition for 2-categories is expressed by $f = \widetilde{f} \cdot \varepsilon_v s_v(f)$ where $\widetilde{f} \in Kers_v$. For more details, see the references [15, 1].

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