Approximate solutions of Volterra-Fredholm integro-differential equations of fractional order

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Abstract

In this study, sinc-collocation method is introduced for solving Volterra-Fredholm integro-differential equations of fractional order. Fractional derivative is described in the Caputo sense. Obtained results are given to literature as a new theorem. Some numerical examples are presented to demonstrate the theoretical results.

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1 Introduction

Many problems, in science and engineering such as earthquake engineering, biomedical engineering, fluid mechanics can be modeled by fractional integro-differential equations. In order to better analysis these systems, it is required to obtain the solution of these equations. But, achieving the analytical solution of these equations can not be possible. Therefore, finding more accurate solutions using numerical schemes can be helpful. Some numerical algorithm for solving integro-differential equation of fractional order can be summarized as follows: but not limited to; Adomian decomposition method [16, 18, 19], Laplace decomposition method [32], Taylor expansion method [9], least squares method [17] differential transform method [5, 21], Spectral collocation method [14], Legendre wavelets method [24, 26], Haar wavelets method [7], Chebyshev wavelets method [29, 33, 37], piecewise collocation methods [23, 36], Chebyshev pseudo-spectral method [10, 31], homotopy analysis method [1, 35, 38], homotopy perturbation method [6, 20, 25] and variational iteration method [6, 20].

The main advantage of the sinc-collocation method than other methods is that sinc-collocation method provides a much better rate of convergence and more efficient results in the presence of singularity. For more details about the sinc-collocation method see [2, 3, 4, 34].

Particulary, in the present paper, as an original contribution to literature, sinc-collocation method is introduced for solving linear Volterra-Fredholm integro-differential equations of fractional order. Examined integro-differential equations in the present paper include singularities at some points. Obtained results are given in the form of a new theorem. Some numerical examples in the form of graphs and tables are given to illustrate the theoretical results.

In this study, Volterra-Fredholm integro-differential equations of fractional order are considered as follows:

$$\mu_2(x)y''+\mu_1(x)y'+\mu_\alpha(x)D_x^\alpha y+\mu_0(x)y=f(x)+\lambda_1\int_a^x K_1(x,t)y(t)dt+\lambda_2\int_a^b K_2(x,t)y(t)dt,\ 0<\alpha\leq 1 \tag{1.1}$$

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in which D_x^α is the Caputo sense fractional derivative. Eq.(1.1) is subject to following homogeneous boundary conditions

$$y(a) = 0, \quad y(b) = 0, \quad a < x < b.$$

The structure of this paper is organized as follows; In section 2, some preliminaries and basic definitions related to fractional calculus and sinc functions are recalled. In the next section, sinc-collocation method is constructed for solving integro-differential equations of fractional order. In section 4, numerical examples are presented. Finally, conclusions and remarks are given in the section 5.

2 Preliminaries

In this section, some preliminaries and notations related to fractional calculus and sinc basis functions are given. For more details see [8, 11, 12, 13, 15, 22, 27, 28, 30].

Definition 2.1. Let $f : [a, b] \to \mathbb{R}$ be a function, α a positive real number, n the integer satisfying $n - 1 \leq \alpha < n$, and Γ the Euler gamma function. Then, the left Caputo fractional derivative of order α of $f(x)$ is given as follows:

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt. \tag{2.1}$$

Definition 2.2. The Sinc function is defined on the whole real line $-\infty < x < \infty$ by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Definition 2.3. For $h > 0$ and $k = 0, \pm 1, \pm 2, ...$ the translated sinc function with space node are given by:

$$S(k,h)(x) = sinc\Big(\frac{x-kh}{h}\Big) = \begin{cases} \frac{\sin\left(\pi \frac{x-kh}{h}\right)}{\pi \frac{x-kh}{h}} & x \neq kh \\ 1 & x = kh. \end{cases}$$

To construct approximation on the interval (a, b) the conformal map

$$\varphi(z) = \ln\left(\frac{z-a}{b-z}\right)$$

is employed. The basis functions on the interval (a, b) are derived from the composite translated sinc functions

$$S_k(z) = S(k,h)(z) \circ \varphi(z) = sinc\Big(\frac{\varphi(z) - kh}{h}\Big).$$

The inverse map of $w = \varphi(z)$ is

$$z = \varphi^{-1}(w) = \frac{a + be^w}{1 + e^w}.$$

The sinc grid points $z_k \in (a, b)$ will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \varphi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \qquad k = 0, \pm 1, \pm 2, ...$$

Definition 2.4. An open set $S \subseteq \mathbb{C}$ is called connected if it cannot be written as the union of two disjoint open sets A and B such that both A and B intersect S. An open set $S \subseteq \mathbb{C}$ is called simply connected if $\overline{\mathbb{C}} \setminus S$, where $\overline{\mathbb{C}}$ is the extended complex plane denoted $\mathbb{C} \cup \{\infty\}$, is connected.

Definition 2.5. Let D_E be a simply connected domain in the complex plane $\mathbb{C}$, and let ∂D_E denote the boundary of D_E. Let a, b be points on ∂D_E and φ be a conformal map D_E onto D_S such that $\varphi(a) = -\infty$ and $\varphi(b) = \infty$. If the inverse map of φ is denoted by φ, define

$$\Gamma = \{\varphi^{-1}(u) \in D_E : -\infty < u < \infty\}$$

and $z_k = \varphi(kh), k = 0, \pm 1, \pm 2, ...$

Definition 2.6. Let $B(D_E)$ be the class of functions F that are analytic in D_E and satisfy

$$\int_{\psi(L+u)} |F(z)| dz \to, \mathrm{as} u = \mp\infty,$$

where

$$L = \Big\{ iy : |y| < d \leq \frac{\pi}{2} \Big\},$$

and those on the boundary of D_E satisfy

$$T(F) = \int_{\partial D_E} |F(z) dz| < \infty.$$

Theorem 2.7. Let Γ be $(0, 1), F \in B(D_E)$, then for $h > 0$ sufficiently small,

$$\int_\Gamma F(z) dz - h \sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\varphi'(z_j)} = \frac{i}{2} \int_{\partial D} \frac{F(z) k(\varphi, h)(z)}{\sin(\pi\varphi(z)/h)} dz \equiv I_F \tag{2.2}$$

where

$$|k(\varphi, h)|_{z \in \partial D} = \Big| e^{\left[\frac{i\pi\varphi(z)}{h} sgn(Im\varphi(z))\right]} \Big|_{z \in \partial D} = e^{\frac{-\pi d}{h}}.$$

For the term of fractional in (1.1), the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

Theorem 2.8. If there exist positive constants α, β and C such that

$$\Big| \frac{F(x)}{\varphi'(x)} \Big| \leq C \begin{cases} e^{-\alpha|\varphi(x)|} & x \in \psi((-\infty, \infty)) \\ e^{-\beta|\varphi(x)|} & x \in \psi((0, \infty)). \end{cases} \tag{2.3}$$

then the error bound for the quadrature rule (2.3) is

$$\Big| \int_\Gamma F(x) dx - h \sum_{j=-M}^{N} \frac{F(x_j)}{\varphi'(x_j)} \Big| \leq C \Big(\frac{e^{-\alpha M h}}{\alpha} + \frac{e^{-\beta N h}}{\beta} \Big) + |I_F| \tag{2.4}$$

The infinite sum in (2.3) is truncated with the use of (2.4) to arrive at the inequality (2.5). Making the selections

$$h = \sqrt{\frac{\pi d}{\alpha M}}$$

$$N \equiv \left[\left\lfloor \frac{\alpha M}{\beta} + 1 \right\rfloor\right]$$

where $[\lfloor . \rfloor]$ is an integer part of the statement and M is the integer value which specifies the grid size, then

$$\int_{\Gamma} F(x)dx = h \sum_{j=-M}^{N} \frac{F(x_j)}{\varphi'(x_j)} + O\left(e^{-(\pi \alpha d M)^{1/2}}\right). \tag{2.5}$$

We used these theorems to approximate the kernel integral and the arising integral in the formulation of the term fractional in (1.1).

Lemma 2.9. Let φ be the conformal one-to-one mapping of the simply connected domain D_E onto D_S, given by (2.2). Then

$$\delta_{jk}^{(0)} = [S(j,h)o\varphi(x)]|_{x=x_k} \begin{cases} 1 & j = k \\ 0 & j \neq k. \end{cases}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\varphi}[S(j,h)o\varphi(x)]\Big|_{x=x_k} \begin{cases} 0 & j = k \\ \frac{(-1)^{k-j}}{k-j} & j \neq k. \end{cases}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\varphi^2}[S(j,h)o\varphi(x)]\Big|_{x=x_k} \begin{cases} -\frac{\pi^2}{3} & j = k \\ \frac{-2(-1)^{k-j}}{(k-j)^2} & j \neq k. \end{cases}$$

3 The sinc-collocation method

Let us assume an approximate solution for $y(x)$ in Eq.(1.1) by finite expansion of sinc basis functions for as follows;

$$y_n(x) = \sum_{k=-M}^{N} c_k S_k(x), \qquad n = M + N + 1 \tag{3.1}$$

where $S_k(x)$ is the function $S(k,h) \circ \varphi(x)$. Here, the unknown coefficients c_k in (3.1) are determined by sinc-collocation method via the following theorems.

Theorem 3.1. The first and second derivatives of $y_n(x)$ are given by

$$\frac{d}{dx}y_n(x) = \sum_{k=-M}^{N} c_k \varphi'(x) \frac{d}{d\varphi} S_k(x) \tag{3.2}$$

$$\frac{d^2}{dx^2}y_n(x) = \sum_{k=-M}^{N} c_k \left(\varphi''(x)\frac{d}{d\varphi}S_k(x) + (\varphi')^2\frac{d^2}{d\varphi^2}S_k(x)\right) \tag{3.3}$$

respectively.

Theorem 3.2. If ξ is a conformal map for the interval $[a,x]$, then α order derivative of $y_n(x)$ for $0<\alpha<1$ is given by

$$D_x^\alpha(y_n(x)) = \sum_{k=-M}^{N} c_k D_x^\alpha(S_k(x)) \tag{3.4}$$

where

$$D_x^\alpha(S_k(x)) \approx \frac{h_L}{\Gamma(1-\alpha)} \sum_{r=-L}^{L} \frac{(x-x_r)S_k'(x_r)}{\xi'(x_r)}$$

Proof. If we use the definition of Caputo fractional derivative given in (2.1), it is written that

$$D_x^\alpha(y_n(x)) = \sum_{k=-M}^{N} c_k D_x^\alpha(S_k(x))$$

where

$$D_x^\alpha(S_k(x)) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} S_k'(t)dt$$

Now we use quadrature rule given by (2.5) to compute the above integral which is divergent on the interval $[a,x]$. For this purpose, a conformal map and its inverse image that denotes the sinc grid points are given by

$$\xi(t) = \ln\left(\frac{t-a}{x-t}\right)$$

and

$$x_r = \xi^{-1}(rh_L) = \frac{a + xe^{rh_L}}{1+e^{rh_L}}$$

where $h_L = \pi/\sqrt{L}$. Then, according to equality (2.5), we can write

$$D_x^\alpha(S_k(x)) \approx \frac{h_L}{\Gamma(1-\alpha)} \sum_{r=-L}^{L} \frac{(x-x_r)S_k'(x_r)}{\xi'(x_r)}$$

This completes the proof. Q.E.D.

Application of equality (2.5) to the kernel integral in (1.1) gives the following two lemmas.

Lemma 3.3. The following relation holds

$$\int_a^{x_j} K_1(x,t)y(t)dt \approx h \sum_{k=-M}^{N} \delta_{jk}^{(-1)} \frac{K_1(x_j,t_k)}{\varphi'(t_k)} y_k \tag{3.5}$$

where

$$\sigma_{jk} = \int_0^{j-k} \frac{\sin \pi t}{\pi t} dt$$

$$\delta_{jk}^{(-1)} = \frac{1}{2} + \sigma_{jk}$$

and y_k denotes an approximate value of $y(t_k)$.

Lemma 3.4. The following relation holds

$$\int_a^b K_2(x,t)y(t)dt \approx h \sum_{k=-M}^{N} \frac{K_2(x,t_k)}{\varphi'(t_k)} y_k \tag{3.6}$$

where y_k denotes an approximate value of $y(t_k)$.

Replacing each term of (1.1) with the approximation given in (3.1)-(3.6), we obtain the following system

$$\sum_{k=-M}^{N} \Bigg[c_k \Big\{ \mu_2(x) \Big(\varphi''(x) \frac{d}{d\varphi} S_k(x) + (\varphi'(x))^2 \frac{d^2}{d\varphi^2} S_k(x) \Big) + \mu_1(x)\varphi'(x) \frac{d}{d\varphi} S_k(x) + \mu_\alpha(x) D_x^\alpha (S_k(x))$$
$$+ \mu_0(x) S_k(x) - \lambda_1 h \delta_{jk}^{(-1)} \frac{K_1(x,t_k)}{\varphi'(t_k)} - \lambda_2 h \frac{K_2(x,t_k)}{\varphi'(t_k)} \Big\} \Bigg] = f(x)$$

Then, multiplying the resulting equation by $\{(1/\varphi'(x))^2\}$, we obtain

$$\sum_{k=-M}^{N} \Bigg[c_k \Big\{ \mu_2(x) \frac{d^2}{d\varphi^2} S_k(x) + \Big(\mu_2(x) \frac{\varphi''(x)}{(\varphi'(x))^2} + \mu_1(x) \frac{1}{\varphi'(x)} \Big) \frac{d}{d\varphi} S_k(x) + \mu_\alpha(x) \Big(\frac{1}{\varphi'(x)} \Big)^2 D_x^\alpha (S_k(x))$$
$$+ \mu_0(x) \Big(\frac{1}{\varphi'(x)} \Big)^2 S_k(x) - \lambda_1 h \delta_{jk}^{(-1)} \frac{K_1(x,t_k)}{\varphi'(x)^2 \varphi'(t_k)} - \lambda_2 h \frac{K_2(x,t_k)}{\varphi'(x)^2 \varphi'(t_k)} \Big\} \Bigg] = f(x) \Big(\frac{1}{\varphi'(x)} \Big)^2$$

Here, by using the following equality

$$\frac{\varphi''(x)}{(\varphi')^2} = -\Big(\frac{1}{\varphi'(x)} \Big)'$$

we can write

$$\sum_{k=-M}^{N} \Bigg[c_k \Big\{ \sum_{i=0}^{2} g_i(x) \frac{d^i}{d\varphi^i} S_k + g_3(x) D_x^\alpha (S_k(x)) + g_4(x) \delta_{jk}^{(-1)} \frac{K_1(x,t_k)}{\varphi'(t_k)} + g_5(x) \frac{K_2(x,t_k)}{\varphi'(t_k)} \Big\} \Bigg]$$
$$= \Bigg(f(x) \Big(\frac{1}{\varphi'(x)} \Big)^2 \Bigg)$$

where

$$g_0(x) = \mu_0(x) \Big(\frac{1}{\varphi'(x)} \Big)^2$$
$$g_1(x) = \Bigg[\mu_1(x) \Big(\frac{1}{\varphi'(x)} \Big) - \mu_2(x) \Big(\frac{1}{\varphi'(x)} \Big)' \Bigg]$$
$$g_2(x) = \mu_2(x)$$
$$g_3(x) = \mu_\alpha(x) \Big(\frac{1}{\varphi'(x)} \Big)^2$$
$$g_4(x) = -\lambda_1 h \Big(\frac{1}{\varphi'(x)} \Big)^2$$
$$g_5(x) = -\lambda_2 h \Big(\frac{1}{\varphi'(x)} \Big)^2$$

It's known from [18] that

$$\delta^{(0)}_{jk} = \delta^{(0)}_{kj}, \qquad \delta^{(1)}_{jk} = -\delta^{(1)}_{kj}, \qquad \delta^{(2)}_{jk} = \delta^{(2)}_{kj}$$

So, we obtain the following theorem.

Theorem 3.5. If the assumed approximate solution of boundary value problem (1.1) is (3.1), then the discrete sinc-collocation system for the determination of the unknown coefficients $\{c_k\}_{k=-M}^{N}$ is given by

$$\sum_{k=-M}^{N}\left[c_k\Big\{\sum_{i=0}^{2}\frac{g_i(x_j)}{h^i}\delta^{(i)}_{jk} + g_3(x_j)D_x^{\alpha}(S_k(x_j)) + g_4(x_j)\delta^{(-1)}_{jk}\frac{K_1(x_j,t_k)}{\varphi'(t_k)} + g_5(x_j)\frac{K_2(x_j,t_k)}{\varphi'(t_k)}\Big\}\right]$$
$$= \left(f(x_j)\Big(\frac{1}{\varphi'(x_j)}\Big)^2\right), \quad j = -M, ..., N \tag{3.7}$$

We now introduce some notations to rewrite in the matrix form for system (3.7). Let $\mathbf{D}(y)$ denotes a diagonal matrix whose diagonal elements are $y(x_{-M}), y(x_{-M+1}), , y(x_N)$ and non-diagonal elements are zero, let

$$\mathbf{G} = D_x^{\alpha}(S_k(x_j))$$

$$\mathbf{E}_1 = \frac{K_1(x_j,t_k)}{(\varphi'(x_j))^2\varphi'(t_k)}$$

and

$$\mathbf{E}_2 = \frac{K_2(x_j,t_k)}{(\varphi'(x_j))^2\varphi'(t_k)}$$

denote a matrix and also let $\mathbf{I}^{(i)}$ denote the matrices

$$\mathbf{I}^{(i)} = [\delta^{(i)}_{jk}], \qquad i = -1, 0, 1, 2$$

where $\mathbf{D}, \mathbf{G}, \mathbf{E}_1, \mathbf{E}_2, \mathbf{I}^{(-1)}, \mathbf{I}^{(0)}, \mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$ are square matrices of order $n \times n$. In order to calculate unknown coefficients c_k in linear system (3.7), we rewrite this system by using the above notations in matrix form as

$$\mathbf{Ac} = \mathbf{B} \tag{3.8}$$

where

$$\mathbf{A} = \sum_{i=0}^{2}\frac{1}{h^i}\mathbf{D}(g_i)\mathbf{I}^{(i)} + \mathbf{D}(g_3)\mathbf{G} + \mathbf{D}(g_4)(\mathbf{E}_1 \circ \mathbf{I}^{(-1)}) + \mathbf{D}(g_5)\mathbf{E}_2$$

$$\mathbf{B} = \mathbf{D}\left(\frac{f}{(\varphi')^2}\right)\mathbf{1}$$

$$\mathbf{c} = (c_{-M}, c_{-M+1}, ..., c_N)^T$$

The notation " $\circ$ " denotes the Hadamard matrix multiplication. Now we have linear system of n equations in the n unknown coefficients given by (3.8). We can find the unknown coefficients c_k by solving this system.

Table 1: Absolute errors for *Example 1* for $N = 32$ and different values of α

x	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
0.1	3.395×10^{-3}	1.945×10^{-3}	1.181×10^{-3}
0.2	6.465×10^{-3}	3.602×10^{-3}	2.170×10^{-3}
0.3	8.649×10^{-3}	4.551×10^{-3}	2.774×10^{-3}
0.4	9.362×10^{-3}	4.411×10^{-3}	2.814×10^{-3}
0.5	8.227×10^{-3}	3.016×10^{-3}	2.205×10^{-3}
0.6	5.285×10^{-3}	5.421×10^{-4}	1.018×10^{-3}
0.7	1.175×10^{-3}	2.394×10^{-3}	4.705×10^{-4}
0.8	2.719×10^{-3}	4.665×10^{-3}	1.728×10^{-3}
0.9	4.167×10^{-3}	4.607×10^{-3}	1.947×10^{-3}

4 Computational examples

In this section, two problems that have homogeneous boundary conditions will be tested by using the present method via *Mathematica10*. In the first example a problem that has the known exact solution for integer order derivative case is considered. So one could compare the obtained results from the proposed numerical algorithm with the exact solution. Then the second example is given to show the efficiency of the proposed method for the singular problems. In the both examples, we take $h = \pi/\sqrt{N}, L = N = M$.

Example 1. Let us first consider the linear fractional integro-differential equation

$$y''(x) + D_x^\alpha y(x) = f(x) - 2\int_0^x K_1(x,t)y(t)dt + \int_0^1 K_2(x,t)y(t)dt$$

subject to the homogeneous boundary conditions

$$y(0) = 0, \qquad y(1) = 0$$

where $f(x) = -\frac{1}{30} - 6x + \frac{181x^2}{20} + 4x^3 - \frac{x^5}{10} + \frac{x^6}{15}$, $K_1(x,t) = x - t$ and $K_2(x,t) = x^2 - t$. The exact solution of this problem for $\alpha = 1$ is $y(x) = x^3(x-1)$. The numerical solutions which are obtained by using the present method for $N = 32$ and different values of α are presented in Table 1. Also, the graphs of approximate solutions for different values of α are given in Figure 1. Graphs in Figure 1 show that when α approaches to 1, the corresponding solutions of fractional order differential equation approach to the solutions of integer order differential equation.

Example 2. Consider the linear singular fractional integro-differential equation

$$y''(x) + \frac{1}{x}D_x^{0.5}y(x) + \frac{1}{x^2}y(x) = f(x) + \int_0^x K_1(x,t)y(t)dt + \int_0^1 K_2(x,t)y(t)dt$$

subject to the homogeneous boundary conditions

$$y(0) = 0, \qquad y(1) = 0$$

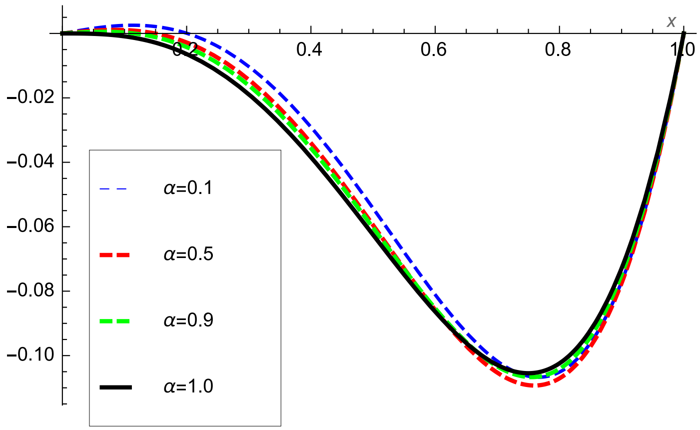
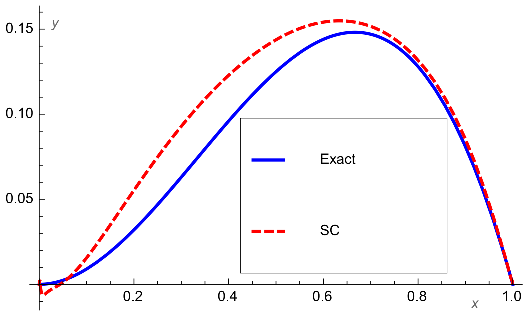


Figure 1: Graphs of approximate solutions for *Example 1* for $N = 32$ and different values of α

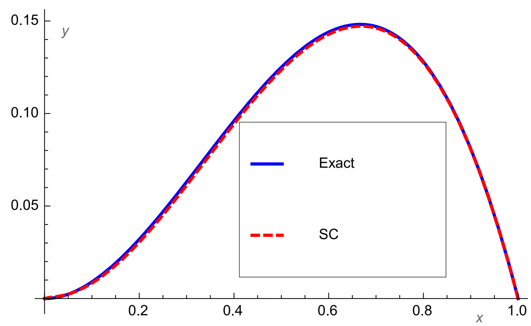
where $f(x) = 5 + 1.50451x^{0.5} - 13x - 1.80541x^{1.5} - x^2 + x^3 - 2.0674\cos(x) + 5.95385\sin(x)$, $K_1(x,t) = \sin(x-t)$ and $K_2(x,t) = \cos(x-t)$. The exact solution of this problem is $y(x) = x^2(1-x)$. The numerical solutions which are obtained by using the present method for different values of N are presented in Table 2. Also, the graphs of approximate solutions for different values of N are given in Figure 2.

Table 2: Absolute errors for *Example 2* for different values of N

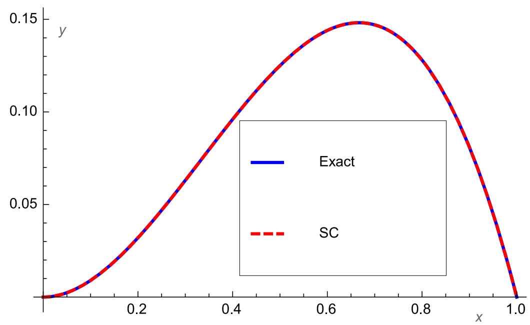
x	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
0.1	6.570×10^{-3}	1.160×10^{-2}	6.951×10^{-4}	7.309×10^{-6}	6.417×10^{-8}
0.2	2.330×10^{-2}	2.277×10^{-2}	1.546×10^{-3}	2.048×10^{-5}	2.931×10^{-7}
0.3	2.989×10^{-2}	2.678×10^{-2}	1.755×10^{-3}	2.606×10^{-5}	3.853×10^{-7}
0.4	2.690×10^{-2}	2.572×10^{-2}	1.632×10^{-3}	2.503×10^{-5}	3.915×10^{-7}
0.5	1.876×10^{-2}	2.158×10^{-2}	1.456×10^{-3}	2.221×10^{-5}	3.449×10^{-7}
0.6	1.011×10^{-2}	1.656×10^{-2}	1.209×10^{-3}	1.789×10^{-5}	2.696×10^{-7}
0.7	4.615×10^{-3}	1.223×10^{-2}	8.495×10^{-4}	1.202×10^{-5}	1.838×10^{-7}
0.8	3.697×10^{-3}	8.652×10^{-3}	5.251×10^{-4}	7.682×10^{-6}	1.011×10^{-7}
0.9	4.256×10^{-3}	4.091×10^{-3}	2.684×10^{-4}	3.034×10^{-6}	3.596×10^{-8}



(a) $N = 4$



(b) $N = 16$



(c) $N = 64$

Figure 2: Graphs of exact and approximate solutions for Example 2

5 Conclusion

In this paper, sinc-collocation method is used to solve a class of fractional Volterra-Fredholm integro differential equation. In order to illustrate the accuracy and effective of the method, it is applied to some examples and obtained results are compared with the exact ones. The comparisons in table and graphical forms show that the approximate solutions converge the exact ones when it is increased that the number of sinc grid points N and the present method is a powerful tool for solving fractional integro-differential equations with boundary conditions.

References

[1] S. Abbasbandy, M. S. Hashemi and I. Hashim, *On convergence of homotopy analysis method and its application to fractional integro-differential equations*, Quaestiones Mathematicae, 36 (2013), No. 1, 93-105.

[2] S. Alkan, *A new solution method for nonlinear fractional integro-differential equations*, Discrete and Continuous Dynamical Systems - Series S, 8 (2015), No. 6, 1065-1077.

[3] S. Alkan and A. Secer, *Solution of nonlinear fractional boundary value problems with nonhomogeneous boundary conditions*, Applied and Computational Mathematics, 14 (2015), No. 3, 284-295.

[4] S. Alkan, K. Yildirim and A. Secer, *An efficient algorithm for solving fractional differential equations with boundary condition*s, Open Physics, 14 (2016), No. 1, 6-14.

[5] A. Arikoglu and I. Ozkol, *Solution of fractional integro-differential equations by using fractional differential transform method*, Chaos, Solitons & Fractals, 40 (2009), No. 2, 521-529.

[6] A. A. Elbeleze, A. Klman and B. M. Taib, *Approximate solution of integro-differential equation of fractional (arbitrary) order*, Journal of King Saud University - Science, 28 (2016), 61-68.

[7] I. Aziz and M. Fayyaz, *A new approach for numerical solution of integro-differential equations via Haar wavelets*, International Journal of Computer Mathematics, 90 (2013), No. 9, 1971-1989.

[8] D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Series on Complexity, Nonlinearity and Chaos in Fractional Calculus Models and Numerical Methods*, World Scientific, 2012.

[9] L. Huang, X. F. Li, Y. Zhao,and X. Y. Duan, *Approximate solution of fractional integro-differential equations by Taylor expansion method*, Computers & Mathematics with Applications, 62 (2011), No. 3, 1127-1134.

[10] M. M. Khader and N. H. Sweilam, *On the approximate solutions for system of fractional integro-differential equations using Chebyshev pseudo-spectral method*, Applied Mathematical Modelling, 37 (2013), No. 24, 9819-9828.

[11] A. Kilbas, H. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.

[12] V. Lakshmikantham, S. Leela and D. J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.

[13] J. Lund, and K. L. Bowers, *Sinc methods for quadrature and differential equations*, SIAM, 1992.

[14] X. Ma and C. Huang, *Spectral collocation method for linear fractional integro-differential equations*, Applied Mathematical Modelling, 38 (2014), No. 4, 1434-1448.

[15] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, 1993.

[16] R. C. Mittal and R. Nigam, *Solution of fractional integro-differential equations by Adomian decomposition method*, The International Journal of Applied Mathematics and Mechanics, 4 (2008), No. 2, 87-94.

[17] D. S. Mohammed, *Numerical Solution of Fractional Integro-Differential Equations by Least Squares Method and Shifted Chebyshev Polynomial*, Mathematical Problems in Engineering 2014 (2014).

[18] S. Momani and M. A. Noor, *Numerical methods for fourth-order fractional integro-differential equations*, Applied Mathematics and Computation, 182 (2006), No. 1, 754-760.

[19] S. Momani and R. Qaralleh, *An efficient method for solving systems of fractional integro-differential equations*, Computers & Mathematics with Applications, 52 (2006), No. 3, 459-470.

[20] Y. Nawaz, *Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations*, Computers & Mathematics with Applications, 61 (2011), No. 8, 2330-2341.

[21] D. Nazari and S. Shahmorad, *Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions*, Journal of Computational and Applied Mathematics, 234 (2010), No. 3, 883-891.

[22] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives*, fractional differential equations, to methods of their solution and some of their applications, Vol. 198. Academic press, 1998.

[23] E. A. Rawashdeh, *Numerical solution of fractional integro-differential equations by collocation method*, Applied Mathematics and Computation, 176 (2006), No. 1, 1-6.

[24] E. A. Rawashdeh, *Legendre wavelets method for fractional integro-differential equations, Applied Mathematical Sciences*, 5 (2011), No. 2, 2467-2474.

[25] R. K. Saeed and H. M. Sdeq, *Solving a system of linear fredholm fractional integro-differential equations using homotopy perturbation method*, Australian Journal of Basic and Applied Sciences, 4 (2010), No. 4, 633-638.

[26] P. K. Sahu and S. S. Ray, *A numerical approach for solving nonlinear fractional VolterraFredholm integro-differential equations with mixed boundary conditions*, International Journal of Wavelets, Multiresolution and Information Processing 14(2016), 1-15.

[27] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[28] A. Secer, S. Alkan, M. A. Akinlar and M. Bayram, *Sinc-Galerkin method for approximate solutions of fractional order boundary value problems*, Boundary Value Problems, 2013 (2013), No. 1, 281.

[29] A. Setia, Y. Liu and A. S. Vatsala, *Numerical solution of Fredholm-Volterra fractional integro-differential equations wth nonlocal boundary conditions*, Journal of Fractional Calculus and Applications 5 (2014), 155-165.

[30] F. Stenger, *Handbook of Sinc numerical methods*, CRC Press, 2010.

[31] N. H. Sweilam and M. M. Khader, *A Chebyshev pseudo-spectral method for solving fractional-order integro-differential equations*, The ANZIAM Journal, 51 (2010), No. 4, 464-475.

[32] C. Yang and J. Hou, *Numerical solution of Volterra Integro-differential equations of fractional order by Laplace decomposition method*, World Academy of Science, Engineering and Technology, International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 7 (2013), 863-867.

[33] L. Yuanlu, *Solving a nonlinear fractional differential equation using Chebyshev wavelets*, Communications in Nonlinear Science and Numerical Simulation, 15 (2010), No. 9, 2284-2292.

[34] M. Zarebnia and Z. Nikpour, *Solution of linear Volterra integro-differential equations via Sinc functions*, International Journal of Applied Mathematics and Computation, 2 (2009), No. 1, 001-010.

[35] X. Zhang, B. Tang, and Y. He, *Homotopy analysis method for higher-order fractional integro-differential equations*, Computers & Mathematics with Applications, 62 (2011), No. 8, 3194-3203.

[36] J. Zhao, J. Xiao, and N. J. Ford, *Collocation methods for fractional integro-differential equations with weakly singular kernels*, Numerical Algorithms, 65 (2014), No. 4, 723-743.

[37] L. Zhu and Q. Fan, *Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet*, Communications in Nonlinear Science and Numerical Simulation, 17 (2012), 2333-2341.

[38] M. Zurigat, S. Momani, A. Alawneh, *Homotopy analysis method for systems of fractional integro-differential equations*, Neural Parallel Sci. Comput., 17 (2009), 169-186.