Fekete-Szegö problem and Second Hankel Determinant for a class of bi-univalent functions

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Abstract

In this paper we define a subclass of bi-univalent functions. Further, we find the estimates on the bounds $|a_2|$ and $|a_3|$, the Fekete-Szegö inequalities and the second Hankel determinant inequality for defined class of bi-univalent functions.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ that are univalent in Δ. It is well known (e.g. see Duren [17]) that every function $f \in \mathcal{S}$ has an inverse map f^{-1}, defined by $f^{-1}(f(z)) = z$, $z \in \Delta$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4})$, where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. We let σ denote the class of bi-univalent functions in Δ given by (1.1). For a further historical account of functions in the class σ, see the work by Srivastava et al. [47]. In fact, judging by the remarkable flood of papers on non-sharp estimates on the first two coefficients a_2 and a_3 of various subclasses of the bi-univalent function class σ (see, for example, [3–8, 10–13, 15, 16, 19, 22, 23, 31–36, 38–45, 48–54] and references therein), the above-cited recent pioneering work of Srivastava et al. [47] has apparently revived the study of analytic and bi-univalent functions in recent years.

We say that a function $\varphi : \Delta \to \mathbb{C}$ is subordinate to a given function $\psi : \Delta \to \mathbb{C}$ and write $\varphi(z) \prec \psi(z)$ (or simply $\varphi \prec \psi$), if there exists a complex-valued function w which maps Δ into itself, $w(0) = 0$ and $\varphi(z) = \psi(w(z))$; $z \in \Delta$. In particular, if ψ is univalent in Δ, then $\varphi(0) = \psi(0)$ and $\varphi(\Delta) \subset \psi(\Delta)$.

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For integers $n \geqq 1$ and $q \geqq 1$, the $q-$th Hankel determinant, defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

The Hankel determinants $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2a_4 - a_2^3$ are well-known as Fekete-Szegö and second Hankel determinant functionals respectively. Further Fekete and Szegö [18] introduced the generalized functional $a_3 - \delta a_2^2$, where δ is some real number. In 1969, Keogh and Merkes [25] studied the Fekete-Szegö problem for the classes $\mathcal{S}^*$ and $\mathcal{K}$. In 2001, Srivastava et al. [46] solved completely the Fekete-Szegö problem for the family $\mathcal{C}_1 := \{f \in \mathcal{A} \ : \ \Re\left(e^{i\eta} f'(z)\right) > 0, \ -\frac{\pi}{2} < \eta < \frac{\pi}{2}, \ z \in \mathbb{D}\}$ and obtained improvement of $|a_3 - a_2^2|$ for the smaller set $\mathcal{C}_1$. Recently, Kowalczyk et al. [26] discussed the developments involving the Fekete-Szegö functional $|a_3 - \delta a_2^2|$, where $0 \leq \delta \leq 1$ as well as the corresponding Hankel determinant for the Taylor-Maclaurin coefficients $\{a_n\}_{n\in\mathbb{N}\setminus\{1\}}$ of normalized univalent functions of the form (1.1). Similarly, several authors have investigated upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions [1,2,14,27,29,30] and the references therein. On the other hand, Zaprawa [53,54] extended the study on Fekete-Szegö problem to some specific classes of bi-univalent functions. Following Zaprawa [53, 54], the Fekete-Szegö problem for functions belonging to various subclasses of bi-univalent functions were obtained in [4, 23, 32, 50]. Very recently, the upper bounds of $H_2(2)$ for the classes $S^*_\sigma(\beta)$ and $K_\sigma(\beta)$ were discussed by Deniz et al. [16]. Later, the upper bounds of $H_2(2)$ for various subclasses of σ were obtained by Altınkaya and Yalçın [6, 7], Çağlar et al. [12], Kanas et al. [24] and Orhan et al. [34] (see also [35]).

Motivated by the recent publications (especially [5, 8, 16, 34]), we define the following subclass of σ.

For $0 \leqq \lambda \leqq 1$ and $0 \leqq \beta < 1$, a function $f \in \sigma$ given by (1.1) is said to be in the class $\mathcal{G}^\lambda_\sigma(\varphi)$, if the following conditions are satisfied:

$$(1-\lambda)f'(z) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z), \qquad 0 \leqq \lambda \leqq 1, \quad z \in \Delta$$

and for $g = f^{-1}$ given by (1.2)

$$(1-\lambda)g'(w) + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \varphi(w), \qquad 0 \leqq \lambda \leqq 1, \quad w \in \Delta,$$

where φ is an analytic and univalent function with positive real part in Δ, $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The Taylor's series expansion of such function is

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots, \tag{1.3}$$

where all coefficients are real and $B_1 > 0$. Throughout this paper, we assume that the function φ satisfies the above conditions unless otherwise stated.

It is interesting to note that the classes

$$\mathcal{G}^0_\sigma(\varphi) := \mathcal{H}_\sigma(\varphi) \qquad \text{and} \qquad \mathcal{G}^1_\sigma(\varphi) := \mathcal{K}_\sigma(\varphi)$$

were introduced and studied by Ali et al. [3],

$$\mathcal{G}_\sigma^\lambda\left(\frac{1+(1-2\beta)z}{1-z}\right) := \mathcal{G}_\sigma^\lambda(\beta) \qquad (0 \leqq \beta < 1)$$

was introduced by Azizi et al. [8],

$$\mathcal{G}_\sigma^0\left(\frac{1+(1-2\beta)z}{1-z}\right) := \mathcal{H}_\sigma^\beta \qquad (0 \leqq \beta < 1) \quad \text{and} \quad \mathcal{G}_\sigma^0\left(\left(\frac{1+z}{1-z}\right)^\beta\right) := \mathcal{H}_\sigma(\beta) \qquad (0 < \beta \leqq 1)$$

were introduced by Srivastava et al. [47] and

$$\mathcal{G}_\sigma^1\left(\frac{1+(1-2\beta)z}{1-z}\right) := \mathcal{K}_\sigma(\beta) \qquad (0 \leqq \beta < 1)$$

was introduced by Brannan and Taha [9].

In this paper, we shall obtain the Fekete-Szegö inequalities for $\mathcal{G}_\sigma^\lambda(\varphi)$ as well as its special classes. Further, we obtain the second Hankel determinant for functions in the class $\mathcal{G}_\sigma^\lambda(\beta)$.

2 Initial Coefficient Bounds

Theorem 2.1. If f given by (1.1) is in the class $\mathcal{G}_\sigma^\lambda(\varphi)$, then

$$|a_2| \leqq \frac{B_1\sqrt{B_1}}{\sqrt{4B_1 + |(3-\lambda)B_1^2 - 4B_2|}} \tag{2.1}$$

and

$$|a_3| \leqq \begin{cases} \left(1 - \frac{4}{3(1+\lambda)B_1}\right)\frac{B_1^3}{4B_1+|(3-\lambda)B_1^2-4B_2|} + \frac{B_1}{3(1+\lambda)}, & \text{if} \quad B_1 \geqq \frac{4}{3(1+\lambda)}; \\ \frac{B_1}{3(1+\lambda)}, & \text{if} \quad B_1 < \frac{4}{3(1+\lambda)}. \end{cases} \tag{2.2}$$

Proof. Suppose that $u(z)$ and $v(z)$ are analytic in the unit disk Δ with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$ and

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \; v(z) = c_1 z + \sum_{n=2}^{\infty} c_n z^n, \qquad |z| < 1. \tag{2.3}$$

It is well known that

$$|b_1| \leqq 1, \; |b_2| \leqq 1 - |b_1|^2, \; |c_1| \leqq 1, \; |c_2| \leqq 1 - |c_1|^2. \tag{2.4}$$

By a simple calculation, we have

$$\varphi(u(z)) = 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + \dots, \quad |z| < 1 \tag{2.5}$$

and

$$\varphi(v(w)) = 1 + B_1 c_1 w + (B_1 c_2 + B_2 c_1^2) w^2 + \dots, \quad |w| < 1. \tag{2.6}$$

Let $f \in \mathcal{G}_\sigma^\lambda(\varphi)$. Then there are analytic functions $u, v : \Delta \to \Delta$ given by (2.3) such that

$$(1-\lambda)f'(z) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(u(z)) \tag{2.7}$$

and

$$(1-\lambda)g'(w) + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) = \varphi(v(w)). \tag{2.8}$$

It follows from (2.5), (2.6), (2.7) and (2.8) that

$$2a_2 = B_1 b_1 \tag{2.9}$$

$$3(1+\lambda)a_3 - 4\lambda a_2^2 = B_1 b_2 + B_2 b_1^2 \tag{2.10}$$

$$-2a_2 = B_1 c_1 \tag{2.11}$$

$$2(\lambda+3)a_2^2 - 3(1+\lambda)a_3 = B_1 c_2 + B_2 c_1^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$b_1 = -c_1. \tag{2.13}$$

By adding (2.10) to (2.12), further, using (2.9) and (2.13), we have

$$(2(3-\lambda)B_1^2 - 8B_2)a_2^2 = B_1^3(b_2 + c_2). \tag{2.14}$$

In view of (2.13) and (2.14), together with (2.4), we get

$$|(2(3-\lambda)B_1^2 - 8B_2)a_2^2| \leqq 2B_1^3(1 - |b_1|^2). \tag{2.15}$$

Substituting (2.9) in (2.15) we obtain

$$|a_2| \leqq \frac{B_1\sqrt{B_1}}{\sqrt{4B_1 + |(3-\lambda)B_1^2 - 4B_2|}}. \tag{2.16}$$

By subtracting (2.12) from (2.10) and in view of (2.13), we get

$$6(1+\lambda)a_3 = 6(1+\lambda)a_2^2 + B_1(b_2 - c_2). \tag{2.17}$$

From (2.4), (2.9), (2.13) and (2.17), it follows that

$$\begin{aligned} |a_3| &\leqq |a_2|^2 + \frac{B_1}{6(1+\lambda)}(|b_2| + |c_2|) \\ &\leqq |a_2|^2 + \frac{B_1}{3(1+\lambda)}(1 - |b_1|^2) \\ &= \left(1 - \frac{4}{3(1+\lambda)B_1}\right)|a_2|^2 + \frac{B_1}{3(1+\lambda)}. \end{aligned} \tag{2.18}$$

Substituting (2.16) in (2.18) we obtain the desired inequality (2.2). Q.E.D.

Remark 2.2. For $\lambda = 0$, the results obtained in the Theorem 2.1 are coincide with results in [36, Theorem 2.1, p.230].

Corollary 2.3. Let $f \in \mathcal{K}_\sigma(\varphi)$. Then

$$|a_2| \leqq \frac{B_1\sqrt{B_1}}{\sqrt{4B_1 + |2B_1^2 - 4B_2|}} \tag{2.19}$$

and

$$|a_3| \leqq \begin{cases} \left(1 - \frac{2}{3B_1}\right) \frac{B_1^3}{4B_1 + |2B_1^2 - 4B_2|} + \frac{B_1}{6} & ; B_1 \geqq \frac{2}{3}; \\ \frac{B_1}{3(1+\lambda)} & ; B_1 < \frac{2}{3}. \end{cases} \tag{2.20}$$

3 Fekete-Szegö inequalities

In order to derive our result, we shall need the following lemma.

Lemma 3.1. (see [17] or [21]) Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}$, where $\mathcal{P}$ is the family of all functions p, analytic in Δ, for which $\Re\{p(z)\} > 0$, $z \in \Delta$. Then

$$|p_n| \leqq 2; \qquad n = 1, 2, 3, ...,$$

and

$$\left|p_2 - \frac{1}{2}p_1^2\right| \leqq 2 - \frac{1}{2}|p_1|^2.$$

Theorem 3.2. Let f of the form (1.1) be in $\mathcal{G}_\sigma^\lambda(\varphi)$. Then

$$|a_2| \leqq \begin{cases} \sqrt{\frac{B_1}{3-\lambda}}, & \text{if} \quad |B_2| \leqq B_1; \\ \sqrt{\frac{|B_2|}{3-\lambda}}, & \text{if} \quad |B_2| \geqq B_1 \end{cases} \tag{3.1}$$

and

$$\left|a_3 - \frac{4\lambda}{3+3\lambda}a_2^2\right| \leqq \begin{cases} \frac{B_1}{3+3\lambda}, & \text{if} \quad |B_2| \leqq B_1; \\ \frac{|B_2|}{3+3\lambda}, & \text{if} \quad |B_2| \geqq B_1. \end{cases} \tag{3.2}$$

Proof. Since $f \in \mathcal{G}_\sigma^\lambda(\varphi)$, there exist two analytic functions $r, s : \Delta \to \Delta$, with $r(0) = 0 = s(0)$, such that

$$(1-\lambda)f'(z) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(r(z)) \tag{3.3}$$

and

$$(1-\lambda)g'(w) + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) = \varphi(s(w)). \tag{3.4}$$

Define the functions p and q by

$$p(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$$

and

$$q(w) = \frac{1+s(w)}{1-s(w)} = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \dots$$

or equivalently,

$$r(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 + \frac{p_1}{2}\left(\frac{p_1^2}{2} - p_2\right) - \frac{p_1 p_2}{2}\right) z^3 + \dots\right) \tag{3.5}$$

and

$$s(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left(q_1 w + \left(q_2 - \frac{q_1^2}{2}\right) w^2 + \left(q_3 + \frac{q_1}{2}\left(\frac{q_1^2}{2} - q_2\right) - \frac{q_1 q_2}{2}\right) w^3 + \dots\right). \tag{3.6}$$

Using (3.5) and (3.6) in (3.3) and (3.4), we have

$$(1-\lambda) f'(z) + \lambda\left(1 + \frac{z f''(z)}{f'(z)}\right) = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) \tag{3.7}$$

and

$$(1-\lambda) g'(w) + \lambda\left(1 + \frac{w g''(w)}{g'(w)}\right) = \varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{3.8}$$

Again using (3.5) and (3.6) along with (1.3), it is evident that

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + \frac{1}{2} B_1 p_1 z + \left(\frac{1}{2} B_1\left(p_2 - \frac{1}{2} p_1^2\right) + \frac{1}{4} B_2 p_1^2\right) z^2 + \dots \tag{3.9}$$

and

$$\varphi\left(\frac{q(w)-1}{q(w)+1}\right) = 1 + \frac{1}{2} B_1 q_1 w + \left(\frac{1}{2} B_1\left(q_2 - \frac{1}{2} q_1^2\right) + \frac{1}{4} B_2 q_1^2\right) w^2 + \dots. \tag{3.10}$$

It follows from (3.7), (3.8), (3.9) and (3.10) that

$$\begin{aligned} 2a_2 &= \frac{1}{2} B_1 p_1 \\ 3(1+\lambda) a_3 - 4\lambda a_2^2 &= \frac{1}{2} B_1\left(p_2 - \frac{1}{2} p_1^2\right) + \frac{1}{4} B_2 p_1^2 \qquad (3.11) \\ -2a_2 &= \frac{1}{2} B_1 q_1 \\ 2(\lambda+3) a_2^2 - 3(1+\lambda) a_3 &= \frac{1}{2} B_1\left(q_2 - \frac{1}{2} q_1^2\right) + \frac{1}{4} B_2 q_1^2. \qquad (3.12) \end{aligned}$$

Dividing (3.11) by $3 + 3\lambda$ and taking the absolute values we obtain

$$\left|a_3 - \frac{4\lambda}{3+3\lambda} a_2^2\right| \leqq \frac{B_1}{6+6\lambda}\left|p_2 - \frac{1}{2} p_1^2\right| + \frac{|B_2|}{12+12\lambda} |p_1|^2.$$

Now applying Lemma 3.1, we have

$$\left|a_3 - \frac{4\lambda}{3+3\lambda}a_2^2\right| \leqq \frac{B_1}{3+3\lambda} + \frac{|B_2| - B_1}{12+12\lambda}|p_1|^2.$$

Therefore

$$\left|a_3 - \frac{4\lambda}{3+3\lambda}a_2^2\right| \leqq \begin{cases} \frac{B_1}{3+3\lambda}, & \text{if} \quad |B_2| \leqq B_1; \\ \\ \frac{|B_2|}{3+3\lambda}, & \text{if} \quad |B_2| \geqq B_1. \end{cases}$$

Adding (3.11) and (3.12), we have

$$(6-2\lambda)a_2^2 = \frac{B_1}{2}(p_2+q_2) - \frac{(B_1-B_2)}{4}(p_1^2+q_1^2). \tag{3.13}$$

Dividing (3.13) by $6-2\lambda$ and taking the absolute values we obtain

$$|a_2|^2 \leqq \frac{1}{6-2\lambda}\left[\frac{B_1}{2}\left|p_2 - \frac{1}{2}p_1^2\right| + \frac{|B_2|}{4}|p_1|^2 + \frac{B_1}{2}\left|q_2 - \frac{1}{2}q_1^2\right| + \frac{|B_2|}{4}|q_1|^2\right].$$

Once again, apply Lemma 3.1 to obtain

$$|a_2|^2 \leqq \frac{1}{6-2\lambda}\left[\frac{B_1}{2}\left(2 - \frac{1}{2}|p_1|^2\right) + \frac{|B_2|}{4}|p_1|^2 + \frac{B_1}{2}\left(2 - \frac{1}{2}|q_1|^2\right) + \frac{|B_2|}{4}|q_1|^2\right].$$

Upon simplification we obtain

$$|a_2|^2 \leqq \frac{1}{6-2\lambda}\left[2B_1 + \frac{|B_2| - B_1}{2}\left(|p_1|^2 + |q_1|^2\right)\right].$$

Therefore

$$|a_2| \leqq \begin{cases} \sqrt{\frac{B_1}{3-\lambda}}, & \text{if} \quad |B_2| \leqq B_1; \\ \\ \sqrt{\frac{|B_2|}{3-\lambda}}, & \text{if} \quad |B_2| \geqq B_1 \end{cases}$$

which completes the proof. Q.E.D.

Remark 3.3. Taking

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\beta} = 1 + 2\beta z + 2\beta^2 z^2 + \ldots, \qquad 0 < \beta \leqq 1 \tag{3.14}$$

the inequalities (3.1) and (3.2) become

$$|a_2| \leqq \sqrt{\frac{2\beta}{3-\lambda}} \qquad \text{and} \qquad \left|a_3 - \frac{4\lambda}{3+3\lambda}a_2^2\right| \leqq \frac{2\beta}{3+3\lambda}. \tag{3.15}$$

For

$$\varphi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \ldots, \qquad 0 \leqq \beta < 1 \tag{3.16}$$

the inequalities (3.1) and (3.2) become

$$|a_2| \leqq \sqrt{\frac{2(1-\beta)}{3-\lambda}} \qquad \text{and} \qquad \left|a_3 - \frac{4\lambda}{3-\lambda}a_2^2\right| \leqq \frac{2(1-\beta)}{3+3\lambda}. \tag{3.17}$$

4 Bounds for the second Hankel determinant of $\mathcal{G}_\sigma^\lambda(\beta)$

Next we state the following lemmas to establish the desired bounds in our study.

Lemma 4.1. [37] If the function $p \in \mathcal{P}$ is given by the series

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \tag{4.1}$$

then the following sharp estimate holds:

$$|p_n| \leqq 2, \qquad n = 1, 2, \cdots. \tag{4.2}$$

Lemma 4.2. [20] If the function $p \in \mathcal{P}$ is given by the series (4.1), then

$$\begin{aligned} 2c_2 &= c_1^2 + x(4-c_1^2) \\ 4c_3 &= c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z \end{aligned}$$

for some x, z with $|x| \leqq 1$ and $|z| \leqq 1$.

The following theorem provides a bound for the second Hankel determinant of the functions in the class $\mathcal{G}_\sigma^\lambda(\beta)$.

Theorem 4.3. Let f of the form (1.1) be in $\mathcal{G}_\sigma^\lambda(\beta)$. Then

$$|a_2a_4 - a_3^2| \leqq \begin{cases} \frac{(1-\beta)^2}{2(1+2\lambda)}\left[(2-\lambda)(1-\beta)^2+1\right]; \\ \qquad \beta \in \left[0, 1 - \frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+18(1+\lambda)^2(2-\lambda)}}{6(1+\lambda)(2-\lambda)}\right] \\ \frac{(1-\beta)^2}{72(1+2\lambda)}\left(\begin{array}{c} 36[8(1+2\lambda)(2-\lambda)-(1+2\lambda)^2](1-\beta)^2 \\ \dfrac{-324(1+\lambda)(1+2\lambda)(1-\beta)+288(1+2\lambda)-729(1+\lambda)^2}{9(1+\lambda)^2(2-\lambda)(1-\beta)^2-6(1+\lambda)(1+2\lambda)(1-\beta)} \\ +8(1+2\lambda)-18(1+\lambda)^2 \end{array}\right); \\ \qquad \beta \in \left(1 - \frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+18(1+\lambda)^2(2-\lambda)}}{6(1+\lambda)(2-\lambda)}, 1\right). \end{cases}$$

Proof. Let $f \in \mathcal{G}_\sigma^\lambda(\beta)$. Then

$$(1-\lambda)f'(z) + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = \beta + (1-\beta)p(z) \tag{4.3}$$

and

$$(1-\lambda)g'(w) + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right) = \beta + (1-\beta)q(w), \tag{4.4}$$

where $p, q \in \mathcal{P}$ and defined by

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \tag{4.5}$$

and

$$q(z) = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \ldots . \tag{4.6}$$

It follows from (4.3), (4.4), (4.5) and (4.6) that

$$2a_2 = (1-\beta)c_1 \tag{4.7}$$
$$3(1+\lambda)a_3 - 4\lambda a_2^2 = (1-\beta)c_2 \tag{4.8}$$
$$4(1+2\lambda)a_4 - 18\lambda a_2 a_3 + 8\lambda a_2^3 = (1-\beta)c_3 \tag{4.9}$$

and

$$-2a_2 = (1-\beta)d_1 \tag{4.10}$$
$$2(3+\lambda)a_2^2 - 3(1+\lambda)a_3 = (1-\beta)d_2 \tag{4.11}$$
$$2(10+11\lambda)a_2 a_3 - 4(5+3\lambda)a_2^3 - 4(1+2\lambda)a_4 = (1-\beta)d_3. \tag{4.12}$$

From (4.7) and (4.10), we find that

$$c_1 = -d_1 \tag{4.13}$$

and

$$a_2 = \frac{1-\beta}{2}c_1. \tag{4.14}$$

Now, from (4.8), (4.11) and (4.14), we have

$$a_3 = \frac{(1-\beta)^2}{4}c_1^2 + \frac{1-\beta}{6(1+\lambda)}(c_2 - d_2). \tag{4.15}$$

Also, from (4.9) and (4.12), we find that

$$a_4 = \frac{5\lambda(1-\beta)^3}{16(1+2\lambda)}c_1^3 + \frac{5(1-\beta)^2}{24(1+\lambda)}c_1(c_2-d_2) + \frac{1-\beta}{8(1+2\lambda)}(c_3-d_3). \tag{4.16}$$

Then, we can establish that

$$\begin{aligned} |a_2 a_4 - a_3^2| = & \left| \frac{(\lambda-2)(1-\beta)^4}{32(1+2\lambda)}c_1^4 + \frac{(1-\beta)^3}{48(1+\lambda)}c_1^2(c_2-d_2) \right. \\ & \left. + \frac{(1-\beta)^2}{16(1+2\lambda)}c_1(c_3-d_3) - \frac{(1-\beta)^2}{36(1+\lambda)^2}(c_2-d_2)^2 \right|. \end{aligned} \tag{4.17}$$

According to Lemma 4.2 and (4.13), we write

$$c_2 - d_2 = \frac{(4-c_1^2)}{2}(x-y) \tag{4.18}$$

$$\begin{aligned} c_3 - d_3 = & \frac{c_1^3}{2} + \frac{c_1(4-c_1^2)(x+y)}{2} - \frac{c_1(4-c_1^2)(x^2+y^2)}{4} \\ & + \frac{(4-c_1^2)[(1-|x|^2)z - (1-|y|^2)w]}{2} \end{aligned} \tag{4.19}$$

for some x, y, z and w with $|x| \leqq 1$, $|y| \leqq 1$, $|z| \leqq 1$ and $|w| \leqq 1$. Using (4.18) and (4.19) in (4.17), we have

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \Bigg| \frac{(\lambda-2)(1-\beta)^4 c_1^4}{32(1+2\lambda)} + \frac{(1-\beta)^3 c_1^2(4-c_1^2)(x-y)}{96(1+\lambda)} + \frac{(1-\beta)^2 c_1}{16(1+2\lambda)} \\
&\quad \times \left[\frac{c_1^3}{2} + \frac{c_1(4-c_1^2)(x+y)}{2} - \frac{c_1(4-c_1^2)(x^2+y^2)}{4} \right. \\
&\quad \left. + \frac{(4-c_1^2)[(1-|x|^2)z - (1-|y|^2)w]}{2} \right] - \frac{(1-\beta)^2(4-c_1^2)^2}{144(1+\lambda)^2}(x-y)^2 \Bigg| \\
&\leqq \frac{(2-\lambda)(1-\beta)^4}{32(1+2\lambda)} c_1^4 + \frac{(1-\beta)^2 c_1^4}{32(1+2\lambda)} + \frac{(1-\beta)^2 c_1 (4-c_1^2)}{16(1+2\lambda)} \\
&\quad + \left[\frac{(1-\beta)^3 c_1^2(4-c_1^2)}{96(1+\lambda)} + \frac{(1-\beta)^2 c_1^2(4-c_1^2)}{32(1+2\lambda)} \right] (|x|+|y|) \\
&\quad + \left[\frac{(1-\beta)^2 c_1^2(4-c_1^2)}{64(1+2\lambda)} - \frac{(1-\beta)^2 c_1(4-c_1^2)}{32(1+2\lambda)} \right] (|x|^2+|y|^2) \\
&\quad + \frac{(1-\beta)^2(4-c_1^2)^2}{144(1+\lambda)^2}(|x|+|y|)^2.
\end{aligned}$$

Since $p \in \mathcal{P}$, so $|c_1| \leqq 2$. Letting $c_1 = c$, we may assume without restriction that $c \in [0,2]$. Thus, for $\gamma_1 = |x| \leqq 1$ and $\gamma_2 = |y| \leqq 1$, we obtain

$$|a_2a_4 - a_3^2| \leqq T_1 + T_2(\gamma_1+\gamma_2) + T_3(\gamma_1^2+\gamma_2^2) + T_4(\gamma_1+\gamma_2)^2 = F(\gamma_1,\gamma_2),$$

$$\begin{aligned}
T_1 &= T_1(c) = \frac{(2-\lambda)(1-\beta)^4}{32(1+2\lambda)} c^4 + \frac{(1-\beta)^2 c^4}{32(1+2\lambda)} + \frac{(1-\beta)^2 c(4-c^2)}{16(1+2\lambda)} \geqq 0 \\
T_2 &= T_2(c) = \frac{(1-\beta)^3 c^2(4-c^2)}{96(1+\lambda)} + \frac{(1-\beta)^2 c^2(4-c^2)}{32(1+2\lambda)} \geqq 0 \\
T_3 &= T_3(c) = \frac{(1-\beta)^2 c^2(4-c^2)}{64(1+2\lambda)} - \frac{(1-\beta)^2 c(4-c^2)}{32(1+2\lambda)} \leqq 0 \\
T_4 &= T_4(c) = \frac{(1-\beta)^2(4-c^2)^2}{144(1+\lambda)^2} \geqq 0.
\end{aligned}$$

Now we need to maximize $F(\gamma_1,\gamma_2)$ in the closed square $\mathbb{S} := \{(\gamma_1,\gamma_2) : 0 \leqq \gamma_1 \leqq 1, 0 \leqq \gamma_2 \leqq 1\}$ for $c \in [0,2]$. We must investigate the maximum of $F(\gamma_1,\gamma_2)$ according to $c \in (0,2)$, $c = 0$ and $c = 2$ taking into account the sign of $F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2$.

Firstly, let $c \in (0,2)$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$ for $c \in (0,2)$, we conclude that

$$F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2 < 0.$$

Thus, the function F cannot have a local maximum in the interior of the square $\mathbb{S}$. Now, we investigate the maximum of F on the boundary of the square $\mathbb{S}$.

For $\gamma_1 = 0$ and $0 \leqq \gamma_2 \leqq 1$ (similarly $\gamma_2 = 0$ and $0 \leqq \gamma_1 \leqq 1$) we obtain

$$F(0,\gamma_2) = G(\gamma_2) = T_1 + T_2\gamma_2 + (T_3 + T_4)\gamma_2^2.$$

(i) The case $T_3 + T_4 \geqq 0$: In this case for $0 < \gamma_2 < 1$ and any fixed c with $0 < c < 2$, it is clear that $G'(\gamma_2) = 2(T_3 + T_4)\gamma_2 + T_2 > 0$, that is, $G(\gamma_2)$ is an increasing function. Hence, for fixed $c \in (0,2)$, the maximum of $G(\gamma_2)$ occurs at $\gamma_2 = 1$ and

$$\max G(\gamma_2) = G(1) = T_1 + T_2 + T_3 + T_4.$$

(ii) The case $T_3 + T_4 < 0$: Since $T_2 + 2(T_3 + T_4) \geqq 0$ for $0 < \gamma_2 < 1$ and any fixed c with $0 < c < 2$, it is clear that $T_2 + 2(T_3 + T_4) < 2(T_3 + T_4)\gamma_2 + T_2 < T_2$ and so $G'(\gamma_2) > 0$. Hence for fixed $c \in (0,2)$, the maximum of $G(\gamma_2)$ occurs at $\gamma_2 = 1$ and also for $c = 2$ we obtain

$$F(\gamma_1,\gamma_2) = \frac{(1-\beta)^2}{2(1+2\lambda)}\left[(2-\lambda)(1-\beta)^2 + 1\right]. \tag{4.20}$$

Taking into account the value (4.20) and the cases i and ii, for $0 \leqq \gamma_2 < 1$ and any fixed c with $0 \leqq c \leqq 2$ we have

$$\max G(\gamma_2) = G(1) = T_1 + T_2 + T_3 + T_4.$$

For $\gamma_1 = 1$ and $0 \leqq \gamma_2 \leqq 1$ (similarly $\gamma_2 = 1$ and $0 \leqq \gamma_1 \leqq 1$), we obtain

$$F(1,\gamma_2) = H(\gamma_2) = (T_3 + T_4)\gamma_2^2 + (T_2 + 2T_4)\gamma_2 + T_1 + T_2 + T_3 + T_4.$$

Similarly, to the above cases of $T_3 + T_4$, we get that

$$\max H(\gamma_2) = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Since $G(1) \leqq H(1)$ for $c \in (0,2)$, $\max F(\gamma_1,\gamma_2) = F(1,1)$ on the boundary of the square $\mathbb{S}$. Thus the maximum of F occurs at $\gamma_1 = 1$ and $\gamma_2 = 1$ in the closed square $\mathbb{S}$.
Let $K : (0,2) \to \mathbb{R}$

$$K(c) = \max F(\gamma_1,\gamma_2) = F(1,1) = T_1 + 2T_2 + 2T_3 + 4T_4. \tag{4.21}$$

Substituting the values of T_1, T_2, T_3 and T_4 in the function K defined by (4.21), yields

$$\begin{aligned} K(c) = \frac{(1-\beta)^2}{288(1+\lambda)^2(1+2\lambda)} &\left\{\left[9(1-\beta)^2(1+\lambda)^2(2-\lambda)\right.\right.\\ &-6(1-\beta)(1+\lambda)(1+2\lambda) - 18(1+\lambda)^2 + 8(1+2\lambda)\big]\, c^4\\ &+\left[24(1-\beta)(1+\lambda)(1+2\lambda) + 108(1+\lambda)^2 - 64(1+2\lambda)\right] c^2\\ &\left.+128(1+2\lambda)\right\}. \end{aligned}$$

Assume that $K(c)$ has a maximum value in an interior of $c \in (0,2)$, by elementary calculation, we find

$$\begin{aligned} K'(c) = \frac{(1-\beta)^2}{72(1+\lambda)^2(1+2\lambda)} &\left\{\left[9(1-\beta)^2(1+\lambda)^2(2-\lambda)\right.\right.\\ &-6(1-\beta)(1+\lambda)(1+2\lambda) - 18(1+\lambda)^2 + 8(1+2\lambda)\big]\, c^3\\ &\left.+\left[12(1-\beta)(1+\lambda)(1+2\lambda) + 54(1+\lambda)^2 - 32(1+2\lambda)\right] c\right\}. \end{aligned}$$

After some calculations we concluded the following cases:

Case 4.4. Let

$$[9(1-\beta)^2(1+\lambda)^2(2-\lambda)-6(1-\beta)(1+\lambda)(1+2\lambda)-18(1+\lambda)^2+8(1+2\lambda)] \geqq 0,$$

that is,

$$\beta \in \left[0, 1-\frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+(2-\lambda)[18(1+\lambda)^2-8(1+2\lambda)]}}{3(1+\lambda)(2-\lambda)}\right].$$

Therefore $K'(c) > 0$ for $c \in (0,2)$. Since K is an increasing function in the interval $(0,2)$, maximum point of K must be on the boundary of $c \in [0,2]$, that is, $c = 2$. Thus, we have

$$\max_{0<c<2} K(c) = K(2) = \frac{(1-\beta)^2}{2(1+2\lambda)}\left[(2-\lambda)(1-\beta)^2+1\right].$$

Case 4.5. Let

$$[9(1-\beta)^2(1+\lambda)^2(2-\lambda)-6(1-\beta)(1+\lambda)(1+2\lambda)-18(1+\lambda)^2+8(1+2\lambda)] < 0,$$

that is,

$$\beta \in \left[1-\frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+(2-\lambda)[18(1+\lambda)^2-8(1+2\lambda)]}}{3(1+\lambda)(2-\lambda)}, 1\right].$$

Then $K'(c) = 0$ implies the real critical point $c_{0_1} = 0$ or

$$c_{0_2} = \sqrt{\frac{-12(1+\lambda)(1+2\lambda)(1-\beta)-54(1+\lambda)^2+32(1+2\lambda)}{9(1-\beta)^2(1+\lambda)^2(2-\lambda)-6(1-\beta)(1+\lambda)(1+2\lambda)-18(1+\lambda)^2+8(1+2\lambda)}}.$$

When

$$\beta \in \left(1-\frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+(2-\lambda)[18(1+\lambda)^2-8(1+2\lambda)]}}{3(1+\lambda)(2-\lambda)}, 1-\frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+18(1+\lambda)^2(2-\lambda)]}}{6(1+\lambda)(2-\lambda)}\right].$$

We observe that $c_{0_2} \geqq 2$, that is, c_{0_2} is out of the interval $(0,2)$. Therefore, the maximum value of $K(c)$ occurs at $c_{0_1} = 0$ or $c = c_{0_2}$ which contradicts our assumption of having the maximum value at the interior point of $c \in [0,2]$. Since K is an increasing function in the interval $(0,2)$, maximum point of K must be on the boundary of $c \in [0,2]$ that is $c = 2$. Thus, we have

$$\max_{0\leqq c\leqq 2} K(c) = K(2) = \frac{(1-\beta)^2}{2(1+2\lambda)}[1+(2-\lambda)(1-\beta)^2].$$

When $\beta \in \left(1-\frac{(1+2\lambda)+\sqrt{(1+2\lambda)^2+18(1+\lambda)^2(2-\lambda)]}}{6(1+\lambda)(2-\lambda)}, 1\right)$, we observe that $c_{0_2} < 2$, that is, c_{0_2} is an interior of the interval $[0,2]$. Since $K''(c_{0_2}) < 0$, the maximum value of $K(c)$ occurs at $c = c_{0_2}$.

Thus, we have

$$\begin{aligned}\max_{0\leqq c\leqq 2} K(c) &= K(c_{0_2})\\ &= \frac{(1-\beta)^2}{72(1+2\lambda)}\left(\frac{\begin{array}{c}36[8(1+2\lambda)(2-\lambda)-(1+2\lambda)^2](1-\beta)^2\\ -324(1+\lambda)(1+2\lambda)(1-\beta)+288(1+2\lambda)-729(1+\lambda)^2\end{array}}{\begin{array}{c}9(1+\lambda)^2(2-\lambda)(1-\beta)^2\\ -6(1+\lambda)(1+2\lambda)(1-\beta)+8(1+2\lambda)-18(1+\lambda)^2\end{array}}\right).\end{aligned}$$

This completes the proof. Q.E.D.

Corollary 4.6. Let f of the form (1.1) be in $\mathcal{H}_\sigma^\beta$. Then

$$|a_2a_4-a_3^2|\leqq\begin{cases}\frac{(1-\beta)^2[1+2(1-\beta)^2]}{2}; & \beta\in\left[0,\frac{11-\sqrt{37}}{12}\right]\\ \frac{(1-\beta)^2[60\beta^2-84\beta-25]}{16(9\beta^2-15\beta+1)}; & \beta\in\left(\frac{11-\sqrt{37}}{12},1\right).\end{cases}$$

Corollary 4.7. Let f of the form (1.1) be in $\mathcal{H}_\sigma$. Then

$$|a_2a_4-a_3^2|\leqq\frac{3}{2}\,.$$

Remark 4.8. For $\lambda=1$, the result obtained in the Theorem 4.3 coincides with results in [16, Theorem 2.3, p.305].

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