# Homological algebra in bivariant K-theory and other triangulated categories. II

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**Abstract**

We use homological ideals in triangulated categories to get a sufficient criterion for a pair of subcategories in a triangulated category to be complementary. We apply this criterion to construct the Baum-Connes assembly map for locally compact groups and torsion-free discrete quantum groups. Our methods are related to the abstract version of the Adams spectral sequence by Brinkmann and Christensen.

*2000 Mathematics Subject Classification.* **18E30**. 19K35, 46L80, 55U35.
*Keywords.* triangulated category, Baum–Connes conjecture, phantom map, spectral sequence.

## 1 Introduction

The framework of *triangulated categories* is ideal to extend basic constructions from homotopy theory to categories of C*-algebras. It provides a uniform setting for various problems in non-commutative topology, including homotopy colimits and Mayer-Vietoris sequences, universal coefficient theorems, and generalisations of the Baum-Connes assembly map (cf. [16, 19, 17, 20, 18]). More specifically, the Baum-Connes assembly map for coactions of certain compact Lie groups, which is studied in [17], is always an isomorphism and it is closely related to a universal coefficient theorem for equivariant Kasparov theory by Jonathan Rosenberg and Claude Schochet ([22]). Universal coefficient theorems for Kirchberg's bivariant K-theory for C*-algebras over certain finite topological spaces are derived in [20, 18].

This article continues [19], which deals with a framework for carrying over familiar notions from homological algebra to general triangulated categories. Before we explain what *this* article is about, we outline some important ideas from [19].

The localisation (or total derived functor) of an additive functor between Abelian categories is a functor between their derived categories. Mapping

*Supported by the German Research Foundation (Deutsche Forschungsgemeinschaft (DFG)) through the Institutional Strategy of the University of Göttingen.

**Tbilisi Mathematical Journal** 1 (2008), pp. 165–210.
Tbilisi Centre for Mathematical Sciences & College Publications.

*Received by the editors:* 17 July 2008; 4 December 2008.
*Accepted for publication:* 15 December 2008.

chain complexes to chain complexes, it belongs to the world of triangulated categories by definition. Although the more classical derived functors originally live in the underlying Abelian categories, they can be carried over to triangulated categories as well.

Both localisations and derived functors require additional structure on a triangulated category to be defined. For the localisation of a functor, we specify the *subcategory* to localise at, consisting of all *objects* on which the localisation vanishes. For its *derived functors*, we specify an *ideal*, consisting of all *morphisms* on which the derived functors vanish.

The idea to use ideals in triangulated categories goes back to Daniel Christensen [7]. Some important related concepts are due to Apostolos Beligiannis [2], who uses a slightly different but equivalent setup, which is inspired by the notion of an exact category in homological algebra.

A *homological ideal* in a triangulated category $\mathfrak{T}$ is, by definition, the kernel of a stable homological functor (cf. [19]). Such an ideal $\mathfrak{I}$ allows us to carry over various notions of homological algebra to $\mathfrak{T}$. The ultimate explanation for this is that a homological ideal generates a canonical homological functor to a certain Abelian category, namely, the *universal* $\mathfrak{I}$-exact stable homological functor $H_{\mathfrak{I}}\colon \mathfrak{T} \to \mathfrak{A}_{\mathfrak{I}}\mathfrak{T}$. All homological notions in $\mathfrak{T}$ defined using the ideal $\mathfrak{I}$ reflect familiar notions in this Abelian category. The homological algebra in the target Abelian category $\mathfrak{A}_{\mathfrak{I}}\mathfrak{T}$ provides a rough *Abelian approximation* to the category $\mathfrak{T}$.

An interesting and typical example is the $G$-equivariant Kasparov category $\mathfrak{KK}^G$ for a countable discrete group $G$. Let $\mathfrak{I}$ be the ideal defined by the K-theory functor, that is, an element of $\mathrm{KK}^G(A, B)$ belongs to the ideal if it induces the zero map $\mathrm{K}_*(A) \to \mathrm{K}_*(B)$. The resulting Abelian approximation $\mathfrak{A}_{\mathfrak{I}}(\mathfrak{KK}^G)$ is the category of all $\mathbb{Z}/2$-graded countable modules over the group ring $\mathbb{Z}[G]$, and the universal functor maps a $\mathrm{C}^*$-algebra $A$ with an action of $G$ to its K-theory, equipped with the induced action of $G$ (this is a special case of a result in [19]).

Notice that the passage to the universal functor adds the group action on $\mathrm{K}_*(A)$. Forgetting this group action does not change the ideal defined by the functor, but it kills most interesting homological algebra. (In § 7, we will actually consider a smaller ideal in $\mathfrak{KK}^G$ that is more closely related to the Baum-Connes conjecture, but leads to a more complicated Abelian approximation.)

The Abelian category $\mathfrak{A}_{\mathfrak{I}}\mathfrak{T}$ is usually not a localisation of $\mathfrak{T}$: we must modify both morphisms and objects to get an Abelian category. Instead, it is described in [2] as a localisation of the Abelian category containing $\mathfrak{T}$ constructed by Peter Freyd. The main innovation in [19] is a concrete criterion for a stable homological functor to be universal, which involves its partially defined left adjoint. Using this criterion, we can often find rather concrete models for the universal functor —as in the example mentioned above— and then compute derived functors associated to the ideal.

What do the derived functors of a homological functor on $\mathfrak{T}$ tell us about the original functor? In general, these derived functors are always related to the original functor by a spectral sequence, whose convergence we will discuss below. This result is mainly of theoretical importance because spectral sequence computations are almost impossible without additional simplifying assumptions. But given how much information is lost by passing to an Abelian category, we cannot hope for much more than a spectral sequence.

The spectral sequence that links a functor to its derived functors was already discovered in the 1960s before triangulated categories became popular. First Frank Adams treated an important special case in stable homotopy theory — the Adams spectral sequence [1]. This was reformulated in an abstract setting by Hans-Berndt Brinkmann [6]. Daniel Christensen [7] formulated the Adams spectral sequence in the setting of triangulated categories, apparently unaware of Brinkmann's work.

Given the sources of the spectral sequence, we call it the *ABC spectral sequence* here. We describe its construction and its higher pages in greater detail than previous authors and weaken the assumptions needed to guarantee its convergence.

I was drawn towards this theory because similar ideas provide an effective method to prove that pairs of subcategories are complementary; this is the most difficult technical aspect of the construction of the Baum-Connes assembly map in [16]. In § 7, we first apply our new criterion to the group case already treated in [16] and then define an analogue of the Baum-Connes assembly map for all "torsion-free" discrete quantum groups. More precisely, we construct an assembly map for all discrete quantum groups, but since this map does not take into account torsion, it is not the right analogue of the Baum-Connes assembly map unless the quantum group in question is torsion-free.

A built-in feature of our new assembly map is that its domain is computed by a spectral sequence —the ABC spectral sequence— whose second page is quite accessible. The spectral sequence computation is very difficult, but an operator algebraist might consider it to be a topological problem, that is, Someone Else's Problem. His own problem is to find out when the assembly map is an isomorphism. Given our experience with the group case, this should happen often but not always. So far —besides classical groups— only the duals of certain compact Lie groups and quantum SU(2) have been treated in [17] and in [24], respectively. For the alternative approach by Aderemi Kuku and Debashish Goswami in [11], it is unclear whether the domain of the assembly map is computable by topological methods.

Our criterion for complementarity of two subcategories is also useful in situations that have nothing to do with bivariant K-theory. The improvement upon similar criteria in [2] is that we can cover categories that are not compactly generated: what we need is an ideal with enough projective objects that is compatible with countable direct sums. This assumption is still satisfied for the ideals that appear in connection with the Baum-Connes assembly map, although the categories in question are probably not compactly generated.

More precisely, the criterion is the following. Let $\mathfrak{I}$ be a homological ideal in a triangulated category $\mathfrak{T}$. We assume that $\mathfrak{T}$ has countable direct sums and that the ideal $\mathfrak{I}$ is compatible with countable direct sums in a suitable sense. Furthermore, we assume that $\mathfrak{I}$ has enough projective objects. Let $\mathfrak{P}_{\mathfrak{I}} \subseteq \mathfrak{T}$ be the class of $\mathfrak{I}$-projective objects in $\mathfrak{T}$ and let $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ be the localising subcategory generated by it, that is, the smallest triangulated subcategory that is closed under countable direct sums and contains $\mathfrak{P}_{\mathfrak{I}}$. Finally, let $\mathfrak{N}_{\mathfrak{I}}$ be the subcategory of $\mathfrak{I}$-contractible objects. Under the assumptions above, the pair of subcategories $(\langle \mathfrak{P}_{\mathfrak{I}} \rangle, \mathfrak{N}_{\mathfrak{I}})$ is complementary, that is, $\mathfrak{T}_*(P, N) = 0$ whenever $P \in\in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ and $N \in\in \mathfrak{N}_{\mathfrak{I}}$, and any object $A \in\in \mathfrak{T}$ is part of an exact triangle $P \to A \to N \to P[1]$ with $P \in\in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ and $N \in\in \mathfrak{N}_{\mathfrak{I}}$. Equivalently, the subcategory $\mathfrak{N}_{\mathfrak{I}}$ is *reflective*, that is, the embedding $\mathfrak{N}_{\mathfrak{I}} \to \mathfrak{T}$ has a right adjoint functor.

Our proof also provides the following structural information on the category $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$. First, we get an increasing chain $(\mathfrak{P}_{\mathfrak{I}}^n)_{n \in \mathbb{N}}$ of subcategories, consisting of the projective objects for the ideals $\mathfrak{I}^n$; these can also be generated iteratively from $\mathfrak{P}_{\mathfrak{I}}$ using exact triangles. We show that any object of $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ is a homotopy colimit of an inductive system $P_n$ with $P_n \in\in \mathfrak{P}_{\mathfrak{I}}^n$.

We write $f \in \mathfrak{C}$ for a morphism and $A \in\in \mathfrak{C}$ for an object of a category $\mathfrak{C}$. We denote the category of Abelian groups by $\mathfrak{Ab}$. We usually write $\mathfrak{T}$ for triangulated, $\mathfrak{A}$ for Abelian, and $\mathfrak{C}$ for additive categories. The translation automorphism in a triangulated category is denoted by $A \mapsto A[1]$.

## 2 Homological ideals, powers, and filtrations

The convergence of a spectral sequence always involves a filtration on the limit group. Hence we expect a homological ideal $\mathfrak{I}$ in $\mathfrak{T}$ to generate filtrations on the category $\mathfrak{T}$ itself and on homological and cohomological functors on $\mathfrak{T}$. After recalling some basic notions, we introduce these filtrations here.

We will use the results and the notation of [19]. In particular, a *stable* category is a category with a *translation* automorphism, denoted $A \mapsto A[1]$, and a *stable* functor is a functor $F$ together with natural isomorphisms $F(A[1]) \cong (FA)[1]$ for all objects $A$.

Let $F\colon \mathfrak{T} \to \mathfrak{A}$ be a stable homological functor from a triangulated category $\mathfrak{T}$ to a stable Abelian category $\mathfrak{A}$. We define an ideal $\ker F$ in $\mathfrak{T}$ by

$$\ker F(A,B) := \{\varphi \in \mathfrak{T}(A,B) \mid F(\varphi) = 0\}.$$

Ideals of this form are called *homological ideals.* A homological ideal is used in [19] to carry over various notions from Abelian to triangulated categories. This includes $\mathfrak{I}$-epimorphisms, $\mathfrak{I}$-exact chain complexes, $\mathfrak{I}$-exact functors, $\mathfrak{I}$-projective objects, and $\mathfrak{I}$-projective resolutions. The first three of these can be tested using the functor $F$; for instance, a chain complex with entries in $\mathfrak{T}$ is $\mathfrak{I}$-exact if and only if its $F$-image is an exact chain complex in the Abelian category $\mathfrak{A}$. Projective objects can only be described in terms of $F$ if $F$ is the *universal* $\mathfrak{I}$-exact stable homological functor, cf. [19]. We also call a morphism an *$\mathfrak{I}$-phantom map* if it belongs to $\mathfrak{I}$.

Most of our constructions require $\mathfrak{T}$ to contain *enough $\mathfrak{I}$-projective objects* — that is, any object should be the range of an $\mathfrak{I}$-epimorphism with $\mathfrak{I}$-projective domain. This is equivalent to the existence of $\mathfrak{I}$-projective resolutions for all objects.

**Remark 2.1.** Daniel Christensen uses a somewhat different terminology in [7]. His *projective classes* $(\mathfrak{I}, \mathfrak{P})$ turn out to be the same as a homological ideal $\mathfrak{I}$ with enough projective objects together with its class $\mathfrak{P} = \mathfrak{P}_{\mathfrak{I}}$ of projective objects. The ideal $\mathfrak{I}$ in a projective class is homological because, in the presence of enough projective objects, the universal homological functor with kernel $\mathfrak{I}$ is well-defined. There are two ways to construct this universal functor, which involve a localisation of categories in one step. Apostolos Beligiannis [2] first embeds the category $\mathfrak{T}$ into an Abelian category and then localises the latter at a Serre subcategory. The authors use the heart of a t-structure on a suitable derived category of chain complexes over $\mathfrak{T}$ in [19, §3.2.1]. In both cases, the morphisms in the relevant localisations can be computed using projective resolutions, so that the localisation is again a category with morphism *sets* instead of morphism classes.

## 2.1 Powers and intersections of ideals

At first, we do not care whether the ideals we are dealing with are homological. Let $\mathfrak{C}$ be an additive category. If $(\mathfrak{I}_\alpha)_{\alpha\in S}$ is a set of ideals, then the intersection $\bigcap \mathfrak{I}_\alpha$ is again an ideal. If $\mathfrak{I}_1, \mathfrak{I}_2 \subseteq \mathfrak{C}$ are ideals, define

$$\mathfrak{I}_1 \circ \mathfrak{I}_2(A,B) := \{f_1 \circ f_2 \mid f_1 \in \mathfrak{I}_1(X,B),\ f_2 \in \mathfrak{I}_2(A,X) \text{ for some } X \in\in \mathfrak{C}\}.$$

This is a subgroup of $\mathfrak{C}(A,B)$ because we can decompose $f_1 \circ f_2 + f_1' \circ f_2'$ as

$$A \xrightarrow{\begin{pmatrix} f_2 \\ f_2' \end{pmatrix}} X \oplus X' \xrightarrow{(f_1\ f_1')} B.$$

Thus $\mathfrak{I}_1 \circ \mathfrak{I}_2$ is an ideal in $\mathfrak{C}$. We have $\mathfrak{I}_1 \circ \mathfrak{I}_2 \subseteq \mathfrak{I}_1 \cap \mathfrak{I}_2$.

Now the *powers* of an ideal $\mathfrak{I} \subseteq \mathfrak{C}$ are defined recursively: we let $\mathfrak{I}^0 := \mathfrak{C}$ consist of all morphisms and define $\mathfrak{I}^n := \mathfrak{I}^{n-1} \circ \mathfrak{I}$ for $1 \leq n < \infty$. The sequence of ideals $(\mathfrak{I}^n)_{n\in\mathbb{N}}$ is decreasing, and we have $\mathfrak{I}^m \circ \mathfrak{I}^n = \mathfrak{I}^{m+n}$ for all $m, n \in \mathbb{N}$. If $\mathfrak{I}^n = \mathfrak{I}^{n+1}$ for some $n \in \mathbb{N}$, then $\mathfrak{I}^n = \mathfrak{I}^N$ for all $N \geq n$.

We also let $\mathfrak{I}^\infty := \bigcap_{n\in\mathbb{N}} \mathfrak{I}^n$ and $\mathfrak{I}^{n\infty} := (\mathfrak{I}^\infty)^n$. In general, the ideals $\mathfrak{I} \circ \mathfrak{I}^\infty$ and $\mathfrak{I}^\infty \circ \mathfrak{I}$ may differ from $\mathfrak{I}^\infty$ (cf. the remark after Proposition 4.2). Theorem 3.19 shows that $\mathfrak{I}^{n\infty} = \mathfrak{I}^{2\infty}$ for all $n \geq 2$ if $\mathfrak{I}$ is compatible with countable direct sums.

Now we replace the additive category $\mathfrak{C}$ by a triangulated category $\mathfrak{T}$ and restrict attention to homological ideals. It is not obvious whether the powers of a homological ideal are again homological. If $\mathfrak{I} = \ker F$, then a functor with kernel $\mathfrak{I}^2 \subset \mathfrak{I}$ contains more information than $F$ because it has a smaller kernel. Therefore, we cannot hope to construct such a functor out of $F$.

Nevertheless, I expect that products and intersections of homological ideals are again homological, at least if the categories in question are small to rule out set theoretic difficulties with localisation of categories. A proof could use Beligiannis' axiomatic characterisation of homological ideals. Since we only need the much easier case where there are enough projective objects, I have not completed the argument. Proposition 2.5 and Theorem 3.1 in [2] show that our "homological ideals" are exactly the "saturated $\Sigma$-stable ideals" in Beligiannis' notation. Clearly, products and intersections of $\Sigma$-stable ideals remain $\Sigma$-stable, and intersections of saturated ideals remain saturated. It is less clear whether products of saturated ideals remain saturated; the proof should involve the octahedral axiom.

Here we only consider the easy case of ideals with enough projective objects, where we can describe which objects are projective for products and intersections:

**Proposition 2.2** ([7, Proposition 3.3])**.** Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be homological ideals in $\mathfrak{T}$ with enough projective objects. Then $\mathfrak{I}_1 \circ \mathfrak{I}_2$ is a homological ideal with enough projective objects. An object $A$ of $\mathfrak{T}$ is $\mathfrak{I}_1 \circ \mathfrak{I}_2$-projective if and only if there are $\mathfrak{I}_j$-projective objects $P_j$ and an exact triangle $P_2 \to P \to P_1 \to P_2[1]$, such that $A$ is a direct summand of $P$.

**Proposition 2.3** ([7, Proposition 3.1])**.** Let $(\mathfrak{I}_\alpha)_{\alpha\in S}$ be a set of homological ideals in $\mathfrak{T}$ with enough projective objects. Suppose that $\mathfrak{T}$ has direct sums of cardinality $|S|$. Then $\mathfrak{I}_S := \bigcap_{\alpha\in S} \mathfrak{I}_\alpha$ is a homological ideal with enough projective objects. An object $A$ of $\mathfrak{T}$ is $\mathfrak{I}_S$-projective if and only if there are $\mathfrak{I}_\alpha$-projective objects $P_\alpha$ such that $A$ is a direct summand of $\bigoplus_{\alpha\in S} P_\alpha$.

We may use Christensen's results because of Remark 2.1.

**Definition 2.4.** Let $\mathfrak{I}$ be a homological ideal in a triangulated category $\mathfrak{T}$ with enough projective objects. We write $\mathfrak{P}_{\mathfrak{I}}$ for the class of $\mathfrak{I}$-projective objects, and $\mathfrak{P}_{\mathfrak{I}}^n$ for the class of $\mathfrak{I}^n$-projective objects for $n \in \mathbb{N} \cup \{\infty\}$.

The class $\mathfrak{P}_{\mathfrak{I}}$ is always closed under direct summands, suspensions, and direct sums that exist in $\mathfrak{T}$.

Propositions 2.2 and 2.3 show that the powers $\mathfrak{I}^n$ for $n \in \mathbb{N} \cup \{\infty\}$ have enough projective objects. Moreover, $\mathfrak{P}_{\mathfrak{I}}^n$ for $n \in \mathbb{N}$ consists of all direct summands of objects $A_n \in\in \mathfrak{T}$ for which there is an exact triangle $A_{n-1} \to A_n \to A_1 \to A_{n-1}[1]$ with $A_{n-1} \in\in \mathfrak{P}_{\mathfrak{I}}^{n-1}$, $A_1 \in\in \mathfrak{P}_{\mathfrak{I}}$; and $\mathfrak{P}_{\mathfrak{I}}^\infty$ consists of all retracts of objects of the form $\bigoplus_{n\in\mathbb{N}} A_n$ with $A_n \in\in \mathfrak{P}_{\mathfrak{I}}^n$. The phantom castle introduced in Definition 3.10 explicitly decomposes objects of $\mathfrak{P}_{\mathfrak{I}}^n$ into objects of $\mathfrak{P}_{\mathfrak{I}}$; essentially, its construction is the proof of Proposition 2.2.

**Example 2.5.** Let $\mathfrak{I}$ be a homological ideal with enough projective objects. If $\mathfrak{I} = \mathfrak{I}^2$, then Proposition 2.2 implies that $\mathfrak{P}_{\mathfrak{I}}$ is closed under extensions. Since this subcategory is always closed under direct summands, suspensions, isomorphism, and direct sums, $\mathfrak{P}_{\mathfrak{I}}$ is a localising subcategory of $\mathfrak{T}$.

Conversely, if $\mathfrak{P}_{\mathfrak{I}}$ is a triangulated subcategory, then $\mathfrak{I}$ and $\mathfrak{I}^2$ have the same projective objects. Since an ideal with enough projective objects is determined by its class of projective objects, this implies $\mathfrak{I} = \mathfrak{I}^2$.

Homological ideals with $\mathfrak{I} = \mathfrak{I}^2$, but possibly without enough projective objects, play an important role in [14] as a substitute for localising subcategories.

We usually know very little about the Abelian approximations generated by $\mathfrak{I}^n$ for $n \geq 2$, even if the situation for $\mathfrak{I}$ itself is rather simple. Derived functors for $\mathfrak{I}$ and $\mathfrak{I}^2$ do not seem closely related. This is particularly obvious in cases where $\mathfrak{I} \neq 0$ and $\mathfrak{I}^2 = 0$. For instance, this happens if $\mathfrak{I}$ is the kernel of the homology functor on the derived category of the category of Abelian groups. Here the universal $\mathfrak{I}$-exact functor is the homology functor to $\mathfrak{Ab}^{\mathbb{Z}}$; the universal $\mathfrak{I}^2$-exact functor is the Freyd embedding of the derived category into an Abelian category.

### 2.2 The phantom filtrations

Let $\mathfrak{I}$ be an ideal in an additive category $\mathfrak{C}$. Since $\mathfrak{I}^\alpha \subseteq \mathfrak{I}^\beta$ for $\alpha \geq \beta$, we get a decreasing filtration

$$\mathfrak{C}(A,B) = \mathfrak{I}^0(A,B) \supseteq \mathfrak{I}^1(A,B) \supseteq \mathfrak{I}^2(A,B) \supseteq \cdots \supseteq \mathfrak{I}^\infty(A,B) \supseteq \{0\},$$

called the *phantom filtration* [2]. We shall also need related filtrations on contravariant and covariant functors on $\mathfrak{C}$.

Let $G\colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{Ab}$ be a *contravariant* functor and $A \in\in \mathfrak{C}$. We define a decreasing filtration

$$G(A) = \mathfrak{I}^0 G(A) \supseteq \mathfrak{I}^1 G(A) \supseteq \mathfrak{I}^2 G(A) \supseteq \cdots \supseteq \mathfrak{I}^\infty G(A) \supseteq \{0\}$$

on $G(A)$ by

$$\mathfrak{I}^\alpha G(A) := \{f^*(\xi) \mid f \in \mathfrak{I}^\alpha(A,B),\, \xi \in G(B) \text{ for some } B \in\in \mathfrak{C}\}.$$

If we apply this construction to the representable functor $\mathfrak{C}(\text{\textvisiblespace}, B)$ we get back the filtration $\mathfrak{I}^\alpha(A,B)$ on $\mathfrak{C}(A,B)$. If $G$ is compatible with direct sums, then (5.2) asserts that $\bigcap_{n\in\mathbb{N}} \mathfrak{I}^n G(A) = \mathfrak{I}^\infty G(A)$.
The functoriality of $G$ restricts to maps

$$\mathfrak{I}^\beta(A,B) \otimes \mathfrak{I}^\alpha G(B) \to \mathfrak{I}^{\alpha+\beta} G(A), \qquad f \otimes x \mapsto f^*(x),$$

for all $\alpha, \beta$. In particular, $\mathfrak{I}^\alpha G$ is a contravariant functor on $\mathfrak{C}$. The ideal $\mathfrak{I}^\beta$ acts trivially on the subquotients $\mathfrak{I}^\alpha G(A)/\mathfrak{I}^{\alpha+\beta} G(A)$, which therefore descend to functors on the quotient category $\mathfrak{C}/\mathfrak{I}^\beta$.

We may also view $G$ as a right module over the category $\mathfrak{C}$ and $\mathfrak{C}/\mathfrak{I}^\alpha$ as a $\mathfrak{C}$-bimodule. The quotient $G/\mathfrak{I}^\alpha G$ corresponds to the right $\mathfrak{C}$-module $G \otimes_{\mathfrak{C}} \mathfrak{C}/\mathfrak{I}^\alpha$.
For a covariant functor $F\colon \mathfrak{C} \to \mathfrak{Ab}$, we define an increasing filtration

$$\{0\} = F : \mathfrak{I}^0(A) \subseteq F : \mathfrak{I}^1(A) \subseteq F : \mathfrak{I}^2(A) \subseteq \cdots \subseteq F : \mathfrak{I}^\infty(A) \subseteq F(A)$$

for any $A \in\in \mathfrak{C}$ by

$$F : \mathfrak{I}^\alpha(A) := \{x \in F(A) \mid f_*(x) = 0 \text{ for all } f \in \mathfrak{I}^\alpha(A,B),\, B \in\in \mathfrak{C}\}.$$

If $\mathfrak{I}$ and $F$ are compatible with direct sums, then $F : \mathfrak{I}^\infty(A) = \bigcup_{n\in\mathbb{N}} F : \mathfrak{I}^n(A)$ (cf. Theorem 5.1), but this need not be the case in general.
The functoriality of $F$ restricts to maps

$$\mathfrak{I}^\beta(A,B) \otimes F : \mathfrak{I}^{\alpha+\beta}(B) \to F : \mathfrak{I}^\alpha(A), \qquad f \otimes x \mapsto f_*(x),$$

for all $\alpha, \beta$. In particular, $F : \mathfrak{I}^\alpha$ is a covariant functor on $\mathfrak{C}$. The ideal $\mathfrak{I}^\beta$ acts trivially on the subquotients $F : \mathfrak{I}^{\alpha+\beta}(A) \bigm/ F : \mathfrak{I}^\alpha(A)$, which therefore descend to functors on $\mathfrak{C}/\mathfrak{I}^\beta$.

We may also view $F$ as a left module over the category $\mathfrak{C}$ and $\mathfrak{C}/\mathfrak{I}^\alpha$ as a $\mathfrak{C}$-bimodule. Then $F : \mathfrak{I}^\alpha$ corresponds to the left $\mathfrak{C}$-module $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{C}/\mathfrak{I}^\alpha, F)$.

The filtration $F : \mathfrak{I}^\alpha(A)$ is closely related to projective resolutions of $A$. In contrast, the filtration

$$\mathfrak{I}^\alpha F(A) := \{f_*(\xi) \in F(A) \mid f \in \mathfrak{I}^\alpha(B,A),\, \xi \in F(B) \text{ for some } B \in\in \mathfrak{C}\}$$

is related to *injective* (co)resolutions. There is also an increasing filtration $G : \mathfrak{I}^n$ for a contravariant functor $G$. The filtrations $\mathfrak{I}^\alpha F$ and $G : \mathfrak{I}^\alpha$ will not be used in this article.

## 3 From projective resolutions to complementary pairs

First we refine a projective resolution by adjoining certain phantom maps. This yields the *phantom tower* over an object (cf. also [2]). We show that the projective resolution determines this tower uniquely up to non-canonical isomorphism. There is another tower over an object, the *cellular approximation tower*. These two towers are related by various commuting diagrams and exact triangles; we call the collection of all these exact or commuting triangles the *phantom castle*.

The goal of this section is to show that the categories $\langle\mathfrak{P}_{\mathfrak{I}}\rangle$ and $\mathfrak{N}_{\mathfrak{I}}$ are complementary if $\mathfrak{I}$ is compatible with direct sums. Before we come to that, we recall the notion of complementary pair of subcategories and define what it means for an ideal to be compatible with countable direct sums. The main ingredients in the proof are the homotopy colimits of the phantom tower and the cellular approximation tower. Finally, we describe a method for checking that a given localising subcategory is reflective, that is, part of a complementary pair.

All results involving infinite direct sums require that the category $\mathfrak{T}$ has countable direct sums. Triangulated categories involving bivariant K-theory have no more than countable direct sums because of built-in separability assumptions that make the analysis behind the scenes work. This is why we only use *countable* direct sums. Of course, everything remains true if we drop the word "countable" or replace it by another cardinality constraint.

A triangulated subcategory of $\mathfrak{T}$ is called *localising* (more precisely, $\aleph_0$*-localising*) if it is closed under countable direct sums. Localising subcategories are automatically thick, that is, closed under direct summands (cf. [21]).

Let $\mathfrak{I}$ be a homological ideal in a triangulated category $\mathfrak{T}$. Recall that an $\mathfrak{I}$*-projective resolution* of an object $A$ of $\mathfrak{T}$ is a chain complex

$$\cdots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} P_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0$$

of $\mathfrak{I}$-projective objects $P_n$, augmented by a map $\pi_0\colon P_0 \to A$, such that the augmented chain complex is $\mathfrak{I}$-exact. If $\mathfrak{I} = \ker F$ for a stable homological functor $F$ to some Abelian category $\mathfrak{A}$, then $\mathfrak{I}$*-exactness* means that the chain complex

$$\cdots \xrightarrow{F(\delta_{n+1})} F(P_n) \xrightarrow{F(\delta_n)} F(P_{n-1}) \xrightarrow{F(\delta_{n-1})} \cdots \xrightarrow{F(\delta_1)} F(P_0) \xrightarrow{F(\pi_0)} F(A)$$

is exact in $\mathfrak{A}$. We say that $\mathfrak{I}$ has *enough projective objects* if each $A \in\in \mathfrak{T}$ has such an $\mathfrak{I}$-projective resolution.

### 3.1 The phantom tower

**Definition 3.1.** A *phantom tower* over an object $A$ of $\mathfrak{T}$ is a diagram

$$
\begin{tikzcd}
A = N_0 \arrow[rr, "\iota_0^1"] & & N_1 \arrow[rr, "\iota_1^2"] \arrow[dl, "\circ" marking, "\varepsilon_0"] & & N_2 \arrow[rr, "\iota_2^3"] \arrow[dl, "\circ" marking, "\varepsilon_1"] & & N_3 \arrow[rr] \arrow[dl, "\circ" marking, "\varepsilon_2"] & & \cdots \arrow[dl, "\circ" marking] \\
& P_0 \arrow[ul, "\pi_0"] & & P_1 \arrow[ll, "\circ" marking, "\delta_1"] \arrow[ul, "\pi_1"] & & P_2 \arrow[ll, "\circ" marking, "\delta_2"] \arrow[ul, "\pi_2"] & & P_3 \arrow[ll, "\circ" marking, "\delta_3"] \arrow[ul, "\pi_3"] & & \cdots \arrow[ll, "\circ" marking] \arrow[ul]
\end{tikzcd}
\tag{3.1}
$$

with $\mathfrak{J}$-phantom maps $\iota_n^{n+1}$ and $\mathfrak{J}$-projective objects $P_n$ for $n \in \mathbb{N}$, such that the triangles

$$P_n \xrightarrow{\pi_n} N_n \xrightarrow{\iota_n^{n+1}} N_{n+1} \xrightarrow{\varepsilon_n} P_n[1]$$

in (3.1) are exact for all $n \in \mathbb{N}$ and the other triangles in (3.1) commute, that is, $\delta_{n+1} = \varepsilon_n \circ \pi_{n+1}$ for all $n \in \mathbb{N}$. Notice our convention that circled arrows are maps of degree 1.

Since the maps $\delta_n$ in the phantom tower have degree 1, we slightly modify our notion of projective resolution, letting the boundary maps have degree 1.

**Lemma 3.2.** The maps $\delta_n$ for $n \in \mathbb{N}_{\geq 1}$ and $\pi_0$ in a phantom tower over $A$ form an $\mathfrak{J}$-projective resolution of $A$. Conversely, any $\mathfrak{J}$-projective resolution can be embedded in a phantom tower, which is unique up to non-canonical isomorphism.

A morphism $f\colon A \to A'$ lifts (non-canonically) to a morphism between two given phantom towers over $A$ and $A'$. A chain map between projective resolutions of $A$ and $A'$ extends to the phantom towers that contain these resolutions.

*Proof.* Let $P_n$, $\pi_0$, and $\delta_n$ be part of a phantom tower over $A$. The objects $P_n$ are $\mathfrak{J}$-projective by definition, and $\delta_n \circ \delta_{n+1} = 0$ for all $n \in \mathbb{N}$ and $\pi_0 \circ \delta_1 = 0$ because these products involve two consecutive arrows in an exact triangle. Hence the maps $\delta_n$ and $\pi_0$ form a chain complex. We claim that it is $\mathfrak{J}$-exact.

Let $F$ be a stable homological functor with $\ker F = \mathfrak{J}$. Recall that a chain complex is $\mathfrak{J}$-exact if and only if its $F$-image is exact in the usual sense by [19, Lemma 3.9]. The exact triangles in the phantom tower yield short exact sequences

$$F_{*+1}(N_{n+1}) \rightarrowtail F_*(P_n) \twoheadrightarrow F_*(N_n)$$

for all $n \in \mathbb{N}$ because $\iota_n^{n+1} \in \mathfrak{J}$; here $F_*(A) := F(A[-n])$. Splicing these extensions as in the definition of the Yoneda product, we get an exact chain complex. Since this chain complex is

$$\cdots \to F_{*+2}(P_2) \to F_{*+1}(P_1) \to F_*(P_0) \to F_*(A) \to 0,$$

we have got an $\mathfrak{J}$-exact chain complex and hence an $\mathfrak{J}$-projective resolution.

Now let $\pi_0 \colon P_0 \to A$ and $\delta_n \colon P_n \to P_{n-1}[1]$ for $n \in \mathbb{N}_{\geq 1}$ form an $\mathfrak{J}$-projective resolution. We recursively construct the triangles that comprise the phantom tower. To begin with, we embed $\pi_0$ in an exact triangle

$$P_0 \xrightarrow{\pi_0} A \xrightarrow{\iota_0^1} N_1 \xrightarrow{\varepsilon_0} P_0[1].$$

Since $\pi_0$ is $\mathfrak{J}$-epic, $\iota_0^1$ is an $\mathfrak{J}$-phantom map and $\varepsilon_0$ is $\mathfrak{J}$-monic. Thus our exact triangle yields a short exact sequence

$$\mathfrak{T}_{*+1}(P, N_1) \rightarrowtail \mathfrak{T}_*(P, P_0) \twoheadrightarrow \mathfrak{T}_*(P, A)$$

for any $\mathfrak{J}$-projective object $P$. In particular, this applies to $P = P_1$ and shows that $\delta_1$ factors uniquely as $\delta_1 = \varepsilon_0 \circ \pi_1$ with $\pi_1 \in \mathfrak{T}_0(P_1, N_1)$.

We claim that $\pi_1$ is $\mathfrak{J}$-epic. Let $F$ be a defining functor for $\mathfrak{J}$ as above. Then $F(P_1) \to F(P_0) \to F(A)$ is exact at $F(P_0)$, and $F(N_1) \rightarrowtail F(P_0) \twoheadrightarrow F(A)$ is a short exact sequence. Hence the range of $F(\delta_1)$ is isomorphic to $F(N_1)$. This implies that $F(\pi_1)$ is an epimorphism, that is, $\pi_1$ is $\mathfrak{J}$-epic.

Thus the maps $\pi_1$ and $\delta_n$ for $n \in \mathbb{N}_{\geq 2}$ form an $\mathfrak{J}$-projective resolution of $N_1$. We may now repeat the above process and recursively construct the phantom tower. Thus any $\mathfrak{J}$-projective resolution embeds in a phantom tower. Furthermore, since the exact triangle containing a given morphism is unique up to isomorphism and the liftings $\pi_1$ above are unique, there is, up to isomorphism, only one phantom tower that contains a given $\mathfrak{J}$-projective resolution. Of course, different resolutions yield different phantom towers.

Finally, it remains to lift a morphism $f \colon A \to A'$ to a transformation between two given phantom towers. First we can lift $f$ to a chain map between the $\mathfrak{J}$-projective resolutions contained in these towers (cf. [19]); let $P_n(f) \colon P_n \to P'_n$ for $n \in \mathbb{N}$ be this chain map. It remains to construct maps $N_n(f) \colon N_n \to N'_n$ that together with the maps $P_n(f)$ intertwine the various maps in the phantom towers. We already have the map $N_0(f) = f$. The triangulated category axioms provide a map $N_1(f) \colon N_1 \to N'_1$ making the diagram

$$\begin{tikzcd}
P_0 \arrow[r, "\pi_0"] \arrow[d, "P_0(f)"] & A \arrow[r, "\iota_0^1"] \arrow[d, "f"] & N_1 \arrow[r, "\varepsilon_0"] \arrow[d, "N_1(f)"] & P_0[1] \arrow[d, "P_0(f)[1]"] \\
P'_0 \arrow[r, "\pi'_0"] & A' \arrow[r, "\iota'^1_0"] & N'_1 \arrow[r, "\varepsilon'_0"] & P'_0[1]
\end{tikzcd}$$

commute. We claim that $N_1(f) \circ \pi_1 = \pi_1' \circ P_1(f)$. As above, we get short exact sequences

$$\mathfrak{T}_{*+1}(P_1, N_1') \rightarrowtail \mathfrak{T}_*(P_1, P_0') \twoheadrightarrow \mathfrak{T}_*(P_1, A').$$

Hence it suffices to check $\varepsilon_0' \circ N_1(f) \circ \pi_1 = \varepsilon_0' \circ \pi_1' \circ P_1(f) = \delta_1' \circ P_1(f)$. But this is true because $\varepsilon_0' \circ N_1(f) = P_0(f) \circ \varepsilon_0$ and the maps $P_n(f)$ form a chain map. Thus the map $N_1(f)$ has all required properties. Iterating this construction, we get the maps $N_n(f)$ for all $n \in \mathbb{N}$. By the way, they need not be unique even if the maps $P_n(f)$ are fixed. Q.E.D.

The following definition formalises an important property of the maps $\iota_n^{n+1}$ in a phantom tower.

**Definition 3.3.** Let $\mathfrak{I} \subseteq \mathfrak{T}$ be an ideal. Let $A, B \in\in \mathfrak{T}$. We call $f \in \mathfrak{I}(A, B)$ $\mathfrak{I}$-*versal* if, for any $C \in\in \mathfrak{T}$, any $g \in \mathfrak{I}(A, C)$ factors as $g = h \circ f$ for some $h \in \mathfrak{T}(B, C)$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[d, dotted, "\exists\ h"] \\ & C \end{tikzcd}$$

We do *not* require this factorisation to be unique.

Since $\mathfrak{I}$ is an ideal, any map of the form $h \circ f$ belongs to $\mathfrak{I}$.

**Lemma 3.4.** The maps $\iota_n^{n+1}$ in a phantom tower are $\mathfrak{I}$-versal for all $n \in \mathbb{N}$.

*Proof.* Let $f \in \mathfrak{I}(N_n, B)$. Since $P_n$ is $\mathfrak{I}$-projective, $\mathfrak{I}_*(P_n, B) = 0$. Thus $f \circ \pi_n = 0$. This forces $f$ to factor through $\iota_n^{n+1}$ because $\mathfrak{T}_*(\llcorner\!\lrcorner, B)$ is cohomological. Q.E.D.

**Lemma 3.5.** Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be ideals in a triangulated category. If $f_1 \in \mathfrak{I}_1(B, C)$ and $f_2 \in \mathfrak{I}_2(A, B)$ are $\mathfrak{I}_1$- and $\mathfrak{I}_2$-versal maps, respectively, then $f_1 \circ f_2 \colon A \to C$ is $\mathfrak{I}_1 \circ \mathfrak{I}_2$-versal.

*Proof.* Let $h \in \mathfrak{I}_1 \circ \mathfrak{I}_2(A, D)$, write $h = h_1 \circ h_2$ with $h_1 \in \mathfrak{I}_1$ and $h_2 \in \mathfrak{I}_2$. Using versality of $f_1$ and $f_2$, we find the maps $h_2'$ and $h'$ in the following diagram:

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[dr, "h_2"'] & B \arrow[r, "f_1"] \arrow[d, dotted, "\exists\ h_2'"] & C \arrow[d, dotted, "\exists\ h'"] \\ & \bullet \arrow[r, "h_1"'] & D. \end{tikzcd}$$

Thus $h$ factors through $f_1 \circ f_2$ as required. Q.E.D.

As a consequence, the maps

$$\iota_n^{n+k} := \iota_{n+k-1}^{n+k} \circ \dots \circ \iota_n^{n+1} \colon N_n \to N_{n+k}$$

in a phantom tower are $\mathfrak{I}^k$-versal for all $n, k \in \mathbb{N}$.

**Lemma 3.6.** A map $f\colon A \to B$ is $\mathfrak{I}^k$-versal if and only if

$$\mathfrak{I}^k(A, C) = \operatorname{range}\bigl(f^*\colon \mathfrak{T}(B, C) \to \mathfrak{T}(A, C)\bigr)$$

for all $C \in\in \mathfrak{T}$.

Let $f\colon A \to B$ be $\mathfrak{I}^k$-versal. If $F\colon \mathfrak{T} \to \mathfrak{Ab}$ is homological, then

$$F : \mathfrak{I}^k(A) = \ker\bigl(f_*\colon F(A) \to F(B)\bigr);$$

if $G\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{Ab}$ is cohomological, then

$$\mathfrak{I}^k G(A) = \operatorname{range}\bigl(f^*\colon G(B) \to G(A)\bigr).$$

*Proof.* This follows immediately from the definitions. Q.E.D.

As a consequence, we can compute the filtrations $F : \mathfrak{I}^k(A)$ and $\mathfrak{I}^k G(A)$ of § 2.2 from the phantom tower.

### 3.2 The phantom castle

Now we extend the phantom tower to the phantom castle, which contains among other things the cellular approximation tower. We start with a phantom tower over some object $A \in\in \mathfrak{T}$. Let

$$\iota^n := \iota_{n-1}^n \circ \dots \circ \iota_1^0\colon A = N_0 \to N_n$$

and embed $\iota^n$ in an exact triangle

$$\tilde{A}_n \xrightarrow{\alpha_n} A \xrightarrow{\iota^n} N_n \xrightarrow{\gamma_n} \tilde{A}_n[1]. \tag{3.2}$$

The octahedral axiom relates the mapping cones $\tilde{A}_{n+1}$, $P_n$, and $\tilde{A}_n$ of the maps $\iota^{n+1}$, $\iota_n^{n+1}$, and $\iota^n$ because $\iota^{n+1} = \iota_n^{n+1} \circ \iota^n$ (cf. [21, Proposition I.4.6] or [16, Proposition A.1]). More precisely, the octahedral axiom allows us to choose maps

$$\tilde{A}_n \xrightarrow{\alpha_n^{n+1}} \tilde{A}_{n+1} \xrightarrow{\sigma_n} P_n \xrightarrow{\kappa_n} \tilde{A}_n[1], \tag{3.3}$$

such that this triangle is exact and the following diagram commutes:

$$
\begin{tikzcd}
N_n \arrow[r, "\iota_n^{n+1}"] \arrow[d, "\gamma_n"', "\circ" marking] & N_{n+1} \arrow[d, "\gamma_{n+1}"', "\circ" marking] \arrow[dr, "\varepsilon_n", "\circ" marking] & & \\
\tilde{A}_n \arrow[r, "\alpha_n^{n+1}"] \arrow[dr, "\alpha_n"'] & \tilde{A}_{n+1} \arrow[r, "\sigma_n"] \arrow[d, "\alpha_{n+1}"] & P_n \arrow[r, "\kappa_n", "\circ" marking] \arrow[d, "\pi_n"'] & \tilde{A}_n \\
& A \arrow[r, "\iota^n"] \arrow[dr, "\iota^{n+1}"'] & N_n \arrow[ur, "\gamma_n"', "\circ" marking] \arrow[d, "\iota_n^{n+1}"] & \\
& & N_{n+1} &
\end{tikzcd}
\tag{3.4}
$$

In addition, we can achieve that the triangle

$$N_n[-1] \to \tilde{A}_{n+1} \xrightarrow{\binom{\alpha_{n+1}}{\sigma_n}} A \oplus P_n \xrightarrow{(\iota^n, \pi_n)} N_n \tag{3.5}$$

is exact, that is, the square in the middle of (3.4) is homotopy Cartesian and the diagonal of the top right square provides its differential.

**Lemma 3.7.** The object $\tilde{A}_n$ is $\mathfrak{I}^n$-projective for each $n \in \mathbb{N}$.

*Proof by induction on $n$.* The case $n = 0$ is clear. Since $P_n \in\in \mathfrak{P}_{\mathfrak{I}}$ for all $n \in \mathbb{N}$, the exact triangles (3.3) and Proposition 2.2 provide the induction step. Q.E.D.

Furthermore, the map $\alpha_n : \tilde{A}_n \to A$ is $\mathfrak{I}^n$-epic because $\iota^n \in \mathfrak{I}^n$, so that it is the first step of an $\mathfrak{I}^n$-projective resolution of $A$. This provides another explanation why the map $\iota^n$ is $\mathfrak{I}$-versal (compare Lemma 3.4).

**Remark 3.8.** The cone of the map $\iota_m^{m+k}$ is $\mathfrak{I}^k$-projective for all $m, k \in \mathbb{N}$ by a similar argument. Hence

$$A = N_0 \xrightarrow{\iota_0^k} N_k \xrightarrow{\iota_k^{2k}} N_{2k} \xrightarrow{\iota_{2k}^{3k}} N_{3k} \to \cdots$$

together with the exact triangles that contain the maps $\iota_{jk}^{jk+k}$ is an $\mathfrak{I}^k$-phantom tower and hence yields an $\mathfrak{I}^k$-projective resolution by Lemma 3.2.

As a result, an $\mathfrak{I}$-phantom tower determines $\mathfrak{I}^k$-phantom towers for all $k \in \mathbb{N}$.

**Definition 3.9.** The sequence of exact triangles (3.3) is called the *cellular approximation tower* over $A$.

The motivation for our terminology is the following. If $A$ has a projective resolution of finite length, then we can choose a phantom tower with $P_n = 0$ for $n \gg 0$. Suppose, in addition, that $A$ belongs to the thick triangulated subcategory generated by $\mathfrak{P}_\mathfrak{J}$. Then the proof of Proposition 4.5 yields $N_n = 0$ for $n \gg 0$. The exact triangles (3.2) mean that the maps $\alpha_n \colon \tilde{A}_n \to A$ become invertible for $n \gg 0$, that is, $\tilde{A}_n \cong A$. Therefore, we think of the objects $N_n$ as "obstructions" that should get smaller for $n \to \infty$, and of the objects $\tilde{A}_n$ as better and better approximations to $A$. They are called "$\mathfrak{P}_\mathfrak{J}$-cellular" because they are constructed out of $\mathfrak{J}$-projective objects —the cells— by iterated exact triangles.

**Definition 3.10.** A *phantom castle* over $A$ is a sequence of objects $N_n$, $P_n$, $\tilde{A}_n$ with maps $\iota_n^{n+1}$, $\pi_n$, $\varepsilon_n$, $\iota^n$, $\alpha_n$, $\gamma_n$, $\sigma_n$, $\kappa_n$ such that the triangles (4.1), (3.2), (3.3), and (3.5) are exact and the diagram (3.4) commutes.

We will use most of the information encoded in this definition to identify the spectral sequences generated by the phantom tower and the cellular approximation tower; only the commutativity of the square in the middle of (3.4) and the exact sequence (3.5) seem irrelevant in the following.

### 3.3 Complementary pairs of subcategories and localisation

We call two thick subcategories $\mathfrak{L}$ and $\mathfrak{N}$ of $\mathfrak{T}$ *complementary* if $\mathfrak{T}_*(L, N) = 0$ for all $L \in\in \mathfrak{L}$, $N \in\in \mathfrak{N}$ and, for any $A \in\in \mathfrak{T}$, there is an exact triangle $L \to A \to N \to L[1]$ with $L \in\in \mathfrak{L}$ and $N \in\in \mathfrak{N}$ (cf. [16, Definition 2.8]). Similar situations have been studied by various authors, under various names, such as localisation pairs, stable t-structures, torsion pairs; a complementary pair is equivalent to a localisation functor $L$ on $\mathfrak{T}$, where $\mathfrak{L}$ is the class of $L$-local objects and $\mathfrak{N}$ is the class of $L$-acyclic objects.

The following assertions are contained in [16, Proposition 2.9]. Let $(\mathfrak{L}, \mathfrak{N})$ be complementary. Then the exact triangle $L \to A \to N \to L[1]$ with $L \in\in \mathfrak{L}$ and $N \in\in \mathfrak{N}$ is unique and functorial, and the resulting functors $L\colon \mathfrak{T} \to \mathfrak{L}$ and $N\colon \mathfrak{T} \to \mathfrak{N}$ mapping $A$ to $L$ and $N$, respectively, are left adjoint to the embedding functor $\mathfrak{L} \to \mathfrak{T}$ and right adjoint to the embedding functor $\mathfrak{N} \to \mathfrak{T}$, respectively. That is, the subcategory $\mathfrak{N}$ is reflective and $\mathfrak{L}$ is coreflective. The composite functors $\mathfrak{L} \to \mathfrak{T} \to \mathfrak{T}/\mathfrak{N}$ and $\mathfrak{N} \to \mathfrak{T} \to \mathfrak{T}/\mathfrak{L}$ are equivalences of categories.

Conversely, let $\mathfrak{N} \subseteq \mathfrak{T}$ be a reflective subcategory and let $N\colon \mathfrak{T} \to \mathfrak{N}$ be the left adjoint of the embedding functor $\mathfrak{N} \to \mathfrak{T}$. Let

$$\mathfrak{L} = \{A \in\in \mathfrak{T} \mid N(A) = 0\}$$

be the *left orthogonal complement* of $\mathfrak{N}$. Then $(\mathfrak{L}, \mathfrak{N})$ is a complementary pair of subcategories, and $\mathfrak{L}$ is the only possible partner for $\mathfrak{N}$. Thus complementary pairs are essentially the same as reflective subcategories. Dually, a subcategory $\mathfrak{L}$ is coreflective if and only if it is part of a complementary pair $(\mathfrak{L},\mathfrak{N})$, and the only candidate for $\mathfrak{N}$ is the right orthogonal complement of $\mathfrak{L}$.

If $F\colon \mathfrak{T} \to \mathfrak{C}$ is a covariant functor, then its *localisation* $\mathbb{L}F$ with respect to $\mathfrak{N}$ is defined by $\mathbb{L}F := F \circ L$, where $L\colon \mathfrak{T} \to \mathfrak{L}$ is the right adjoint of the embedding $\mathfrak{L} \to \mathfrak{T}$. The natural maps $L(A) \to A$ provide a natural transformation $\mathbb{L}F \Rightarrow F$. If $G\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{C}$ is a contravariant functor, then the localisation $G \circ L$ is denoted by $\mathbb{R}G$. It comes together with a natural transformation $G \Rightarrow \mathbb{R}G$.

This localisation process is an important tool to construct functors. Special cases are derived functors in homological algebra and the domain of the Baum-Connes assembly map (cf. [16]).

Although the definition of a complementary pair is symmetric, the subcategories $\mathfrak{L}$ and $\mathfrak{N}$ have a rather different nature in most examples. Usually, one of them —here it is always $\mathfrak{N}$— is defined directly and the other one is only described by generators. This makes it hard to tell which objects it contains and to find the exact triangles needed for complementarity.

Here homological ideals help. Let $\mathfrak{I}$ be a homological ideal with enough projective objects in a triangulated category $\mathfrak{T}$. Let $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ be the localising subcategory generated by $\mathfrak{P}_{\mathfrak{I}}$, that is, the smallest localising subcategory of $\mathfrak{T}$ that contains $\mathfrak{P}_{\mathfrak{I}}$. Since the name "projective" is already taken, we call objects of $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ $\mathfrak{P}_{\mathfrak{I}}$*-cellular*. We have $\mathfrak{P}_{\mathfrak{I}}^n \subseteq \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ for all $n \in \mathbb{N} \cup \{\infty, 2\infty, \dots\}$ by Propositions 2.2 and 2.3.

**Definition 3.11.** Let $\mathfrak{N}_{\mathfrak{I}}$ be the full subcategory of $\mathfrak{I}$-contractible objects, that is, objects $N$ with $\mathrm{id}_N \in \mathfrak{I}(N,N)$.

An object $N$ is $\mathfrak{I}$-contractible if and only if $0 \to N$ is an $\mathfrak{I}$-projective resolution. Thus all $\mathfrak{I}$-derived functors vanish on $\mathfrak{N}_{\mathfrak{I}}$.

Now the following question arises: is the pair of subcategories $(\langle \mathfrak{P}_{\mathfrak{I}} \rangle, \mathfrak{N}_{\mathfrak{I}})$ complementary? It is evident that $\mathfrak{T}(P,N) = 0$ if $P \in \mathfrak{P}_{\mathfrak{I}}$ and $N \in \mathfrak{N}_{\mathfrak{I}}$. This extends to $P \in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ because the left orthogonal complement of $\mathfrak{N}_{\mathfrak{I}}$ is localising. This is the easy half of the definition of a complementary pair. The other, non-trivial half requires an additional condition on the ideal $\mathfrak{I}$.

### 3.4 Compatibility with direct sums

**Definition 3.12.** An ideal $\mathfrak{I}$ is called *compatible with countable direct sums* if, for any countable family $(A_i)_{i \in I}$ of objects of $\mathfrak{T}$, the canonical isomorphism

$$\mathfrak{T}\Bigl(\bigoplus_{i \in I} A_i, B\Bigr) \cong \prod_{i \in I} \mathfrak{T}(A_i, B)$$

restricts to an isomorphism $\mathfrak{I}\bigl(\bigoplus_{i \in I} A_i, B\bigr) \cong \prod_{i \in I} \mathfrak{I}(A_i, B)$.

An ideal $\mathfrak{I}$ is compatible with countable direct sums if and only if the following holds: given countable families of objects $(A_i)$, $(B_i)$ and maps $f_i \in \mathfrak{I}(A_i, B_i)$ for $i \in I$, we have $\bigoplus f_i \in \mathfrak{I}\big(\bigoplus A_i, \bigoplus B_i\big)$.

Recall that direct sums of exact triangles are again exact (cf. [21]) and that the ideal determines and is determined by the classes of $\mathfrak{I}$-epimorphisms or of $\mathfrak{I}$-monomorphisms. Therefore, $\mathfrak{I}$ is compatible with direct sums if and only if direct sums of $\mathfrak{I}$-monomorphisms are again $\mathfrak{I}$-monomorphisms, if and only if direct sums of $\mathfrak{I}$-epimorphisms are again $\mathfrak{I}$-epimorphisms.

Moreover, if $\mathfrak{I}$ is compatible with countable direct sums, then a direct sum of $\mathfrak{I}$-equivalences is again an $\mathfrak{I}$-equivalence, and $\mathfrak{N}_{\mathfrak{I}}$ is a localising subcategory of $\mathfrak{T}$; and a direct sum of phantom castles over $A_i$ is a phantom castle over $\bigoplus A_i$.

**Example 3.13.** Let $F$ be a stable homological functor or an exact functor to another triangulated category, and suppose that $F$ commutes with countable direct sums. Then $\ker F$ is a homological ideal and compatible with countable direct sums.

This example is, in fact, already the most general case:

**Proposition 3.14.** Let $\mathfrak{T}$ be a triangulated category with countable direct sums and let $\mathfrak{I}$ be a homological ideal in $\mathfrak{T}$. Let $F\colon \mathfrak{T} \to \mathfrak{A}_{\mathfrak{I}}\mathfrak{T}$ be a universal $\mathfrak{I}$-exact stable homological functor. The ideal $\mathfrak{I}$ is compatible with countable direct sums if and only if the Abelian category $\mathfrak{A}_{\mathfrak{I}}\mathfrak{T}$ has exact countable direct sums and the functor $F\colon \mathfrak{T} \to \mathfrak{A}_{\mathfrak{I}}\mathfrak{T}$ commutes with countable direct sums.

*Proof.* One direction is just the assertion in Example 3.13. The other direction requires some description of the universal functor $F$. We use the description in [19], which starts with the homotopy category $\mathrm{Ho}(\mathfrak{T})$ of chain complexes with entries in $\mathfrak{T}$. Since $\mathfrak{T}$ has countable direct sums, so has $\mathrm{Ho}(\mathfrak{T})$. The $\mathfrak{I}$-exact chain complexes form a thick subcategory $\mathcal{E}$ of $\mathrm{Ho}(\mathfrak{T})$; it is closed under countable direct sums because $\mathfrak{I}$ is compatible with countable direct sums. Hence the localisation $\mathrm{Ho}(\mathfrak{T})/\mathcal{E}$ still has countable direct sums.

The Abelian approximation $\mathfrak{A}_{\mathfrak{I}}\mathfrak{T}$ is equivalent to the heart of a canonical truncation structure on $\mathrm{Ho}(\mathfrak{T})/\mathcal{E}$ described in [19] and consists of chain complexes that are exact in degrees not equal to 0. The universal functor $F$ is the obvious one, viewing an object of $\mathfrak{T}$ as a chain complex supported in degree 0. It is evident that the subcategory $\mathfrak{A}_{\mathfrak{I}}\mathfrak{T} \subseteq \mathrm{Ho}(\mathfrak{T})/\mathcal{E}$ is closed under countable direct sums. Countable direct sums of extensions in $\mathfrak{A}_{\mathfrak{I}}\mathfrak{T}$ remain extensions because the analogous assertion holds for direct sums of exact triangles in any triangulated category (cf. [21]) and extensions in the heart are related to exact triangles in the ambient triangulated category. Clearly, the functor $\mathfrak{T} \to \mathfrak{A}_{\mathfrak{I}}\mathfrak{T}$ preserves countable direct sums. Q.E.D.

**Example 3.15.** The ideal of finite rank operators on the category of vector spaces is an ideal that is not compatible with countable direct sums.

### 3.5 Complementarity and structure of cellular objects

The results in this section generalise results of Apostolos Beligiannis (cf. [2, Theorem 6.5], [2, Corollary 5.12]) in the case where $\mathfrak{P}_{\mathfrak{I}}$ is generated by a single compact object.

**Theorem 3.16.** Let $\mathfrak{T}$ be a triangulated category with countable direct sums, and let $\mathfrak{I}$ be a homological ideal in $\mathfrak{T}$ with enough projective objects. Suppose that $\mathfrak{I}$ is compatible with countable direct sums. Then the pair of localising subcategories $(\langle\mathfrak{P}_{\mathfrak{I}}\rangle, \mathfrak{N}_{\mathfrak{I}})$ in $\mathfrak{T}$ is complementary.

We will present two independent proofs, one using phantom towers, the other cellular approximation towers. Both require homotopy colimits:

**Definition 3.17.** Let $(D_n, \varphi_n^{n+1})$ be an inductive system in $\mathfrak{T}$. Define the *shift*

$$S\colon \bigoplus D_n \to \bigoplus D_n, \qquad S|_{D_n} : D_n \xrightarrow{\varphi_n^{n+1}} D_{n+1} \subseteq \bigoplus D_n.$$

The *homotopy colimit* $\text{ho-}\varinjlim (D_n, \varphi_n^{n+1})$ is the third leg in the exact triangle

$$\bigoplus D_n \xrightarrow{\text{id}-S} \bigoplus D_n \longrightarrow \text{ho-}\varinjlim (D_n, \varphi_n^{n+1}) \longrightarrow \bigoplus D_n[1].$$

Recall that $\text{id} - S$ determines this triangle uniquely up to isomorphism.

*Proof of Theorem 3.16.* Since the class of $A \in\in \mathfrak{T}$ with $\mathfrak{T}_*(A, B) = 0$ for all $B \in\in \mathfrak{N}_{\mathfrak{I}}$ is localising, we have $\mathfrak{T}_*(A, B) = 0$ if $A \in\in \langle\mathfrak{P}_{\mathfrak{I}}\rangle$ and $B \in\in \mathfrak{N}_{\mathfrak{I}}$. It remains to construct, for each $A \in\in \mathfrak{T}$, an exact triangle $\tilde{A} \to A \to N \to \tilde{A}[1]$ with $N \in\in \mathfrak{N}_{\mathfrak{I}}$ and $\tilde{A} \in\in \langle\mathfrak{P}_{\mathfrak{I}}\rangle$.

Construct a phantom castle over $A$ and let $N := \text{ho-}\varinjlim (N_n, \iota_n^{n+1})$ be the homotopy colimit of the phantom tower. We also use the homotopy colimit of the constant inductive system $(A, \text{id}_A)$. This is just $A$ because of the split exact triangle

$$\bigoplus A \xrightarrow{\text{id}-S} \bigoplus A \xrightarrow{\nabla} A \xrightarrow{0} \bigoplus A[1], \tag{3.6}$$

where $\nabla$ is the codiagonal map. By [3, Proposition 1.1.11] (and a rotation), we can find $\tilde{A}$ and the dotted arrows in the following diagram

$$
\begin{tikzcd}
\bigoplus \tilde{A}_n \arrow[r, dotted] \arrow[d, "\bigoplus \alpha_n"] & \bigoplus \tilde{A}_n \arrow[r, dotted] \arrow[d, "\bigoplus \alpha_n"] & \tilde{A} \arrow[r, dotted, "\circ" marking] \arrow[d, dotted] & \bigoplus \tilde{A}_n \arrow[d, "\bigoplus \alpha_n"] \\
\bigoplus A \arrow[r, "\mathrm{id}-S"] \arrow[d, "\bigoplus \iota^n"] & \bigoplus A \arrow[r, "\nabla"] \arrow[d, "\bigoplus \iota^n"] & A \arrow[r, "0", "\circ" marking] \arrow[d, dotted] & \bigoplus A \arrow[d, "\bigoplus \iota^n"] \\
\bigoplus N_n \arrow[r, "\mathrm{id}-S"] \arrow[d, "\bigoplus \gamma_n", "\circ" marking] & \bigoplus N_n \arrow[r] \arrow[d, "\bigoplus \gamma_n", "\circ" marking] & N \arrow[r, "\circ" marking] \arrow[d, dotted, "\circ" marking] \arrow[dr, phantom, "-"] & \bigoplus N_n \arrow[d, "-\bigoplus \gamma_n", "\circ" marking] \\
\bigoplus \tilde{A}_n \arrow[r, dotted] & \bigoplus \tilde{A}_n \arrow[r, dotted] & \tilde{A}[1] \arrow[r, dotted, "\circ" marking] & \bigoplus \tilde{A}_n
\end{tikzcd}
\tag{3.7}
$$

so that the rows and columns are exact triangles and the squares commute except for the one marked with a minus sign, which anti-commutes.

Lemma 3.7 yields $\tilde{A}_n \in\in \mathfrak{P}_{\mathfrak{I}}^n$ for all $n \in \mathbb{N}$. Hence $\bigoplus \tilde{A}_n \in\in \mathfrak{P}_{\mathfrak{I}}^\infty \subseteq \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ by Proposition 2.3. The exactness of the first row in (3.7) implies $\tilde{A} \in\in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$. We claim that $N \in\in \mathfrak{N}_{\mathfrak{I}}$. Hence the third column in (3.7) is the kind of exact triangle we need for $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ and $\mathfrak{N}_{\mathfrak{I}}$ to be complementary.

Let $F$ be a stable homological functor with $\ker F = \mathfrak{I}$. We must show $F(N) = 0$. The map $S$ factors through $\bigoplus \iota_n^{n+1}$; this map belongs to $\mathfrak{I} = \ker F$ because $\mathfrak{I}$ is compatible with direct sums. Hence $F(\mathrm{id} - S) = F(\mathrm{id})$ is invertible. By a long exact sequence, this implies $F(N) = 0$, that is, $N \in\in \mathfrak{N}_{\mathfrak{I}}$. Q.E.D.

Suppose from now on that we are in the situation of Theorem 3.16. Since $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ and $\mathfrak{N}_{\mathfrak{I}}$ are complementary, there is a unique exact triangle $\tilde{A} \to A \to N \to \tilde{A}[1]$ with $\mathfrak{I}$-contractible $N$ and $\mathfrak{P}_{\mathfrak{I}}$-cellular $\tilde{A}$; we call $\tilde{A}$ the $\mathfrak{P}_{\mathfrak{I}}$-*cellular approximation* of $A$. Even more, $\tilde{A}$ and $N$ depend functorially on $A$, so that we get two functors $L\colon \mathfrak{T} \to \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ and $N\colon \mathfrak{T} \to \mathfrak{N}_{\mathfrak{I}}$.

The proof of Theorem 3.16 above provides an explicit model for $N(A)$: it is the homotopy colimit of the phantom tower of $A$.

**Proposition 3.18.** Let $A \in\in \mathfrak{T}$ and construct a phantom castle over $A$. Then $L(A) = \tilde{A}$ is the homotopy colimit of the cellular approximation tower $(\tilde{A}_n)_{n\in\mathbb{N}}$.

This does not yet follow from (3.7) because we cannot control the dotted maps.

*Proof.* Let $\tilde{A} := \text{ho-}\varinjlim (\tilde{A}_n, \alpha_n^{n+1})$. We compare the exact triangle that defines the homotopy colimit $\tilde{A}$ with the triangle (3.6). The triangulated category axioms provide $f \in \mathfrak{T}(\tilde{A}, A)$ that makes the following diagram commute:

$$
\begin{tikzcd}
\bigoplus \tilde{A}_n \arrow[r, "\mathrm{id}-S"] \arrow[d, "\bigoplus \alpha_n"] & \bigoplus \tilde{A}_n \arrow[r] \arrow[d, "\bigoplus \alpha_n"] & \tilde{A} \arrow[r, circ] \arrow[d, dotted, "f"] & \bigoplus \tilde{A}_n \arrow[d, "\bigoplus \alpha_n"] \\
\bigoplus A \arrow[r, "\mathrm{id}-S"] & \bigoplus A \arrow[r, "\nabla"] & A \arrow[r, circ, "0"] & \bigoplus A.
\end{tikzcd}
\tag{3.8}
$$

We claim that $f$ is an $\mathfrak{I}$-equivalence. Equivalently, the cone of $f$ is $\mathfrak{I}$-contractible, so that the mapping cone triangle for $f$ has entries in $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ and $\mathfrak{N}_{\mathfrak{I}}$; this implies that $L(A) = \tilde{A}$.

Let $F$ be a stable homological functor with $\ker F = \mathfrak{I}$. We check that $F(f)$ is invertible. The direct sum of the triangles (3.2) for $n \in \mathbb{N}$ is again an exact triangle. On the long exact homology sequence

$$
\cdots \longrightarrow F_{m+1}\Bigl(\bigoplus N_n\Bigr) \\
\longrightarrow F_m\Bigl(\bigoplus \tilde{A}_n\Bigr) \longrightarrow F_m\Bigl(\bigoplus A\Bigr) \longrightarrow F_m\Bigl(\bigoplus N_n\Bigr) \longrightarrow \cdots
$$

for this exact triangle, consider the operator induced by $\mathrm{id} - S$ on each entry. Since $\mathfrak{I}$ is compatible with direct sums, the shift map $S$ on $\bigoplus N_n$ is a phantom map, so that $F(\mathrm{id} - S)$ acts identically on $F\bigl(\bigoplus N_n\bigr)$. On $F\bigl(\bigoplus A\bigr)$, the map $\mathrm{id} - S$ induces a split monomorphism with cokernel $F(\nabla)\colon F\bigl(\bigoplus A\bigr) \to F(A)$. Now a diagram chase shows that the map on $F\bigl(\bigoplus \tilde{A}_n\bigr)$ induced by $\mathrm{id} - S$ is injective and has the cokernel

$$
F\Bigl(\bigoplus \tilde{A}_n\Bigr) \xrightarrow{F\left(\bigoplus \alpha_n\right)} F\Bigl(\bigoplus A\Bigr) \xrightarrow{F(\nabla)} F(A).
$$

Comparing the long exact homology sequences for the two rows in (3.8), we conclude that $F(f)$ is indeed invertible. Q.E.D.

We have proved Proposition 3.18 by constructing an $\mathfrak{I}$-equivalence between an arbitrary object of $\mathfrak{T}$ and the homotopy colimit of its cellular approximation tower. This provides another, independent proof of Theorem 3.16. Since the exact triangle $\tilde{A} \to A \to N \to \tilde{A}[1]$ with $\tilde{A} \in\in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ and $N \in\in \mathfrak{N}_{\mathfrak{I}}$ is unique up to isomorphism, both proofs construct the same exact triangle. The first proof shows that $N$ is the homotopy colimit of the phantom tower, the second one shows that $\tilde{A}$ is the homotopy colimit of the cellular approximation tower. We conclude, therefore, that the object $\tilde{A}$ in (3.7) is the homotopy colimit of the phantom tower and that the map $\tilde{A} \to A$ in (3.7) agrees with the map $f$ from (3.8).

**Theorem 3.19.** Let $\mathfrak{T}$ be a triangulated category with countable direct sums, and let $\mathfrak{I}$ be a homological ideal in $\mathfrak{T}$ that is compatible with countable direct sums and has enough projective objects. Let $A \in\in \mathfrak{T}$ and construct a phantom castle over $A$. The following are equivalent:

(1) $A$ is $\mathfrak{P}_{\mathfrak{I}}$-cellular, that is, $A \in\in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$;

(2) $A$ is isomorphic to the homotopy colimit of its cellular approximation tower;

(3) $A$ is isomorphic to the homotopy colimit of an inductive system $(P_n, \varphi_n)$ with $P_n \in\in \bigcup_{k\in\mathbb{N}} \mathfrak{P}_{\mathfrak{I}}^k$ for all $n \in \mathbb{N}$;

(4) $A$ is $\mathfrak{I}^{2\infty}$-projective.

As a consequence, $\mathfrak{I}^{2\infty} = \mathfrak{I}^{n\infty}$ for all $n \geq 2$.

*Proof.* Since $L(A) \cong A$ if and only if $A \in\in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$, Proposition 3.18 yields the equivalence of (1) and (2). The implication (2)$\Longrightarrow$(3) is trivial: the cellular approximation tower provides an inductive system of the required kind.

We check that (3) implies (4). Let $(P_n, \varphi_n)$ be an inductive system as in (3). First, Proposition 2.3 shows that $\bigoplus P_n$ is $\mathfrak{I}^{\infty}$-projective. Then Proposition 2.2 shows that the homotopy colimit is $\mathfrak{I}^{2\infty}$-projective.

Propositions 2.3 and 2.2 show recursively that all $\mathfrak{I}^{\alpha}$-projective objects belong to $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ for $n = 0, 1, 2, 3, \dots, \infty, 2 \cdot \infty$. Hence (4) implies (1), so that all four conditions are equivalent.

Finally, since $\mathfrak{P}_{\mathfrak{I}}^{2\infty} = \langle \mathfrak{P}_{\mathfrak{I}} \rangle$, it follows from Proposition 2.2 that the powers $\mathfrak{I}^{n\infty}$ for $n \geq 2$ have the same projective objects. Therefore, they are all equal. Q.E.D.

**Proposition 3.20.** The $\mathfrak{P}_{\mathfrak{I}}$-cellular approximation functor $L\colon \mathfrak{T} \to \langle \mathfrak{P}_{\mathfrak{I}} \rangle$ maps a phantom castle over $A \in\in \mathfrak{T}$ to a phantom castle over $L(A)$. A morphism $f \in \mathfrak{T}(A, B)$ belongs to $\mathfrak{I}^{\alpha}$ for some $\alpha$ if and only if $L(f) \in \mathfrak{T}\bigl(L(A), L(B)\bigr)$ does.

*Proof.* Since $L$ is an exact functor, it preserves the commuting diagrams and exact triangles required for a phantom castle. It also maps $\mathfrak{P}_{\mathfrak{I}}$ to itself because $L(B) \cong B$ for all $B \in\in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$. It remains to check that $L(f) \in \mathfrak{I}^{\alpha}\bigl(L(B), L(B')\bigr)$ if and only if $f \in \mathfrak{I}^{\alpha}(B, B')$. Let $F$ be a stable homological functor with $\mathfrak{I}^{\alpha} = \ker F$. Then $F(N) = 0$ for all $N \in\in \mathfrak{N}_{\mathfrak{I}}$. Therefore, $F$ descends to the localisation $\mathfrak{T}/\mathfrak{N}_{\mathfrak{I}}$; equivalently, the natural transformation $F \circ L \Rightarrow F$ is an isomorphism. In particular, $F(f) = 0$ if and only if $F\bigl(L(f)\bigr) = 0$. Q.E.D.

As a result, it makes almost no difference whether we work in $\mathfrak{T}$ or $\mathfrak{T}/\mathfrak{N}_{\mathfrak{J}}$. We work in $\mathfrak{T}$ most of the time and allow $\mathfrak{N}_{\mathfrak{J}}$ to be non-trivial in order to formulate Theorem 3.16.

The direct sums $\bigoplus \tilde{A}_n$ in (3.8) are $\mathfrak{J}^\infty$-projective by Proposition 2.3. The map $\nabla \circ \bigoplus \alpha_n \colon \bigoplus \tilde{A}_n \to A$ is $\mathfrak{J}^\infty$-epic because it is $\mathfrak{J}^n$-epic for all $n \in \mathbb{N}$. We may replace $A$ by $\tilde{A}$ in this statement by Proposition 3.20. Thus the top row in (3.8) is an $\mathfrak{J}^\infty$-exact triangle. This means that the chain complex

$$\cdots \to 0 \to \bigoplus \tilde{A}_n \xrightarrow{\mathrm{id}-S} \bigoplus \tilde{A}_n \to \tilde{A}$$

is $\mathfrak{J}^\infty$-exact and hence an $\mathfrak{J}^\infty$-projective resolution of $\tilde{A}$. Once again, Proposition 3.20 allows us to replace $\tilde{A}$ by $A$ in this statement, that is, we get an $\mathfrak{J}^\infty$-projective resolution of length 1

$$\cdots \to 0 \to \bigoplus \tilde{A}_n \xrightarrow{\mathrm{id}-S} \bigoplus \tilde{A}_n \to A. \tag{3.9}$$

This will allow us to analyse the convergence of the ABC spectral sequence.

### 3.6 Complementarity via partially defined adjoints

Suppose that we are given a thick subcategory $\mathfrak{N}$ of a triangulated category $\mathfrak{T}$ and that we want to use Theorem 3.16 to show that it is reflective, that is, there is another thick subcategory $\mathfrak{L}$ such that $(\mathfrak{L}, \mathfrak{N})$ is complementary.

To have a chance of doing so, $\mathfrak{T}$ must have countable direct sums, and the subcategory $\mathfrak{N}$ must be localising, that is, closed under countable direct sums: this happens whenever Theorem 3.16 applies. By [16, Proposition 2.9], the only candidate for $\mathfrak{L}$ is the left orthogonal complement

$$\mathfrak{L} := \{A \in\in \mathfrak{T} \mid \mathfrak{T}(A, N) = 0 \text{ for all } N \in\in \mathfrak{N}\}$$

of $\mathfrak{N}$, which is another localising subcategory.

The starting point of our method is a stable additive functor $F \colon \mathfrak{T} \to \mathfrak{C}$ with

$$\mathfrak{N} = \mathfrak{N}_F := \{A \in\in \mathfrak{T} \mid F(A) = 0\}.$$

This functor yields a stable ideal $\mathfrak{J}_F := \ker F$. In applications, $F$ is either a stable homological functor to a stable Abelian category or an exact functor to another triangulated category; in either case, the ideal $\ker F$ is homological and $\mathfrak{N}_F$ is the class of all $\mathfrak{J}_F$-contractible objects. In addition, we assume $F$ to *commute with countable direct sums*, so that $\mathfrak{J}_F$ is compatible with countable direct sums.

In order to apply Theorem 3.16, it remains to prove that there are enough $\mathfrak{J}_F$-projective objects in $\mathfrak{T}$. Then the pair of subcategories $(\langle \mathfrak{P}_{\mathfrak{J}_F} \rangle, \mathfrak{N})$ is complementary. For a *good* choice of $F$, this may be much easier than proving directly that $(\mathfrak{L}, \mathfrak{N})$ is complementary. The choice in the following example never helps. But often, there is another choice for $F$ that does.

**Example 3.21.** The localisation functor $\mathfrak{T} \to \mathfrak{T}/\mathfrak{N}$ is a possible choice for $F$ —that is, it has the right kernel on objects— but it should be a bad one because it tells us nothing new about $\mathfrak{N}$. In fact, the $\ker F$-projective objects are precisely the objects of $\mathfrak{L}$, so that we have not gained anything.

We now discuss a sufficient condition for enough projective objects from [19]. The *left adjoint* of $F \colon \mathfrak{T} \to \mathfrak{C}$ is defined on an object $A \in\in \mathfrak{C}$ if the functor $B \mapsto \mathfrak{C}\bigl(A, F(B)\bigr)$ on $\mathfrak{T}$ is representable, that is, there is an object $F^{\vdash}(A)$ of $\mathfrak{T}$ and a natural isomorphism $\mathfrak{T}(F^{\vdash}(A), B) \cong \mathfrak{C}\bigl(A, F(B)\bigr)$ for all $B \in\in \mathfrak{T}$. We say that $F^{\vdash}$ is *defined on enough objects* if, for any object $B$ of $\mathfrak{C}$ there is an epimorphism $B' \to B$ such that $F^{\vdash}$ is defined on $B'$.

The following theorem asserts that $\mathfrak{I}_F$ has enough projective objects if $F^{\vdash}$ is defined on enough objects. The statement is somewhat more involved because it is often useful to shrink the domain of definition of $F^{\vdash}$ to a sufficiently big subcategory $\mathfrak{PC}$.

**Theorem 3.22.** Let $\mathfrak{T}$, $\mathfrak{C}$, and $F$ be as above, that is, $\mathfrak{T}$ is a triangulated category with countable direct sums, $\mathfrak{C}$ is either a stable Abelian category or a triangulated category, and $F \colon \mathfrak{T} \to \mathfrak{C}$ is a stable functor commuting with countable direct sums and either homological (if $\mathfrak{C}$ is Abelian) or exact (if $\mathfrak{C}$ is triangulated). Let $\mathfrak{I}_F := \ker F$ and let $\mathfrak{N}_F$ be the class of $F$-contractible objects as above.

Let $\mathfrak{PC} \subseteq \mathfrak{C}$ be a subcategory with two properties: first, for any $A \in\in \mathfrak{T}$, there exists an epimorphism $P \to F(A)$ with $P \in\in \mathfrak{PC}$; secondly, the left adjoint functor $F^{\vdash}$ of $F$ is defined on $\mathfrak{PC}$, that is, for each $P \in \mathfrak{PC}$, there is an object $F^{\vdash}(P)$ in $\mathfrak{T}$ with $\mathfrak{T}(F^{\vdash}(P), B) \cong \mathfrak{T}\bigl(P, F(B)\bigr)$ naturally for all $B \in\in \mathfrak{T}$.

Then $\mathfrak{I}_F$ has enough projective objects, the subcategory $\mathfrak{N}_F$ is reflective, and the pair of localising subcategories $\bigl(\langle F^{\vdash}(\mathfrak{PC})\rangle, \mathfrak{N}_F\bigr)$ is complementary.

*Proof.* [19, Proposition 3.37] shows that $\mathfrak{I}_F$ has enough projective objects and that any projective object is a direct summand of $F^{\vdash}(P)$ for some $P \in \mathfrak{PC}$. Now Theorem 3.16 yields the assertions. Q.E.D.

Theorem 3.22 is non-trivial even if $\mathfrak{N}_F$ contains only zero objects, that is, if $F(A) = 0$ implies $A = 0$. Then it asserts $\langle F^{\vdash}(\mathfrak{PC})\rangle = \mathfrak{T}$.

In the situation of Theorem 3.22, we also understand how objects of $\langle F^{\vdash}(\mathfrak{PC})\rangle$ are to be constructed from the building blocks in $F^{\vdash}(\mathfrak{PC})$.

Let $\mathfrak{C}_1 \star \mathfrak{C}_2$ for subcategories $\mathfrak{C}_1, \mathfrak{C}_2 \subseteq \mathfrak{T}$ be the subcategory of all objects $A$ for which there is an exact sequence $A_1 \to A \to A_2 \to A_1[1]$ with

$A_1 \in\in \mathfrak{C}_1$ and $A_2 \in\in \mathfrak{C}_2$. We abbreviate $\mathfrak{P}_F := F^\vdash(\mathfrak{P}\mathfrak{C})$ and recursively define $\mathfrak{P}_F^{\star n}$ for $n \in \mathbb{N}$ by $\mathfrak{P}_F^{\star 0} := \{0\}$ and $\mathfrak{P}_F^{\star n} := \mathfrak{P}_F^{\star n-1} \star \mathfrak{P}_F$ for $n \geq 1$.

**Theorem 3.23.** In the situation of Theorem 3.22, any object of $\langle F^\vdash(\mathfrak{P}\mathfrak{C})\rangle$ is a homotopy colimit of an inductive system $(A_n)_{n\in\mathbb{N}}$ with $A_n \in \mathfrak{P}_F^{\star n}$.

*Proof.* This follows from Theorem 3.19. But an extra observation is needed here because we do not adjoin direct summands of objects in the definition of the subcategories $\mathfrak{P}_F^{\star n}$, so that they do not necessarily contain all $\mathfrak{I}_F^n$-projective objects.

There is an $\mathfrak{I}_F$-projective resolution with entries in $\mathfrak{P}_F$, which we embed in a phantom castle. The resulting cellular approximation tower satisfies $\tilde{A}_n \in\in \mathfrak{P}_F^{\star n}$, so that Theorem 3.19 yields an inductive system of the required form. Q.E.D.

We have considered two cases above: homological and exact functors. For homological functors with values in the category of Abelian groups, our results were obtained previously by Apostolos Beligiannis [2]. Let $F\colon \mathfrak{T} \to \mathfrak{Ab}^{\mathbb{Z}}$ be a stable homological functor that commutes with direct sums. Suppose that $F$ is defined on sufficiently many objects. Then there must be a surjective map $A \twoheadrightarrow \mathbb{Z}$ for which $F^\vdash(A)$ is defined. Since $\mathbb{Z}$ is projective, $\mathbb{Z}$ is a retract of $A$. Since we assume $\mathfrak{T}$ to have direct sums, idempotent morphisms in $\mathfrak{T}$ have range objects. Thus $F^\vdash(\mathbb{Z})$ is defined as well. By definition, $F^\vdash(\mathbb{Z})$ is a representing object for $F$. Conversely, if $F$ is representable, then $F^\vdash$ can be defined on all free Abelian groups. Hence the adjoint $F^\vdash$ is defined on sufficiently many objects if and only if $F$ is representable. Furthermore, we can take $\mathfrak{P}_F$ to be the set of all direct sums of translated copies of the representing object $F^\vdash(\mathbb{Z})$. The assumption that $F$ commute with direct sums means that $F^\vdash(\mathbb{Z})$ is a compact object.

Summing up, if $F$ is a stable homological functor to $\mathfrak{Ab}^{\mathbb{Z}}$, then our methods apply if and only if $F(A) \cong \mathfrak{T}_*(D, A)$ for a compact generator $D$ of $\mathfrak{T}$. This situation is considered already in [2].

## 4 The ABC spectral sequence

When we apply a homological or cohomological functor to the phantom tower, we get first an exact couple and then a spectral sequence. We call it the ABC spectral sequence after Adams, Brinkmann, and Christensen. Its second page only involves derived functors. The higher pages can be described in terms of the phantom tower, but are more complicated. It is remarkable that the ABC spectral sequence is well-defined and functorial on the level of triangulated categories, that is, all the higher boundary maps are uniquely determined and functorial without introducing finer structure like, say, model categories.

Several results in this section are already know to the experts or can be extracted from [2, 4, 7]. We have included them, nevertheless, to give a reasonably self-contained account.

### 4.1 A spectral sequence from the phantom tower

We are going to construct exact couples out of the phantom tower, extending results of Daniel Christensen [7]. We fix $A \in\in \mathfrak{T}$ and a phantom tower (3.1) over $A$. In addition, we let

$$P_n := 0, \qquad N_n := A, \quad \text{and} \quad \iota_n^{n+1} := \mathrm{id}_A$$

for $n < 0$. Thus the triangles

$$P_n \xrightarrow{\pi_n} N_n \xrightarrow{\iota_n^{n+1}} N_{n+1} \xrightarrow{\varepsilon_n} P_n[1] \tag{4.1}$$

are exact for all $n \in \mathbb{Z}$. Of course, $\iota_n^{n+1}$ rarely belongs to $\mathfrak{I}$ for $n < 0$.

Let $F\colon \mathfrak{T} \to \mathfrak{Ab}$ be a homological functor. Define bigraded Abelian groups

$$D := \sum_{p,q\in\mathbb{Z}} D_{pq}, \qquad D_{pq} := F_{p+q+1}(N_{p+1}),$$
$$E := \sum_{p,q\in\mathbb{Z}} E_{pq}, \qquad E_{pq} := F_{p+q}(P_p),$$

and homogeneous group homomorphisms

$$\begin{tikzcd}[column sep=small] D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd} \qquad \begin{aligned} i_{pq} &:= (\iota_{p+1}^{p+2})_*\colon D_{p,q} \to D_{p+1,q-1}, & \deg i &= (1,-1),\\ j_{pq} &:= (\varepsilon_p)_*\colon \quad D_{p,q} \to E_{p,q}, & \deg j &= (0,0),\\ k_{pq} &:= (\pi_p)_*\colon \quad E_{p,q} \to D_{p-1,q}, & \deg k &= (-1,0). \end{aligned}$$

Since $F$ is homological and the triangles (4.1) are exact, the chain complexes

$$\cdots \longrightarrow F_m(P_n) \xrightarrow{\pi_{n*}} F_m(N_n) \xrightarrow{\iota_{n*}^{n+1}} F_m(N_{n+1}) \xrightarrow{\varepsilon_{n*}} F_{m-1}(P_n) \longrightarrow \cdots$$

are exact for all $m \in \mathbb{Z}$. Hence the data $(D, E, i, j, k)$ above is an exact couple (cf. [15, § XI.5]).

We briefly recall how an exact couple yields a spectral sequence, cf. also [15, pp. 336–337] or [4]. The first step is to form *derived exact couples*. Let

$$D^r := i^{r-1}(D) \subseteq D, \qquad E^r := k^{-1}(D^r) \;/\; j(\ker i^{r-1}),$$

for all $r \geq 1$. Let $i^{(r)}\colon D^r \to D^r$ be the restriction of $i$; let $k^{(r)}\colon E^r \to D^r$ be induced by $k\colon E \to D$; and let $j^{(r)}\colon D^r \to E^r$ be induced by the relation $j \circ i^{1-r}$. It is shown in [15] that $E^{r+1} \cong H(E^r, d^{(r)})$ for all $r \in \mathbb{N}$, where $d^{(r)} = j^{(r)}k^{(r)}$; the map $d^{(r)}$ has bidegree $(-r, r-1)$. We call this spectral sequence the *ABC spectral sequence* for $F$ and $A$.

We are going to describe the derived exact couples explicitly. First, we claim that

$$D^{r+1}_{p-1,q} \cong \begin{cases} F_{p+q}(A) & \text{for } p \leq 0, \\ F_{p+q}(A) \bigm/ F_{p+q} : \mathfrak{I}^p(A) & \text{for } 0 \leq p \leq r, \\ F_{p+q}(N_{p-r}) \bigm/ F_{p+q} : \mathfrak{I}^r(N_{p-r}) & \text{for } r \leq p. \end{cases} \tag{4.2}$$

By definition, $D^{r+1}_{p-1,q}$ for $r \in \mathbb{N}$ is the range of the map $i^r\colon D_{p-1-r,q+r} \to D_{p-1,q}$. This is the identity map on $F_{p+q}(A)$ for $p \leq 0$, the map $\iota^p_*\colon F_{p+q}(A) \to F_{p+q}(N_p)$ for $0 \leq p \leq r$, and the map $(\iota^p_{p-r})_*\colon F_{p+q}(N_{p-r}) \to F_{p+q}(N_p)$ for $r \leq p$. Now recall that the maps $\iota^n_m$ are $\mathfrak{I}^{n-m}$-versal for all $n \geq m \geq 0$ and use Lemma 3.6.

**Proposition 4.1.** Let $1 \leq r < \infty$. Then we have $E^{r+1}_{pq} \cong 0$ for $p \leq -1$; for $0 \leq p \leq r$, there is an exact sequence

$$\begin{aligned} 0 \to \frac{F_{p+q+1}(N_p)}{F_{p+q+1} : \mathfrak{I}^{r+1}(N_p)} &\xrightarrow{(\iota^{p+1}_p)_*} \frac{F_{p+q+1}(N_{p+1})}{F_{p+q+1} : \mathfrak{I}^r(N_{p+1})} \\ &\to E^{r+1}_{pq} \to \frac{F_{p+q} : \mathfrak{I}^{p+1}(A)}{F_{p+q} : \mathfrak{I}^p(A)} \to 0; \end{aligned}$$

for $p \geq r$, there is an exact sequence

$$\begin{aligned} 0 \to \frac{F_{p+q+1}(N_p)}{F_{p+q+1} : \mathfrak{I}^{r+1}(N_p)} &\xrightarrow{(\iota^{p+1}_p)_*} \frac{F_{p+q+1}(N_{p+1})}{F_{p+q+1} : \mathfrak{I}^r(N_{p+1})} \\ &\to E^{r+1}_{pq} \to \frac{F_{p+q} : \mathfrak{I}^{r+1}(N_{p-r})}{F_{p+q} : \mathfrak{I}^r(N_{p-r})} \to 0. \end{aligned}$$

Finally, for $r = \infty$, let

$$\mathsf{Bad}_{pq} := F_q(N_p) \Bigm/ \bigcup_{r \in \mathbb{N}} F_q : \mathfrak{I}^r(N_p);$$

then we get $E^\infty_{pq} \cong 0$ for $p \leq -1$, and exact sequences

$$0 \to \mathsf{Bad}_{p,p+q+1} \to \mathsf{Bad}_{p+1,p+q+1} \to E^\infty_{pq} \to \frac{F_{p+q} : \mathfrak{I}^{p+1}(A)}{F_{p+q} : \mathfrak{I}^p(A)} \to 0.$$

Therefore, if $\bigcup_{r\in\mathbb{N}} F_q : \mathfrak{I}^r(N_p) = F_q(N_p)$ for all $p \in \mathbb{N}$, then

$$E^\infty_{pq} \cong \frac{F_{p+q} : \mathfrak{I}^{p+1}(A)}{F_{p+q} : \mathfrak{I}^p(A)},$$

that is, the ABC spectral sequence converges towards $F_m(A)$, and the induced increasing filtration on the limit is $\bigl(F_m : \mathfrak{I}^r(A)\bigr)_{r\in\mathbb{N}}$.

*Proof.* We have $E^{r+1}_{pq} \cong 0$ for $p \le -1$ because already $E_{pq} = E^1_{pq} = 0$. Let $p \ge 0$. We use (4.2) and the exactness of the derived exact couple $(D^{r+1}, E^{r+1})$ to compute $E^{r+1}$ by an extension involving the kernel and cokernel of the restriction of $i$ to $D^{r+1}$. Since $\iota_m^{m+1}$ is $\mathfrak{I}$-versal, $x \in F(N_m)$ satisfies $(\iota_m^{m+1})_*(x) \in F : \mathfrak{I}^r(N_{m+1})$ if and only if $x \in F : \mathfrak{I}^{r+1}(N_m)$. Plugging this into the extension that describes $E^{r+1}$, we get the assertion, at least for finite $r$.

The case $r = \infty$ is similar. Now

$$E^\infty := \bigcap_{r\in\mathbb{N}} k^{-1}(i^r D) \Bigm/ \bigcup_{r\in\mathbb{N}} j(\ker i^r). \tag{4.3}$$

The injectivity of the map $\mathsf{Bad}_{p,p+q+1} \to \mathsf{Bad}_{p+1,p+q+1}$ follows from the exactness of colimits of Abelian groups. Using $i(D) = \ker j$, $\ker i = k(E)$, and (4.3), we get a short exact sequence

$$\begin{aligned} 0 \to D_{pq} \Bigm/ \Bigl(i(D_{p-1,q+1}) + \bigcup \ker i^r\Bigr) &\to E^\infty_{pq} \\ &\to D_{p-1,q} \cap \ker i \cap \bigcap i^r(D) \to 0; \end{aligned} \tag{4.4}$$

the first map is induced by $j$, the second one by $k$.

The intersection $\bigcap i^r(D)$ is described by (4.2) for $r = \infty$, so that the third case in (4.2) is missing. Hence the quotient in (4.4) is

$$\ker i \cap \bigcap i^r(D) \cong F_{p+q} : \mathfrak{I}^{p+1}(A) \bigm/ F_{p+q} : \mathfrak{I}^p(A).$$

The versality of the maps $\iota_m^n$ yields $\bigcup \ker i^r \cap D_{p-1,q} = \bigcup_r F_{p+q} : \mathfrak{I}^r(N_p)$. Hence the kernel in (4.4) is $\mathsf{Bad}_{p+1,p+q+1}/\mathsf{Bad}_{p,p+q+1}$. If $\mathsf{Bad} = 0$, then the groups $E^\infty_{pq}$ for $p + q = m$ are the subquotients of the filtration $F_m : \mathfrak{I}^p(A)$ on $F_m(A)$. Moreover, since $\mathsf{Bad}_{0m} = F_m(A) \Bigm/ \bigcup_{p\in\mathbb{N}} F_m : \mathfrak{I}^p(A)$, our hypothesis includes the statement that $F_m(A) = \bigcup_{p\in\mathbb{N}} F_m : \mathfrak{I}^p(A)$. Q.E.D.

Dual constructions apply to a cohomological functor $G\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{Ab}$. Equation (4.1) yields a sequence of exact chain complexes

$$\cdots \longrightarrow G^{m-1}(P_n) \xrightarrow{\varepsilon_n^*} G^m(N_{n+1}) \xrightarrow{(\iota_n^{n+1})^*} G^m(N_n) \xrightarrow{\pi_n^*} G^m(P_n) \longrightarrow \cdots.$$

Therefore, the following defines an exact couple:

$$\tilde{D}^{pq} := G^{p+q+1}(N_{p+1}), \qquad \tilde{E}^{pq} := G^{p+q}(P_p),$$

$$\begin{tikzcd} \tilde{D} \arrow[rr, "i"] & & \tilde{D} \arrow[dl, "j"] \\ & \tilde{E} \arrow[ul, "k"] & \end{tikzcd} \qquad \begin{aligned} i^{pq} &:= (\iota_p^{p+1})^* \colon \tilde{D}^{p,q} \to \tilde{D}^{p-1,q+1}, & \deg i &= (-1,1), \\ j^{pq} &:= \pi_{p+1}^* \colon \quad \tilde{D}^{p,q} \to \tilde{E}^{p+1,q}, & \deg j &= (1,0), \\ k^{pq} &:= \varepsilon_p^* \colon \qquad \tilde{E}^{p,q} \to \tilde{D}^{p,q}, & \deg k &= (0,0). \end{aligned}$$

Again we form derived exact couples $(\tilde{D}_r, \tilde{E}_r, i_r, j_r, k_r)$, and $(\tilde{E}_r, d_r)$ with $d_r := j_r k_r$ is a spectral sequence. The map $d_r$ has bidegree $(r, 1-r)$. We call this spectral sequence the *ABC spectral sequence* for $G$ and $A$.

We can describe the derived exact couples as above. To begin with,

$$\tilde{D}_{r+1}^{p-1,q} = \begin{cases} G^{p+q}(A) & \text{for } p \le 0, \\ \mathfrak{I}^p G^{p+q}(A) & \text{for } 0 \le p \le r, \\ \mathfrak{I}^r G^{p+q}(N_{p-r}) & \text{for } r \le p. \end{cases} \tag{4.5}$$

**Proposition 4.2.** Let $1 \le r < \infty$. Then $\tilde{E}_{r+1}^{pq} \cong 0$ for $p \le -1$, and there are exact sequences

$$0 \to \frac{\mathfrak{I}^p G^{p+q}(A)}{\mathfrak{I}^{p+1} G^{p+q}(A)} \to \tilde{E}_{r+1}^{pq} \to \mathfrak{I}^r G^{p+q+1}(N_{p+1}) \xrightarrow{(\iota_p^{p+1})^*} \mathfrak{I}^{r+1} G^{p+q+1}(N_p) \to 0 \tag{4.6}$$

for $0 \le p \le r$ and

$$0 \to \frac{\mathfrak{I}^r G^{p+q}(N_{p-r})}{\mathfrak{I}^{r+1} G^{p+q}(N_{p-r})} \to \tilde{E}_{r+1}^{pq} \to \mathfrak{I}^r G^{p+q+1}(N_{p+1}) \xrightarrow{(\iota_p^{p+1})^*} \mathfrak{I}^{r+1} G^{p+q+1}(N_p) \to 0 \tag{4.7}$$

for $p \ge r$. For $r = \infty$, let

$$\widetilde{\mathsf{Bad}}^{pq} := \bigcap_{r \in \mathbb{N}} \mathfrak{I}^r G^q(N_p),$$

then we get $E_{pq}^\infty \cong 0$ for $p \le -1$, and exact sequences

$$0 \to \frac{\mathfrak{I}^p G_{p+q}(A)}{\mathfrak{I}^{p+1} G_{p+q}(A)} \to \tilde{E}_\infty^{pq} \to \widetilde{\mathsf{Bad}}^{p+1,p+q+1} \to \widetilde{\mathsf{Bad}}^{p,p+q+1}.$$

Therefore, if $\bigcap_{r\in\mathbb{N}} \mathfrak{I}^r G^q(N_p) = 0$ for all $p, q$, then

$$\tilde{E}_\infty^{pq} \cong \frac{\mathfrak{I}^p G_{p+q}(A)}{\mathfrak{I}^{p+1} G_{p+q}(A)},$$

that is, the ABC spectral sequence converges towards $G^m(A)$, and the induced decreasing filtration on the limit is $\bigl(\mathfrak{I}^r G^m(A)\bigr)_{r\in\mathbb{N}}$.

*Proof.* This is proved exactly as in the homological case. Q.E.D.

Notice that we do not claim that the maps $\widetilde{\mathsf{Bad}}^{p+1,p+q+1} \to \widetilde{\mathsf{Bad}}^{p,p+q+1}$ are surjective. If $G$ is representable, that is, $G(A) \cong \mathfrak{T}(A,B)$ for some $B \in\in \mathfrak{T}$, then the question whether the maps $\widetilde{\mathsf{Bad}}^{p+1,p+q+1} \to \widetilde{\mathsf{Bad}}^{p,p+q+1}$ are surjective is related to the question whether $\mathfrak{I}^\infty \cdot \mathfrak{I} = \mathfrak{I}^\infty$. In this case, $\widetilde{\mathsf{Bad}}^{p,*} = \mathfrak{I}^\infty(N_p[*], B)$. Since the maps $\iota_n^{n+1}$ are $\mathfrak{I}$-versal,

$$\operatorname{range}\bigl(\widetilde{\mathsf{Bad}}^{p+1,*} \to \widetilde{\mathsf{Bad}}^{p,*}\bigr) = \mathfrak{I}^\infty \circ \mathfrak{I}(N_p[*], B) \subseteq \mathfrak{I}^\infty(N_p[*], B).$$

**Theorem 4.3.** Starting with the second page, the ABC spectral sequences for homological and cohomological functors are independent of auxiliary choices and functorial in $A$. Their second pages contain the derived functors:

$$E^2_{pq} \cong \mathbb{L}_p F_q(A), \qquad \tilde{E}^{pq}_2 \cong \mathbb{R}^p G^q(A).$$

*Proof.* We only formulate the proof for homological functors; the cohomological case is similar. The map $d := jk\colon E \to E$ is induced by the maps $\delta_p\colon P_{p+1} \to P_p[1]$ in the phantom tower. By Lemma 3.2, these maps form a $\mathfrak{P}$-projective resolution of $A$. This together with counting of suspensions yields the description of $E^2_{pq}$.

Let $f\colon A \to A'$ be a morphism in $\mathfrak{T}$. By Lemma 3.2, it lifts to a morphism between the phantom towers over $A$ and $A'$. This induces a morphism of exact couples and hence a morphism of spectral sequences. The maps $P_n \to P'_n$ form a chain map between the $\mathfrak{I}$-projective resolutions embedded in the phantom towers. This chain map lifting of $f$ is unique up to chain homotopy (cf. [19, Proposition 3.36]). Hence the induced map on $E^2$ is unique and functorial. We get $E^r$ for $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ as subquotients of $E^2$. Since our map on $E^2$ is part of a morphism of exact couples, it descends to these subquotients in a unique and functorial way. Thus $E^r$ is functorial for all $r \geq 2$. Q.E.D.

The naturality of the ABC spectral sequence does not mean that the exact sequences in Propositions 4.1 and 4.2 are natural. They use the exact couple underlying the ABC spectral sequence, and this exact couple is not natural. It is easy to check that the maps $E^{r+1}_{pq} \to F_{p+q} : \mathfrak{I}^{p+1}(A) \bigm/ F_{p+q} : \mathfrak{I}^p(A)$ in Proposition 4.1 are canonical for $0 \leq p \leq r \leq \infty$. But $F_{p+q} : \mathfrak{I}^{r+1}(N_{p-r}) \bigm/ F_{p+q} : \mathfrak{I}^r(N_{p-r})$ depends on auxiliary choices.

Our results so far only *formulate* the convergence problem for the ABC spectral sequence. It remains to check whether the relevant obstructions vanish. The easy special case where the projective resolutions have finite length is already dealt with in [7]. Recall that $\mathfrak{P}^n_{\mathfrak{I}}$ denotes the class of $\mathfrak{I}^n$-projective objects.

**Lemma 4.4.** Let $k \in \mathbb{N}$ and $A \in\in \mathfrak{P}^k_{\mathfrak{I}}$. Then $\iota_m^{m+k} = 0$ and $N_m \in\in \mathfrak{P}^k_{\mathfrak{I}}$ for all $m \in \mathbb{N}$.

*Proof.* Since $\iota_m^{m+k}$ is $\mathfrak{I}^k$-versal, we have $\iota_m^{m+k} = 0$ if and only if $N_m \in\in \mathfrak{P}^k_{\mathfrak{I}}$. We prove $\iota_m^{m+k} = 0$ by induction on $m$. The case $m = 0$ is clear because $N_0 = A$. If $\iota_m^{m+k} = 0$, then $\iota_{m+1}^{m+k}$ factors through the map $\varepsilon_m \colon N_{m+1} \to P_m[1]$ by the long exact homology sequence for the triangles (4.1). If we compose the resulting map $P_m[1] \to N_{m+k}$ with $\iota_{m+k}^{m+k+1} \in \mathfrak{I}$, we get zero because $P_m[1] \in\in \mathfrak{P}$. Thus $\iota_{m+1}^{m+k+1}$ vanishes as well. Q.E.D.

**Proposition 4.5.** Let $F\colon \mathfrak{T} \to \mathfrak{Ab}$ be a homological functor and let $m \in \mathbb{N}$.

If $A \in\in \mathfrak{P}^{m+1}_{\mathfrak{I}}$, then the ABC spectral sequence for $F$ and $A$ collapses at $E^{m+2}$ and converges towards $F_*(A)$, and its limiting page $E^\infty = E^{m+2}$ is supported in the region $0 \le p \le m$.

If, instead, $A$ has a $\mathfrak{P}$-projective resolution of length $m$, then the ABC spectral sequence for $F$ and $A$ is supported in the region $0 \le p \le m$ from the second page onward, so that it collapses at $E^{m+1}$. If, in addition, $A$ belongs to the localising subcategory of $\mathfrak{T}$ generated by $\mathfrak{P}_{\mathfrak{I}}$, then the spectral sequence converges towards $F_*(A)$. Similar assertions hold in the cohomological case.

*Proof.* We only write down the argument in the homological case. If $A \in\in \mathfrak{P}^{m+1}_{\mathfrak{I}}$, then $N_p \in\in \mathfrak{P}^{m+1}_{\mathfrak{I}}$ for all $p \in \mathbb{N}$ by Lemma 4.4. Therefore, $F : \mathfrak{I}^r(N_p) = F(N_p)$ for all $r \ge m+1$. Plugging this into Proposition 4.1, we get $E^r_{pq} = 0$ for $r \ge m+2$ and $p \ge m+1$. This forces the boundary maps $d^r$ to vanish for $r \ge m+2$, so that $E^\infty_{pq} = E^{m+2}_{pq}$.

Suppose now that $A$ has a projective resolution of length $m$. Embed such a resolution in a phantom tower by Lemma 3.2, so that $P_p = 0$ and $N_p \cong N_{p+1}$ for $p > m$. Then $E^r$ is supported in the region $0 \le p \le m$ for all $r \ge 1$. For $r \ge 2$, this holds for any choice of phantom tower by Theorem 4.3. As a consequence, $d^r = 0$ for $r \ge m+1$ and hence $E^\infty = E^{m+1}$.

Suppose, in addition, that $A$ belongs to the localising subcategory of $\mathfrak{T}$ generated by $\mathfrak{P}_{\mathfrak{I}}$. We claim that $N_p \cong 0$ for $p > m$. This implies that the ABC spectral sequence converges towards $F_*(A)$. If $D \in \mathfrak{P}_{\mathfrak{I}}$, then there are exact sequences

$$\mathfrak{T}_{*+1}(D, N_p) \rightarrowtail \mathfrak{T}_*(D, P_{p-1}) \twoheadrightarrow \mathfrak{T}_*(D, N_{p-1})$$

for all $p \in \mathbb{N}_{\ge 1}$ (cf. the proof of Lemma 3.2). Therefore, $\mathfrak{T}_*(D, N_p) = 0$ for $p > m$ if $D \in\in \mathfrak{P}_{\mathfrak{I}}$. The class of objects $D$ with this property is localising, that is, closed under suspensions, direct sums, direct summands, and exact triangles. Hence it contains the localising subcategory generated by $\mathfrak{P}_{\mathfrak{I}}$. This includes $A = N_0$ by assumption. Since it contains all $P_n$ as well, it contains $N_n$ for all $n \in \mathbb{N}$ because of the exact triangles in the phantom tower, so that $\mathfrak{T}_*(N_n, N_p)$ vanishes for all $n \in \mathbb{N}$. Thus $\mathfrak{T}_*(N_p, N_p) = 0$, forcing $N_p = 0$. Q.E.D.

If $A$ belongs to the localising subcategory generated by the $\mathfrak{J}$-projective objects, then the existence of a projective resolution of length $n$ implies that $A$ is $\mathfrak{J}^n$-projective. This fails without an additional hypothesis because $\mathfrak{J}$-contractible objects have projective resolutions of length 0.

The converse assertion is usually far from true (cf. [7]). Proposition 2.2 shows that an object $A$ of $\mathfrak{T}$ is $\mathfrak{J}^2$-projective if and only if there is an exact triangle $P_2 \to P_1 \to A \to P_2[1]$ with $\mathfrak{J}$-projective objects $P_1$ and $P_2$. The resulting chain complex $0 \to P_2 \to P_1 \to A$ is an $\mathfrak{J}$-projective resolution if and only if the map $A \to P_2[1]$ is an $\mathfrak{J}$-phantom map, if and only if the map $P_2 \to P_1$ is $\mathfrak{J}$-monic. But this need not be the case in general.

Recall that the derived functors of the contravariant functor $A \mapsto \mathfrak{T}_*(A, B)$ are the *extension groups* $\operatorname{Ext}^*_{\mathfrak{T},\mathfrak{J}}(A, B)$. These agree with extension groups in the Abelian approximation, that is, the target category of the universal $\mathfrak{J}$-exact stable homological functor. In particular, $\operatorname{Ext}^0_{\mathfrak{T},\mathfrak{J}}(A, B)$ is the space of morphisms between the images of $A$ and $B$ in the Abelian approximation (cf. [19]). Theorem 4.3 and Proposition 4.2 yield exact sequences

$$0 \to \frac{\mathfrak{T}(A,B)}{\mathfrak{J}(A,B)} \to \operatorname{Ext}^0_{\mathfrak{T},\mathfrak{J}}(A,B) \to \mathfrak{J}(N_1[-1],B) \xrightarrow{(\iota_0^1)^*} \mathfrak{J}^2(A[-1],B) \to 0 \tag{4.8}$$

and

$$0 \to \frac{\mathfrak{J}(A,B)}{\mathfrak{J}^2(A,B)} \to \operatorname{Ext}^1_{\mathfrak{T},\mathfrak{J}}(A,B) \to \mathfrak{J}(N_2[-1],B) \xrightarrow{(\iota_1^2)^*} \mathfrak{J}^2(N_1[-1],B) \to 0. \tag{4.9}$$

In particular, we get injective maps

$$\mathfrak{T}/\mathfrak{J}(A,B) \to \operatorname{Ext}^0_{\mathfrak{T},\mathfrak{J}}(A,B),$$
$$\mathfrak{J}/\mathfrak{J}^2(A,B) \to \operatorname{Ext}^1_{\mathfrak{T},\mathfrak{J}}(A,B).$$

But these maps need not be surjective. What they do is easy to understand. The first map is simply the functor from $\mathfrak{T}$ to the Abelian approximation. The second map embeds a morphism in $\mathfrak{J}$ in an exact triangle. This triangle is $\mathfrak{J}$-exact because it involves a phantom map, and hence provides an extension in the Abelian approximation.

The higher quotients $\mathfrak{J}^n/\mathfrak{J}^{n+1}(A,B)$ are also related to $\operatorname{Ext}^n_{\mathfrak{T},\mathfrak{J}}(A,B)$, but this is merely a relation in the formal sense, that is, it is no longer a map. To construct this relation, we use the $\mathfrak{J}^n$-versality of the map $\iota^n\colon A \to N_n$. Thus any map $f \in \mathfrak{J}^n(A,B)$ factors through a map $\hat{f}\colon N_n \to B$, and we can choose $\hat{f} \in \mathfrak{I}(N_n, B)$ if $f \in \mathfrak{I}^{n+1}(A, B)$. Now we use the map

$$\mathfrak{T}/\mathfrak{I}(N_n, B) \to \mathrm{Ext}^n_{\mathfrak{T},\mathfrak{I}}(A, B)$$

provided by Theorem 4.3 and Proposition 4.2. Since $f$ does not determine the class of $\hat{f}$ in $\mathfrak{T}/\mathfrak{I}(N_n, B)$ uniquely, we only get a relation. The ambiguity in this relation disappears on the $n$th page of the ABC spectral sequence by Proposition 4.2.

Once we have chosen $\hat{f}$ as above, we can also extend it to a morphism between the phantom towers of $A$ and $B$ that shifts degrees down by $n$ — the extension to the left of $N_n$ is induced by $\hat{f} \circ \iota^n_m \colon N_m \to B$ for $m < n$ and vanishes on $P_m$ for $m < n$. Thus we get a morphism between the ABC spectral sequences for $A$ and $B$ for any homological or cohomological functor — shifting degrees down by $n$, of course.

### 4.2 An equivalent exact couple

The cellular approximation tower produces a spectral sequence in the same way as the phantom tower.

We extend the phantom tower to $n < 0$ by $\tilde{A}_n = 0$ and $P_n = 0$ for all $n < 0$. The triangles (3.3) are exact for all $n \in \mathbb{Z}$. A homological functor $F$ maps these exact triangles to exact chain complexes

$$\cdots \longrightarrow F_m(\tilde{A}_n) \xrightarrow{\alpha^{n+1}_{n*}} F_m(\tilde{A}_{n+1}) \xrightarrow{\sigma_{n*}} F_m(P_n) \xrightarrow{\kappa_{n*}} F_{m-1}(\tilde{A}_n) \longrightarrow \cdots .$$

As above, these amount to an exact couple

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd} \qquad \begin{aligned} D'_{pq} &:= F_{p+q}(\tilde{A}_{p+1}), \\ E'_{pq} &:= F_{p+q}(P_p), \end{aligned} \qquad \begin{aligned} i'_{pq} &:= (\alpha^{p+2}_{p+1})_* \colon & D'_{p,q} &\to D'_{p+1,q-1}, \\ j'_{pq} &:= (\sigma_p)_* \colon & D'_{p,q} &\to E'_{p,q}, \\ k'_{pq} &:= (\kappa_p)_* \colon & E'_{p,q} &\to D'_{p-1,q}. \end{aligned}$$

Part of the commuting diagram (3.4) asserts that the identity maps $E \to E'$ and the maps $\gamma_{p+1,*} \colon D_{pq} \to D'_{pq}$ form a morphism of exact couples between the exact couples from the phantom tower and the cellular approximation tower. This induces a morphism between the resulting spectral sequences. Since this morphism acts identically on $E^1$, the induced morphisms on $E^r$ must be invertible for all $r \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. Hence our new spectral sequence is isomorphic to the ABC spectral sequence.

Although the spectral sequences are isomorphic, the underlying exact couples are different and thus provide isomorphic but different descriptions of $E^\infty$.

An important difference between the two exact couples is that $D'_{pq} = 0$ for $p \leq 0$. Hence any element of $D'_{pq}$ is annihilated by a sufficiently high power of $i$. Therefore, the kernel in (4.4) vanishes and

$$E^{\infty}_{pq} \cong D'_{p-1,q} \cap \ker i' \cap \bigcap_{r\in\mathbb{N}} (i')^r(D').$$

Let

$$L_{pq} := \varinjlim \operatorname{range}\bigl(\alpha^r_{p*}\colon F_q(\tilde{A}_p) \to F_q(\tilde{A}_r)\bigr);$$

these spaces define an increasing filtration $(F_{pq})_{p\in\mathbb{N}}$ on $\varinjlim F_q(\tilde{A}_r)$ — we form the limit with the maps $\alpha^n_m$. Using (4.4) and the exactness of colimits of Abelian groups, we get isomorphisms

$$E^{\infty}_{pq} \cong \frac{L_{p+1,p+q}}{L_{p,p+q}}.$$

Hence $E^{\infty}_{pq}$ converges towards $\varinjlim F_q(\tilde{A}_r)$ and induces the filtration $(L_{p,p+q})$ on its limit — without any assumption on the ideal or the homological functor.

In the cohomological case, the exact triangles (3.3) yield an exact couple as well, and the morphism between the two exact couples from the cellular approximation tower and the phantom tower induces an isomorphism between the associated spectral sequences. Again, we get a new description of $\tilde{E}_\infty$.

But the result is not as simple as in the homological case because the projective limit functor for Abelian groups is not exact. Let

$$\tilde{L}^{pq} := \bigcap_{r\geq p} \operatorname{range}\bigl(\alpha^{r*}_p\colon G^q(\tilde{A}_r) \to G^q(\tilde{A}_p)\bigr).$$

Then

$$\tilde{E}^{pq}_\infty \cong \ker\bigl(\tilde{L}^{p+1,p+q} \to \tilde{L}^{p,p+q}\bigr).$$

In general, we cannot say much more than this. If $\tilde{L}^{pq}$ is the range of the map $\varprojlim_r G^q(\tilde{A}_r) \to G^q(\tilde{A}_p)$ for all $p$ and $q$, then the ABC spectral sequence converges towards $\varprojlim_r G^q(\tilde{A}_r)$ and induces on this limit the decreasing filtration by the subspaces $\ker\bigl(\varprojlim_r G^q(\tilde{A}_r) \to G^q(\tilde{A}_p)\bigr)$ for $p \in \mathbb{N}$.

## 5 Convergence of the ABC spectral sequence

There is an obvious obstruction to the convergence of the ABC spectral sequence: the subcategory $\mathfrak{N}_\mathfrak{J}$ of $\mathfrak{J}$-contractible objects. Since $\mathfrak{J}$-derived functors vanish on $\mathfrak{N}_\mathfrak{J}$, the spectral sequence cannot converge towards the original functor unless it vanishes on $\mathfrak{N}_\mathfrak{J}$ as well. At best, the ABC spectral sequence may converge to the *localisation* of the given functor at $\mathfrak{N}_\mathfrak{J}$. We show that this is indeed the case for homological functors that commute with direct sums, provided the ideal $\mathfrak{I}$ is compatible with direct sums. The situation for cohomological functors is less satisfactory because the projective limit functor for Abelian groups is not exact.

We continue to assume throughout this section that the category $\mathfrak{T}$ has countable direct sums. Various notions of convergence of spectral sequences are discussed in [4]. The following results deal only with strong convergence.

**Theorem 5.1.** Let $\mathfrak{I}$ be a homological ideal compatible with direct sums in a triangulated category $\mathfrak{T}$; let $F\colon \mathfrak{T} \to \mathfrak{Ab}$ be a homological functor that commutes with countable direct sums, and let $A \in\in \mathfrak{T}$; let $\mathbb{L}F$ be the localisation of $F$ at $\mathfrak{N}_{\mathfrak{I}}$. Then

$$F : \mathfrak{I}^{2\infty}(A) = F : \mathfrak{I}^{\infty}(A) = \bigcup_{r\in\mathbb{N}} F : \mathfrak{I}^{r}(A) = \operatorname{range}\bigl(\mathbb{L}F(A) \to F(A)\bigr),$$

and the ABC spectral sequence for $F$ and $A$ converges towards $\mathbb{L}F(A)$ with the filtration $\bigl(\mathbb{L}F : \mathfrak{I}^k(A)\bigr)_{k\in\mathbb{N}}$. We have $\bigcup_{k\in\mathbb{N}} \mathbb{L}F : \mathfrak{I}^k(A) = \mathbb{L}F(A)$.

*Proof.* Lemma 3.6 implies that $F : \mathfrak{I}^r(A)$ is the range of $\alpha_{r*}\colon F(\tilde{A}_r) \to F(A)$ for all $r \in \mathbb{N}$ and that $F : \mathfrak{I}^{\infty}(A)$ is the range of $F\bigl(\bigoplus \tilde{A}_r\bigr) \to F(A)$ because (3.9) is an $\mathfrak{I}^{\infty}$-projective resolution. Now $F\bigl(\bigoplus \tilde{A}_r\bigr) = \bigoplus F(\tilde{A}_r)$ shows that $F : \mathfrak{I}^{\infty}(A)$ is the union of $F : \mathfrak{I}^r(A)$ for $r \in \mathbb{N}$.

Let $\tilde{A}$ be as in (3.8), so that $\tilde{A} = L(A)$ and $\mathbb{L}F(A) = F(\tilde{A})$. Since the inductive limit functor for Abelian groups is exact, the map $\mathrm{id} - S$ on $\bigoplus F(\tilde{A}_r)$ is injective and has cokernel $\varinjlim F(\tilde{A}_r)$. Since the top row in (3.8) is an exact triangle, the long exact sequence yields $F(\tilde{A}) \cong \varinjlim F(\tilde{A}_r)$. As a consequence, the range of $f_*\colon F(\tilde{A}) \to F(A)$ is equal to $F : \mathfrak{I}^{\infty}(A)$. Since $\tilde{A}$ is $\mathfrak{I}^{2\infty}$-projective by Theorem 3.19 and $f\colon \tilde{A} \to A$ is an $\mathfrak{I}$-equivalence, this map is an $\mathfrak{I}^{2\infty}$-projective resolution of $A$. Hence the range of $f_*$ also agrees with $F : \mathfrak{I}^{2\infty}(A)$ by Lemma 3.6.

Especially, $F : \mathfrak{I}^{\infty}(A) = F(A)$ if $A \in\in \langle \mathfrak{P}_{\mathfrak{I}} \rangle$. For such $A$, all objects that occur in the phantom castle belong to $\langle \mathfrak{P}_{\mathfrak{I}} \rangle$ as well, so that $F(N_p) = \bigcup_{r\in\mathbb{N}} F : \mathfrak{I}^r(N_p)$ for all $p \in \mathbb{N}$. Hence Proposition 4.1 yields the convergence of the ABC spectral sequence to $F(A)$ as asserted. Since the ABC spectral sequences for $A$ and $\tilde{A}$ are isomorphic by Proposition 3.20, we get convergence towards $\mathbb{L}F(A)$ for general $A$. Q.E.D.

The convergence of the ABC spectral sequences is more problematic for a cohomological functor $G\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{Ab}$ because projective limits of Abelian groups are not exact. In the following, we *assume that $G$ maps direct sums to direct products* — this is the correct compatibility with direct sums for contravariant functors.

The exactness of the first row in (3.8) yields an exact sequence

$$\varprojlim{}^1 G^{*-1}(\tilde{A}_n) \rightarrowtail G^*(\tilde{A}) \twoheadrightarrow \varprojlim G^*(\tilde{A}_n) \tag{5.1}$$

for any $A$ (this also follows from [19, Theorem 4.4] applied to the ideal $\mathfrak{I}^\infty$). Furthermore, $G^*(\tilde{A}) = \mathbb{R}G^*(A)$. Since (3.8) is an $\mathfrak{I}^\infty$-projective resolution, we have

$$\varprojlim{}^1 G^{-1}(\tilde{A}_n) \cong \mathfrak{I}^\infty G(\tilde{A}).$$

The same argument as in the homological case yields

$$\mathfrak{I}^\infty G(A) = \bigcap_{n\in\mathbb{N}} \mathfrak{I}^n G(A). \tag{5.2}$$

Using compatibility of $G$ with direct sums, we can also rewrite the obstructions to the convergence of the ABC spectral sequence in Proposition 4.2:

$$\widetilde{\mathsf{Bad}}^{pq} \cong \mathfrak{I}^\infty G^q(\tilde{N}_p),$$

where $\tilde{N}_p$ is the $p$th object in a phantom tower over $\tilde{A}$ instead of $A$. The spectral sequence converges towards $\mathbb{R}G(A)$ if these obstructions all vanish.

**Proposition 5.2.** Let $\mathfrak{I}$ be a homological ideal with enough projectives that is compatible with direct sums, and let $G\colon \mathfrak{T}^{\mathrm{op}} \to \mathfrak{Ab}$ be a cohomological functor that maps direct sums to direct products. Let $A \in\in \mathfrak{T}$ and let $L(A) \in\in \langle \mathfrak{P}_\mathfrak{I} \rangle$ be its $\mathfrak{P}_\mathfrak{I}$-cellular approximation. If $L(A)$ is $\mathfrak{I}^\infty$-projective, then the ABC spectral sequence for $A$ and $G$ converges towards $\mathbb{R}G(A) = G \circ L(A)$.

*Proof.* Proposition 3.20 implies that $A$ and $L(A)$ have canonically isomorphic ABC spectral sequences. Hence we may replace $A$ by $L(A)$ and assume that $A$ itself is $\mathfrak{I}^\infty$-projective. By Proposition 2.3, $A$ is a direct summand of $\bigoplus_{n\in\mathbb{N}} A_n$ with $\mathfrak{I}^n$-projective objects $A_n$. The ABC spectral sequence for each $A_n$ converges by Proposition 4.5.

Since $\mathfrak{I}$ is compatible with countable direct sums, a direct sum of phantom castles over $A_n$ is a phantom castle over $\bigoplus A_n$. Thus the ABC spectral sequence for $\bigoplus_{n\in\mathbb{N}} A_n$ is the direct product of the ABC spectral sequences for $A_n$; here we use that $G$ maps direct sums to direct products. Hence the ABC spectral sequence for $\bigoplus_{n\in\mathbb{N}} A_n$ converges towards $\prod G(A_n) = G\bigl(\bigoplus A_n\bigr)$. Since the ABC spectral sequence is an additive functor on $\mathfrak{T}$, this implies that the ABC spectral sequence for any direct summand of $\bigoplus A_n$ converges. This yields the convergence of the ABC spectral sequence for $L(A)$, as desired. Q.E.D.

## 6 A classical special case

Before we apply our results to equivariant bivariant K-theory, we briefly discuss a more classical application in homological algebra, where we recover results by Marcel Bökstedt and Amnon Neeman [5] and where the

ABC sectral sequence specialises to a spectral sequence due to Alexander Grothendieck.

Let $\mathfrak{A}$ be an Abelian category with enough projective objects and exact countable direct sums. Let $\mathrm{Ho}(\mathfrak{A})$ be the homotopy category of chain complexes over $\mathfrak{A}$. We require no finiteness conditions, so that $\mathrm{Ho}(\mathfrak{A})$ is a triangulated category with countable direct sums. We are interested in the derived category of $\mathfrak{A}$ and therefore want to localise at the full subcategory $\mathfrak{N} \subseteq \mathrm{Ho}(\mathfrak{A})$ of exact chain complexes. This subcategory is localising because countable direct sums of exact chain complexes are again exact by assumption.

The obvious functor defining this subcategory $\mathfrak{N}$ is the functor

$$H\colon \mathrm{Ho}(\mathfrak{A}) \to \mathfrak{A}^{\mathbb{Z}}$$

that maps a chain complex to its homology. The functor $H$ is a stable homological functor that commutes with direct sums. Hence its kernel $\mathfrak{I}_H$ is a homological ideal that is compatible with direct sums.

Let $\mathfrak{P}\mathfrak{A}^{\mathbb{Z}} \subseteq \mathfrak{A}^{\mathbb{Z}}$ be the full subcategory of projective objects. Since we assume $\mathfrak{A}$ to have enough projective objects, any object of $\mathfrak{A}^{\mathbb{Z}}$ admits an epimorphism from an object in $\mathfrak{P}\mathfrak{A}^{\mathbb{Z}}$. It is easy to see that the left adjoint of the homology functor is defined on $\mathfrak{P}\mathfrak{A}^{\mathbb{Z}}$ and maps a sequence $(P_n)$ of projective objects to the chain complex $(P_n)$ with vanishing boundary map. Since this functor is clearly fully faithful, we use it to view $\mathfrak{P}\mathfrak{A}^{\mathbb{Z}}$ as a full subcategory of $\mathrm{Ho}(\mathfrak{A})$, omitting the functor $H^{\vdash}$ from our notation. Using the criterion of [19], it is easy to check that the functor $H$ above is the universal $\mathfrak{I}_H$-exact homological functor.

Theorems 3.22 and 3.23 apply here. The first one shows that $(\langle \mathfrak{P}\mathfrak{A}^{\mathbb{Z}} \rangle, \mathfrak{N})$ is a complementary pair of subcategories. Thus $\langle \mathfrak{P}\mathfrak{A}^{\mathbb{Z}} \rangle$ is equivalent to the derived category of $\mathfrak{A}$. Furthermore, any object of the derived category is a homotopy colimit of a diagram with entries in $(\mathfrak{P}\mathfrak{A}^{\mathbb{Z}})^{\star n}$ for $n \in \mathbb{N}$.

Let $F\colon \mathfrak{A} \to \mathfrak{Ab}$ be an additive covariant functor that commutes with direct sums. We extend $F$ to an exact functor $\mathrm{Ho}(F)\colon \mathrm{Ho}(\mathfrak{A}) \to \mathrm{Ho}(\mathfrak{Ab})$. Let

$$\bar{F}_q = H_q \circ \mathrm{Ho}(F)\colon \mathrm{Ho}(\mathfrak{A}) \to \mathfrak{Ab}$$

be the functor that maps a chain complex $C_\bullet$ to the $q$th homology of $\mathrm{Ho}(F)(C_\bullet)$. This is a homological functor. Its derived functors with respect to $\mathfrak{I}_H$ are computed in [19]: for a chain complex $C_\bullet$, we have

$$\mathbb{L}_p \bar{F}_q(C_\bullet) = (\mathbb{L}_p F)\big(H_q(C_\bullet)\big),$$

that is, we apply the usual derived functors of $F$ to the homology of $C_\bullet$. Thus the ABC spectral sequence computes the homology of the total derived functor of $F$ applied to $C_\bullet$ in terms of the derived functors of $F$, applied to $H_*(C_\bullet)$. Such a spectral sequence was already constructed by Alexander Grothendieck.

## 7 Construction of the Baum-Connes assembly map

Finally, we apply our general machinery to construct the Baum-Connes assembly map with coefficients first for locally compact groups and then for certain discrete quantum groups. In the group case, we get a simpler argument than in [16].

### 7.1 The assembly map for locally compact groups

Let $G$ be a second countable locally compact group and let $\mathfrak{KK}^G$ be the $G$-equivariant Kasparov category; its objects are the separable C*-algebras with a strongly continuous action of $G$, its morphism space $A \to B$ is $\mathrm{KK}_0^G(A,B)$. It is shown in [16] that this category is triangulated (we must exclude $\mathbb{Z}/2$-graded C*-algebras for this).

The category $\mathfrak{KK}^G$ has countable direct sums — they are just direct sums of C*-algebras. But uncountable direct sums usually do not exist because of the separability assumption in the definition of $\mathfrak{KK}^G$, which is needed to make the analysis work. Alternative definitions of bivariant K-theory by Joachim Cuntz [8] still work for non-separable C*-algebras, but it is not clear whether direct sums of C*-algebras remain direct sums in this category because the definition of the Kasparov groups for inseparable C*-algebras involves colimits, which do not commute with the direct products that appear in the definition of the direct sum.

With enough effort, it should be possible to extend $\mathfrak{KK}^G$ to a category with uncountable direct sums. But it seems easier to avoid this by imposing cardinality restrictions on direct sums.

For any closed subgroup $H \subseteq G$, we have induction and restriction functors

$$\mathrm{Ind}_H^G\colon \mathfrak{KK}^H \to \mathfrak{KK}^G, \qquad \mathrm{Res}_G^H\colon \mathfrak{KK}^G \to \mathfrak{KK}^H;$$

the latter functor is quite trivial and simply forgets part of the group action. These functors give rise to two subcategories of $\mathfrak{KK}^G$, which play a crucial role in [16].

**Definition 7.1.** Let $\mathcal{F}$ be the set of all compact subgroups of $G$.

$$\mathcal{CC} := \{A \in\in \mathfrak{KK}^G \mid \mathrm{Res}_G^H(A) = 0 \text{ for all } H \in \mathcal{F}\},$$
$$\mathcal{CI} := \{\mathrm{Ind}_H^G(A) \mid A \in\in \mathfrak{KK}^H \text{ and } H \in \mathcal{F}\}.$$

Whereas the subcategory $\mathcal{CC}$ is localising by definition, $\mathcal{CI}$ is not. Therefore, the localising subcategory it generates, $\langle\mathcal{CI}\rangle$, plays an important role as well. Since $G$ acts properly on objects of $\mathcal{CI}$, they satisfy the Baum-Connes conjecture, that is, the Baum-Connes assembly map is an isomorphism for coefficients in $\mathcal{CI}$. Since domain and target of the assembly map are exact functors on $\mathfrak{KK}^G$, this extends to the category $\langle\mathcal{CI}\rangle$. On objects of $\mathcal{CC}$ the domain of the Baum-Connes assembly map is known to vanish, so that the Baum-Connes conjecture predicts $\mathrm{K}_*(G \ltimes_{\mathrm{r}} A) = 0$ for $A \in\in \mathcal{CC}$.

On a technical level, the main tool in [16] is that the pair of subcategories $(\langle\mathcal{CI}\rangle, \mathcal{CC})$ is complementary. Hence the Baum-Connes assembly map is determined by what it does on these two subcategories. This implies that its domain is the localisation of the functor $A \mapsto \mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$ at $\mathcal{CC}$ and that the assembly map is the natural transformation from this localisation to the original functor.

Put differently, the Baum-Connes assembly map is the only natural transformation from an exact functor on $\mathfrak{KK}^G$ to the functor $\mathrm{K}_*(G \ltimes_{\mathrm{r}} \text{␣})$ that is an isomorphism on $\mathcal{CI}$ and whose domain vanishes on $\mathcal{CC}$ (we give some more details about this argument in the related quantum group case below).

In order to prove that $(\langle\mathcal{CI}\rangle, \mathcal{CC})$ is complementary, we introduce the following ideal:

**Definition 7.2.** Let $\mathcal{I} = \bigcap_{H\in\mathcal{F}} \ker \mathrm{Res}_G^H$.

This ideal consists of the morphisms that *vanish for compact subgroups* in the notation of [16]. Clearly, an object belongs to $\mathcal{CC}$ if and only if its identity map belongs to $\mathcal{I}$, that is, $\mathfrak{N}_{\mathcal{I}} = \mathcal{CC}$. Moreover, [16, Proposition 4.4] implies that objects of $\mathcal{CI}$ are $\mathcal{I}$-projective; even more, $f \in \mathcal{I}(A, B)$ if and only if $f$ induces the zero map $\mathrm{KK}_*^G(D, A) \to \mathrm{KK}_*^G(D, B)$ for all $D \in\in \mathcal{CI}$.

We can also describe $\mathcal{I}$ as the kernel of a single functor:

$$F = (\mathrm{Res}_G^H)_{H\in\mathcal{F}} \colon \mathfrak{KK}^G \to \prod_{H\in\mathcal{F}} \mathfrak{KK}^H.$$

The functor $F$ commutes with direct sums because each functor $\mathrm{Res}_G^H$ clearly does so. Hence $\mathcal{I}$ is compatible with countable direct sums.

The following theorem contains the main assertion in [16, Theorem 4.7]. We will provide a simpler proof here than in [16].

**Theorem 7.3.** The projective objects for $\mathcal{I}$ are the retracts of direct sums of objects in $\mathcal{CI}$, and the ideal $\mathcal{I}$ has enough projective objects. Hence the pair of subcategories $(\langle\mathcal{CI}\rangle, \mathcal{CC})$ is complementary.

*Proof.* As in Theorem 3.22, we study the partially defined left adjoint of the functor $F$ above or, equivalently, of the functors $\mathrm{Res}_G^H$ for $H \in \mathcal{F}$.

The discrete case is particularly simple because then all $H \in \mathcal{F}$ are open subgroups. If $H \subseteq G$ is open, then $\mathrm{Ind}_H^G$ is left adjoint to $\mathrm{Res}_G^H$. Thus we may take $\mathfrak{PC} = \prod_{H\in\mathcal{F}} \mathfrak{KK}^H$ in Theorem 3.22 and get $F^\vdash\big((A_H)_{H\in\mathcal{F}}\big) = \bigoplus_{H\in\mathcal{F}} \operatorname{Ind}_H^G(A_H)$. Notice that the set $\mathcal{F}$ is countable if $G$ is discrete, so that this definition is legitimate. It follows that $\mathcal{I}$ has enough projective objects and that they are all direct summands of $\bigoplus_{H\in\mathcal{F}} \operatorname{Ind}_H^G(A_H)$ for suitable families $(A_H)$, as asserted.

For locally compact $G$, the argument gets more complicated because the functor $\operatorname{Res}_G^H$ does not always have a left adjoint, and if it has, it need not be simply $\operatorname{Ind}_H^G$. But there are still enough compact subgroups $H$ for which the left adjoint is defined on enough $H$-$\mathrm{C}^*$-algebras and close enough to the induction functor for the argument above to go through.

A good way to understand this is the duality theory developed in [9, 10]. This is relevant because the induction functor provides an equivalence of categories $\mathfrak{KK}^H \simeq \mathfrak{KK}^{G\ltimes G/H}$, where we use the groupoid $G \ltimes G/H$, that is, we consider $G$-equivariant bundles of $\mathrm{C}^*$-algebras over $G/H$. This equivalence of categories reflects the equivalence between the groupoids $H$ and $G \ltimes G/H$.

Identifying $\mathfrak{KK}^H \simeq \mathfrak{KK}^{G\ltimes G/H}$, the restriction functor $\operatorname{Res}_G^H$ becomes the functor $p^*_{G/H}\colon \mathfrak{KK}^G \to \mathfrak{KK}^{G\ltimes G/H}$ that pulls back a $G$-$\mathrm{C}^*$-algebra to a trivial bundle of $G$-$\mathrm{C}^*$-algebras over $G/H$. Following [12], it is shown in [9] that the left adjoint of $p^*_{G/H}$ is defined on all trivial bundles if $G/H$ is a smooth manifold. We will see that this is enough for our purposes.

As in [16], we call a compact subgroup *large* if it is a maximal compact subgroup in an open, almost connected subgroup of $G$.

Let $H$ be large. Then $G/H$ is a smooth manifold and any compact subgroup is contained in a large one by [16, Lemma 3.1]. Furthermore, since $G$ is second countable there is a sequence $(U_n)_{n\in\mathbb{N}}$ of almost connected open subgroups of $G$ such that any other one is contained in $U_n$ for some $n \in \mathbb{N}$. Pick a maximal compact subgroup $H_n \subseteq U_n$ for each $n \in \mathbb{N}$. Then any compact subgroup of $G$ is subconjugate to $H_n$ for some $n \in \mathbb{N}$. Therefore, we already have $\mathcal{I} = \bigcap_{n\in\mathbb{N}} \operatorname{Res}_G^{H_n}$ because $\operatorname{Res}_G^K$ factors through $\operatorname{Res}_G^H$ if $K$ is subconjugate to $H$.

For a compact subgroup $H \subseteq G$, let $\mathfrak{RKK}^G(G/H) \subseteq \mathfrak{KK}^{G\ltimes G/H}$ be the full subcategory of trivial bundles over $G/H$ or, equivalently, the essential range of the functor $p^*_{G/H}$. We do not care whether this category is triangulated, it is certainly additive. We replace the functors $\operatorname{Res}_G^H$ by $p^*_{G/H}\colon \mathfrak{KK}^G \to \mathfrak{RKK}^G(G/H)$ for $H \in \mathcal{F}$. For the large compact subgroups $H_n$ selected above, the results in [10] show that the left adjoint of $p^*_{G/H}$ is defined on all of $\mathfrak{RKK}^G(G/H)$ and maps the trivial bundle with fibre $A$ to $\mathrm{C}_0(T\,G/H) \otimes A$ with the diagonal action of $G$; here $T\,G/H$ denotes the tangent space of $G/H$ (we are not allowed to use the Clifford algebra dual considered in [9] because it involves $\mathbb{Z}/2$-graded $\mathrm{C}^*$-algebras, which do not belong to our category).

Thus we have verified the hypotheses of Theorem 3.22 and can conclude that $\mathcal{I}$ has enough projective objects and that $\mathcal{CC}$ is reflective. It remains to observe that the projective objects are precisely the direct summands of countable direct sums of objects of $\mathcal{CI}$. We have already observed that objects of $\mathcal{CI}$ are $\mathcal{I}$-projective. Conversely, Theorem 3.22 shows that the projective objects are retracts of $\bigoplus_{n\in\mathbb{N}} \mathrm{C}_0(T\,G/H_n)\otimes A_n$ for suitable $G$-C*-algebras $A_n$. The summands are isomorphic to $\mathrm{Ind}_{H_n}^G(\mathrm{C}_0(T_1\,G/H_n)\otimes A_n)$ where $T_1\,G/H_n$ denotes the tangent space of $G/H_n$ at the base point $1\cdot H_n$. Hence all projective objects are of the required form. Q.E.D.

Since the stable homological functor $F_*(A) := \mathrm{K}_*(G \ltimes_{\mathrm{r}} A)$ commutes with direct sums, Theorem 5.1 applies to it and shows that the ABC spectral sequence for the ideal $\mathcal{I}$ converges towards the domain of the Baum-Connes assembly map — which is the localisation of $F_*$ at $\mathcal{CC}$ by [16, Theorem 5.2].

It turns out that for a totally disconnected group $G$ the ABC spectral sequence agrees with a known spectral sequence that we get from the older definition of the Baum-Connes assembly map and the skeletal filtration of a $G$-CW-model for the universal proper $G$-space $\mathcal{E}G$ (cf. [13]). We omit the proof of this statement, which requires some work.

### 7.2 An assembly map for torsion-free discrete quantum groups

Before we turn to the assembly map, we must discuss some open problems that lead us to restrict attention to the torsion-free case.

The first issue is the correct definition of "torsion" for locally compact quantum groups. The torsion in a locally compact group is the family of compact subgroups. Quantum groups exhibit some torsion phenomena that do not appear for groups, and it is conceivable that we have not yet found all of them. First, compact quantum subgroups are not enough: they should be replaced by proper quantum homogeneous spaces, so that open subgroups provide torsion in $\mathrm{C}^*(G)$ whenever $G$ is disconnected. Secondly, projective representations of compact groups with a non-trivial cocycle also provide torsion (in their discrete dual); for instance, $\mathrm{C}^*\bigl(\mathrm{SO}(3)\bigr)$ is not torsion-free because of its projective representation on $\mathbb{C}^2$.

If we considered $\mathrm{C}^*\bigl(\mathrm{SO}(3)\bigr)$ to be torsion-free, then the Baum-Connes assembly map for it (which we describe below) would fail to be an isomorphism. The correct formulation of the Baum-Connes conjecture for $\mathrm{C}^*\bigl(\mathrm{SO}(3)\bigr)$ turns out to be equivalent to the Baum-Connes conjecture for $\mathrm{C}^*\bigl(\mathrm{SU}(2)\bigr)$ —which is torsion-free— so that there is no need to discuss it in its own right in [17].

I propose to approach torsion in *discrete* quantum groups by studying actions of its compact dual quantum group on finite-dimensional C*-algebras.

A discrete quantum group is *torsion-free* if any such action is a direct sum of actions that are Morita equivalent to the trivial action on $\mathbb{C}$.

The above definition of torsion gives the expected results in simple cases. First, $\mathrm{C}_0(G)$ for a discrete group $G$ is torsion-free if and only if $G$ contains no finite subgroups. Secondly, $\mathrm{C}^*(G)$ for a compact group is torsion-free if and only if $G$ is connected and has torsion-free fundamental group; this is exactly the generality in which Universal Coefficient Theorems for equivariant Kasparov theory work (cf. [17, 22]). Christian Voigt shows in [24] that the quantum deformations of simply connected Lie groups such as $\mathrm{SU}_q(n)$ are torsion-free.

Another issue is to find analogues of the restriction and induction functors for the non-classical torsion that may appear, and to prove analogues of the adjointness relations used in the proof of Theorem 7.3. For honest quantum subgroups, the restriction functor is evident, and Stefaan Vaes has constructed induction functors for actions of quantum group $\mathrm{C}^*$-algebras in [23]. I expect restriction to be left adjoint to induction for open quantum subgroups and, in particular, for quantum subgroups of discrete quantum groups.

For the time being, we avoid these problems and limit our attention to the torsion-free case. More precisely, we consider arbitrary discrete quantum groups, but disregard torsion. The resulting assembly map should not be an isomorphism for quantum groups with torsion.

The discrete quantum groups are precisely the duals of compact quantum groups; we use reduced duals here because these appear also in the Baum-Connes conjecture. It is useful to reformulate results about a discrete quantum group in terms of its compact dual as in [19, Remark 2.9]. Let $G$ be a compact quantum group and let $\widehat{G}$ be its discrete dual. Since we pretend that $\widehat{G}$ is torsion-free, there is only one “restriction functor” to consider: the forgetful functor $\mathfrak{KK}^{\widehat{G}} \to \mathfrak{KK}$ that forgets the action of $\widehat{G}$ altogether. The category $\mathfrak{KK}^{\widehat{G}}$ is equivalent to $\mathfrak{KK}^{G}$ by Baaj-Skandalis duality. Under this equivalence, the forgetful functor $\mathfrak{KK}^{\widehat{G}} \to \mathfrak{KK}$ corresponds to the crossed product functor

$$G \ltimes \text{␣}\colon \mathfrak{KK}^{G} \to \mathfrak{KK}, \qquad A \mapsto G \ltimes A.$$

The induction functor from the trivial subgroup to $\widehat{G}$ corresponds under Baaj-Skandalis duality to the functor $\tau\colon \mathfrak{KK} \to \mathfrak{KK}^{G}$ that equips a $\mathrm{C}^*$-algebra with the trivial action of $G$. This functor is left adjoint to the crossed product functor.

Hence the relevant subcategories $\mathcal{CI}$, $\mathcal{CC}$ and the ideal $\mathcal{I}$ correspond to

$$\mathcal{CI} = \{\tau(A) \mid A \in\in \mathfrak{KK}\}, \qquad \mathcal{CC} = \{A \in\in \mathfrak{KK}^{G} \mid G \ltimes A \simeq 0\},$$

where $\simeq$ means KK-equivalence, that is, isomorphism in $\mathfrak{KK}$, and

$$\mathcal{I} = \{f \in \mathfrak{KK}^G \mid G \ltimes f = 0\}.$$

The ideal $\mathcal{I}$ is already studied in [19, § 5]. It is shown there that $\mathcal{I}$ has enough projective objects, and the universal homological functor for it is described. The target category involves actions of the representation ring $\mathrm{Rep}(G)$ of the compact quantum group $G$ on objects of $\mathfrak{KK}$; such an action on $A$ is, by definition, a ring homomorphism $\mathrm{Rep}(G) \to \mathrm{KK}_0(A, A)$. The category $\mathfrak{KK}[\mathrm{Rep}(G)]$ of $\mathrm{Rep}(G)$-modules in $\mathfrak{KK}$ is not yet Abelian because $\mathfrak{KK}$ is not Abelian. To remedy this, we must replace $\mathfrak{KK}$ by its Freyd category of coherent functors $\mathfrak{KK} \to \mathfrak{Ab}$. But this completion does not affect homological algebra much because $\mathfrak{KK}[\mathrm{Rep}(G)]$ is an exact subcategory that contains all projective objects; hence we can compute derived functors without leaving the subcategory $\mathfrak{KK}[\mathrm{Rep}(G)]$.

We could modify the ideal $\mathcal{I}$ and consider all $f$ for which $G \ltimes f$ induces the zero map on K-theory. This leads to a simpler Abelian approximation, namely, the category of all countable $\mathbb{Z}/2$-graded $\mathrm{Rep}(G)$-modules. But this larger ideal no longer leads to the subcategories $\mathcal{CC}$ and $\mathcal{CI}$ above.

**Theorem 7.4.** Let $G$ be any compact quantum group. Then the ideal $\mathcal{I}$ is compatible with countable direct sums and has enough projective objects. The pair of subcategories $(\langle\mathcal{CI}\rangle, \mathcal{CC})$ is complementary.

*Proof.* The ideal $\mathcal{I}$ has enough $\mathcal{I}$-projective objects by [16, Lemma 5.2], which also shows that the $\mathcal{I}$-projective objects are precisely the direct summands of objects in $\mathcal{CI}$. The ideal $\mathcal{I}$ is compatible with direct sums because the crossed product functor commutes with direct sums. Now Theorem 3.16 shows that the pair of subcategories $(\langle\mathcal{CI}\rangle, \mathcal{CC})$ is complementary. Q.E.D.

**Definition 7.5.** Let $F: \mathfrak{KK}^G \to \mathfrak{A}$ be some homological functor. The *assembly map* for $F$ with coefficients in $A$ is the map $\mathbb{L}F(A) \to F(A)$, where the localisation $\mathbb{L}F$ is formed with respect to the subcategory $\mathcal{CC}$.

To get an analogue of the Baum-Connes assembly map, we should consider the functor $F(A) := \mathrm{K}_*(A)$ because it corresponds to the functor $B \mapsto \mathrm{K}_*(G \ltimes_{\mathrm{r}} B)$ under Baaj-Skandalis duality. A torsion-free discrete quantum group has the *Baum-Connes property* with coefficients if the assembly map $\mathbb{L}F(A) \to F(A)$ is an isomorphism for all $A$ for this functor.

**Proposition 7.6.** Let $F: \mathfrak{KK}^G \to \mathfrak{A}$ be a homological functor that commutes with direct sums. The assembly map $\mathbb{L}F \Rightarrow F$ is the unique natural transformation from a functor $\tilde{F}$ to $F$ with the following properties:

- $\tilde{F}$ is homological and commutes with direct sums;

- the natural transformation is an isomorphism for objects in $\mathcal{CI}$;
- $\tilde{F}$ vanishes on $\mathcal{CC}$.

*Proof.* Let $\tilde{F} \Rightarrow F$ be a natural transformation with the required properties. Since both functors involved are homological, the Five Lemma implies that the class of objects for which the natural transformation $\tilde{F} \Rightarrow F$ is an isomorphism is triangulated. It is also closed under direct sums because both functors commute with direct sums. Hence the natural transformation $\tilde{F} \Rightarrow F$ is an isomorphism for all objects in $\langle \mathcal{CI} \rangle$ because this holds for objects in $\mathcal{CI}$.

Since $\tilde{F}$ vanishes on $\mathcal{CC}$ and is homological, the universal property of the localisation shows that the natural transformation $\tilde{F} \Rightarrow F$ factors uniquely through the assembly map: $\tilde{F} \Rightarrow \mathbb{L}F \Rightarrow F$. Both $\tilde{F}$ and $\mathbb{L}F$ descend to the category $\mathfrak{KK}^G/\mathcal{CC}$, which is equivalent to $\langle \mathcal{CI} \rangle$. Since both natural transformations $\tilde{F} \Rightarrow F$ and $\mathbb{L}F \Rightarrow F$ are invertible on objects of $\langle \mathcal{CI} \rangle$, we get the desired natural isomorphism $\tilde{F} \cong \mathbb{L}F$. Q.E.D.

The critical property in Proposition 7.6 is the vanishing on $\mathcal{CC}$. This cannot be expected if $\widehat{G}$ has torsion. The Baum-Connes assembly map is an isomorphism for $F$ if and only if $F$ vanishes on $\mathcal{CC}$: one direction is trivial, and the other follows by taking $\tilde{F} = F$ in Proposition 7.6. While this reformulation of the Baum-Connes conjecture came too late to be used in verifying the conjecture for groups, it is quite helpful for duals of compact groups (cf. [17]) and probably also for their deformations.

## 8 Conclusion

The idea of localisation —central both in homological algebra and in homotopy theory— is becoming more important in non-commutative topology as well. When refined using homological ideals, it unifies various new and old universal coefficient theorems, the Baum-Connes conjecture, and its extensions to quantum groups.

Homological ideals provide some basic topological tools in the general setting of triangulated categories. This includes

- important notions from homological algebra like projective resolutions and derived functors (these were already dealt with in [19]);
- an efficient method to check that pairs of subcategories in a triangulated category are complementary;
- some control on how objects of the category are constructed from generators, that is, from the projective objects for the ideal;

- a natural spectral sequence that computes the localisation of a homological functor from its values on generators.

Since the assumptions on the underlying category are quite weak, all this applies to equivariant bivariant K-theory.

We have applied this general machinery to construct the Baum-Connes assembly map for torsion-free quantum groups, whose domain is of topological nature in the sense that it can be computed by topological techniques such as spectral sequences. But much remains to be done here. The three main issues are to understand torsion in locally compact quantum groups, to adapt the reduction and induction functors to exotic torsion phenomena, and to check whether the assembly map is an isomorphism. These problems are mainly *analytical* in nature.

## References


[1] J. F. Adams. *Stable homotopy and generalised homology.* University of Chicago Press, 1974. Chicago Lectures in Mathematics.

[2] A. Beligiannis. Relative homological algebra and purity in triangulated categories. *Journal of Algebra*, 227(1):268–361, 2000.

[3] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analyse et topologie sur les espaces singuliers. I. Proceedings of the colloquium held at Luminy, July 6–11, 1981*, volume 100 of *Astérisque*, pages 5–171, Paris, 1982. Société Mathématique de France.

[4] J. M. Boardman. Conditionally convergent spectral sequences. In J.-P. Meyer, J. Morava, and W. S. Wilson, editors, *Homotopy invariant algebraic structures, Proceedings of the AMS Special Session on Homotopy Theory held at the Annual Meeting of the American Mathematical Society, Baltimore, MD, January 7–10, 1998*, volume 239 of *Contemporary Mathematics*, pages 49–84, Providence, RI, 1999. American Mathematical Society.

[5] M. Bökstedt and A. Neeman. Homotopy limits in triangulated categories. *Compositio Mathematica*, 86(2):209–234, 1993.

[6] H.-B. Brinkmann. Relative homological algebra and the Adams spectral sequence. *Archiv der Mathematik*, 19:137–155, 1968.

[7] J. D. Christensen. Ideals in triangulated categories: phantoms, ghosts and skeleta. *Advances of Mathematics*, 136(2):284–339, 1998.

[8] J. Cuntz, R. Meyer, and J. M. Rosenberg. *Topological and bivariant K-theory*, volume 36 of *Oberwolfach Seminars*. Birkhäuser Verlag, 2007.

[9] H. Emerson and R. Meyer. Euler characteristics and Gysin sequences for group actions on boundaries. *Mathematische Annalen*, 334(4):853–904, 2006.

[10] H. Emerson and R. Meyer. Dualities in equivariant Kasparov theory, 2007. preprint arXiv `0711.0025`.

[11] D. Goswami and A. O. Kuku. A complete formulation of the Baum–Connes conjecture for the action of discrete quantum groups. *K-Theory*, 30(4):341–363, 2003.

[12] G. G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Inventiones Mathematicae*, 91(1):147–201, 1988.

[13] G. G. Kasparov and G. Skandalis. Groups acting on buildings, operator K-theory, and Novikov's conjecture. *K-Theory*, 4(4):303–337, 1991.

[14] H. Krause. Cohomological quotients and smashing localizations. *American Journal of Mathematics*, 127(6):1191–1246, 2005.

[15] S. Mac Lane. *Homology.* Classics in Mathematics. Springer, 1995. Reprint of the 1975 edition.

[16] R. Meyer and R. Nest. The Baum–Connes conjecture via localisation of categories. *Topology*, 45(2):209–259, 2006.

[17] R. Meyer and R. Nest. An analogue of the Baum–Connes isomorphism for coactions of compact groups. *Math. Scand.*, 100(2):301–316, 2007.

[18] R. Meyer and R. Nest. $C^*$-algebras over topological spaces: filtrated K-theory, 2007. preprint arXiv `0810.0096`.

[19] R. Meyer and R. Nest. Homological algebra in bivariant K-theory and other triangulated categories. I, 2007. preprint arXiv `math.KT/0702146`.

[20] R. Meyer and R. Nest. $C^*$-algebras over topological spaces: the bootstrap class. *Münster Journal of Mathematics*, to appear.

[21] A. Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, 2001.

[22] J. Rosenberg and C. Schochet. The Künneth theorem and the universal coefficient theorem for equivariant K-theory and KK-theory. *Memoirs of the American Mathematical Society*, 62(348), 1986.

[23] S. Vaes. A new approach to induction and imprimitivity results. *Journal of Functional Analysis*, 229(2):317–374, 2005.

[24] C. Voigt. An analogue of the Baum–Connes conjecture for the dual of $\mathrm{SU_q}(2)$, 2008. preprint.