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SUBMODULES OF MULTIPLICATION MODULES

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Abstract. Let R be a commutative ring with identity (zero-divisors admitted). Various properties of submodules a multiplication module are considered. In fact, our aim here is to generalize some of the results in the paper listed as [1], from finitely generated faithful multiplication ideals to finitely generated faithful multiplication modules.

1. INTRODUCTION

Throughout this paper all rings will be commutative with identity (zero-divisors admitted) and all modules will be unitary. Let M be an R-module. Then M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N : M) = \{r \in R : rM \subseteq N\}$. Examples of multiplication ideals (i.e., ideals of a ring R that are multiplication R-modules) include invertible ideals, principal ideals, and ideals generated by idempotents. An R-module M is called a weak cancellation module whenever $AM = BM$ for ideals A and B of R, then $A + \text{Ann}(M) = B + \text{Ann}(M)$. In particular, if $\text{Ann}(M) = 0$, then we call M a cancellation module.

Let M be an R-module. The idealization of R and M is the commutative ring with identity $R(M) = R \oplus M$ with addition $(r, m) + (r', m') = (r + r', m + m')$ and multiplication $(r, m)(r', m') = (rr', rm' + r'm)$. Note that $0 \oplus M$ ia an ideal of $R(M)$ satisfying $(0 \oplus M)^2 = 0$ and that the structure of $0 \oplus M$ as $R(M)$-module (i.e., an ideal of $R(M)$) is essentially the same as the R-module structure of M. Let N be a submodule of M. Then $0 \oplus N$ is an ideal of $R(M)$ contained in $0 \oplus M$. A good reference for the basic facts about idealization is [10, Section 25].

Throughout this paper we shall assume unless otherwise stated, that M is a finitely generated faithful multiplication module, $\mathbf{S(M)}$ is the set finitely generated faithful multiplication submodules of M and $\mathbf{S(R)}$ is the semi-group of finitelty generated faithful multiplication ideals of R.

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2. GCD AND LCM OF MULTIPLICATION MODULES

Let M be an R-module and N a submodule of M with $N = IM$ for some ideal I of R. Then we say that I is a presentation ideal of N (for short a presentation of N). It is possible that for a submodule N no such presentation exist. For example:

(1) Assume that M is a vector space over an arbitrary field of F with $\dim_F M = k \geq 2$ and let N be a proper subspace of M such that $N \neq 0$. Then M is finite length (so M is artinian, noetherian, and pure-injective), but M is not multiplication and N has not any presentation.

(2) Let R be a local Dedekind domain with maximal ideal $P = Rp$. The module $E = E(R/P)$, the injective hull of R/P, is pure-injective, secondary and Artinian (see [8, Theorem 1.1]). Set $A_n = (0 :_E P^n)$ $(n \geq 1)$. Then every non-zero proper submodule L of E is of the form $L = A_m$ for some m and $E \cong P^n E$ $(n \geq 1)$, so L has not any presentation (see [9, p. 324]), and hence E is not multiplication.

Clearly, for every submodule of M has a presentation ideal if and only if M is multiplication module. In particular, every submodule N of a multiplication module M, $(N : M)$ is a presentation for N.

Let M be a multiplication module, and let $N = I_1 M$, $K = I_2 M$ and $T = I_3 M$ for some ideals I_1, I_2 and I_3 of R. The product of N and K denoted by NK is defined by $I_1 I_2 M$. Moreover, the product of N and K is independent of presentation ideals of N and K (see [2, Theorem 3.4]). Clearly, NK is a submodule of M and contained in $N \cap K$. Also it is clear that if $N \subseteq K$ then $NT \subseteq KT$.

Remark 1. Clearly, $M^2 = MM = M$ (so M is idempotent). Note that this definition is different from the definition of ordinary ideal multiplication. To see that the two definitions are actually different, let $R = Z$ be the ring of integers, and let $M = 2Z$ and $N = K = 4Z$. Then NK is $16Z$ by the usual definition and is $8Z$ by the our definition. In fact, when I is idempotent - in that case, the two definitions coincide.

Definition 2.1. Let N and K be submodules of an R-module M and let $R(M)$ be the idealization of M. We say that N divides K, denoted by $N \mid K$, if there exists an ideal J of $R(M)$ such that $0 \oplus K = J(0 \oplus N)$.

Let N and K be submodules of an R-module M. A submodule G of M is called a **greatest common divisor** of N and K, or $\gcd(N, K)$, if and only if:

(i) $G \mid N$ and $G \mid K$,

(ii) If G' is a submodule of M with $G' \mid N$ and $G' \mid K$, then $G' \mid G$. We say that N is relatively prime to K if $\gcd(N, K) = M$.

Similarly, a submodule L of M is called a **least common multiple** of N and K, or $\operatorname{lcm}(N, K)$, if and only if:

(i) $N \mid L$ and $K \mid L$,

(ii) If L' is a submodule of M with $N \mid L'$ and $K \mid L'$, then $L \mid L'$.

Proposition 2.2. *Let N and K be submodules of an R-module M and let $R(M)$ be the idealization of M. Then $N \mid K$ if and only if there exists an ideal I of R such that $K = IN$. In particular, if $N \mid K$ then $K \subseteq N$.*

Proof. Suppose first that $N \mid K$. Then $0 \oplus K = J(0 \oplus N)$ for some ideal J in $R(M)$. By [10, Theorem 25.1 (1)], if

$$I = \{r \in R : \ (r, m) \in J \text{ for some } m \in M\}$$

and if

$$N' = \{n' \in M : \ (r, n') \in J \text{ for some } r \in R\}$$

then I is an ideal of R, N' is an R-submodule of M, $IM \subseteq N'$ and $J = I \oplus N'$. It follows that $0 \oplus K = (I \oplus N')(0 \oplus N) = 0 \oplus IN$, and hence $K = IN$. Conversely, suppose that $K = IN$ for some ideal I in R. Then $(I \oplus M)(0 \oplus N) = 0 \oplus IN = 0 \oplus K$, and hence $0 \oplus N \mid 0 \oplus K$, as required. ■

Remark 2. (i) By Proposition 2.2, it is clear that if N is a multiplication submodule of M then $N \mid K$ if and only if $K \subseteq N$.

(ii) Let N_1, N_2 and N_3 be multiplication submodules of an R-module M. Then by (i) and definition, 1) $N_1 \mid N_1$, 2) if $N_1 \mid N_2$ and $N_2 \mid N_1$ then $N_1 = N_2$ and 3) if $N_1 \mid N_2$ and $N_2 \mid N_3$ then $N_1 \mid N_3$. Moreover, it is clear that $\gcd(N_1, N_2)$ and $\operatorname{lcm}(N_1, N_2)$ are unique if they exist.

(iii) Let N be a submodule of M. 1) If gcd$(M, N) = G$ then $G = M$. 2) If $\operatorname{lcm}(0, N) = L$ then $L = 0$.

Lemma 2.3. *Let M be a finitely generated faithful multiplication R module and N a submodule of M. Then $N = IM \in S(M)$ if and only if $I \in S(R)$.*

Proof. This follows from [13, Theorem 10] and [11, Lemma 1.4]. ■

Lemma 2.4. *The set $S(M)$ is a multiplicative semi-group.*

Proof. Let $N, K \in S(M)$. Then $N = IM$ and $K = JM$ for some ideals I, J in $S(R)$ by Lemma 2.3. If $r(NK) = (rI)K = 0$ then $rI = 0$. It follows that $r = 0$, so NK is faithful.

Suppose that $H \subseteq NK \subseteq K$. Then $H = J'K$ for some ideal J' in R. As $J'K \subseteq IK$, we have $J' \subseteq I$ by [13, p. 231 Corollary], so $J' = J_1 I$ for some ideal J_1 of R, and hence $H = J_1 IK = J_1 NK$. Thus NK is multiplication. Finally, since $NK = IJM = IK$, from Lemma 2.3, we get NK is finitely generated, as rquired. ■

Corollary 2.5. *Let $N, T \in S(M)$.*

(i) *If* K *is a submodule of* M *and* $K \mid N$, *then* $K \in S(M)$.

(ii) *If* $\mathrm{lcm}(N, T) = K$ *exists*, *then* $K \in S(M)$.

Proof. (i) There exists an ideal I in $S(R)$ such that $N = IM$ by Lemma 2.3. Therefore $K = JM$ and $N = IM = J_1JM$ for some ideals J, J_1 of R. It follows that $I = J_1J$ by [12, Theorem 6.1], so $J \mid I$. Thus by [1, Lemma 1.4], $J \in S(R)$, and hence $K = JM \in S(M)$ by Lemma 2.3.

(ii) Since $N, T \in S(M)$, we have $NT \in S(M)$ by Lemma 2.4. As NT is a common multiple of N and T, we have $K \mid NT$, and by (i), $K \in S(M)$. ■

Theorem 2.6. *Let* $N, K \in S(M)$. *If* $\mathrm{lcm}(N, K) = T$ *exists*, *then so too does* $\mathrm{gcd}(N, K)$. *In particular*, $NK = \mathrm{lcm}(N, K)\mathrm{gcd}(N, K)$.

Proof. We can write $N = I_1M$, $K = I_2M$ and $T = I_3M$ for some ideals I_1, I_2 and I_3 of R, so $NK = I_1K = I_2N$ and $T \mid I_1K$. Then there exists an ideal J_1 in R such that $I_1K = J_1T = J_1I_3M = (J_1M)(I_3M)$. Since $T \in S(M)$, we have

$$(I_1K : T)M = (J_1T : T)M = J_1M$$

by [13, Theorem 10 (ii)]. It is enough to show that $\mathrm{gcd}(N, K) = J_1M$. Since $K \mid T$, there exists an ideal J_2 of R such that $T = J_2K$. It follows that $J_1T = J_1J_2K = I_1K$ and by [12, Theorem 6.1], $J_1J_2 = I_1$, and hence $J_2J_1M = I_1M$. Thus $J_1M \mid N$. Similarly, $J_1M \mid K$. Assume that T_1 is a submodule of M such that $T_1 \mid K$ and $T_1 \mid N$. Then there exists ideals J_3, J_4 and J_5 of R such that $K = J_3T_1$, $N = J_4T_1$ and $T_1 = J_5M$. Therefore $NK = J_3J_4J_5M$, so $J_5M \mid NK$, and hence $T_1 = J_5M \in S(M)$ by Proposition 2.5. By [13, Theorem 10 (ii)], we have $(NK : T_1)M = (J_3J_4T_1 : T_1)M = J_3J_4M$. It follows that $K, N \mid (NK : T_1)M$. Therefore $T \mid (NK : T_1)M$, and hence there is an ideal I_4 of R such that $(NK : T_1)M = I_4T$. But $NK = J_3J_4J_5M \subseteq J_5M = T_1$ and T_1 is a multiplication module. Thus

$$J_1T = NK = (NK : T_1)T_1 = (NK : T_1)J_5M = I_4J_5T$$

It follows that $J_1 = I_4J_5$ since T is cancellation module. Thus $J_1M = I_4J_5M = I_4T_1$, so $T_1 \mid J_1M$, as required. ■

Lemma 2.7. *Let* $I, J \in S(R)$, *and let* M *be a finitely generated faithful multiplication* R*-module such that* $\mathrm{lcm}(IM, JM)$ *exists. Then the following statements are true*:

(i) $\mathrm{lcm}(I, J)$ *exists and* $\mathrm{lcm}(I, J)M = \mathrm{lcm}(IM, JM)$.

(ii) $\mathrm{gcd}(I, J)$ *exists and* $\mathrm{gcd}(I, J)M = \mathrm{gcd}(IM, JM)$.

(iii) $(I : J) = (IM : JM)$.

Proof. (i) Let $\mathrm{lcm}(IM, JM) = U$. Then $U = I_1M = J_1IM = J_2JM$ for some ideals I_1, J_1 and J_2 of R. It follows that $I_1 = J_1I = J_2J$ since M is a cancellation module, and hence I_1 is a common multiple of I and J. Assume that J_3 is an ideal of R with $I \mid J_3$ and $J \mid J_3$. Thus J_3M is a common multiple of IM and JM, so there exists an ideal J_4 in R such that $J_3M = J_4I_1M$ and therefore $J_3 = J_4I_1$. It follows that $I_1 \mid J_3$, so $\mathrm{lcm}(I, J) = I_1$, as required.

(ii) This proof is similar to that in case (i) and we omit it.

(iii) Clearly, $(I : J) \subseteq (IM : JM)$. Suppose that $r \in (IM : JM)$. Then $rJM \subseteq IM$, so $rJ \subseteq I$ by [13, p. 231 Corollary], as required. ■

Corollary 2.8. *Let $N, K, T \in S(M)$. Then the following statements are true*:

(i) $\mathrm{lcm}(N, K)$ *exists if and only if* $\mathrm{lcm}(TN, TK)$ *exists, in which case*

$$\mathrm{lcm}(TN, TK) = T\mathrm{lcm}(N, K)$$

(ii) *If* $\gcd(TN, TK)$ *exists, then so too does* $\gcd(N, K)$, *and*

$$\gcd(TN, TK) = T\gcd(N, K)$$

Proof. (i) We can write $N = I_1M$, $K = I_2M$ and $T = I_3M$ for some ideals I_1, I_2 and $I_3 \in S(R)$. Suppose first that $\mathrm{lcm}(N, K)$ exists. As $I_3 \in S(R)$, we get $\mathrm{lcm}(I_3I_1, I_3I_2) = I_3\mathrm{lcm}(I_1, I_2)$ by [1, Theorem 2.2] and Lemma 2.7. It follows from Lemma 2.7 that

$$\mathrm{lcm}(TN, TK) = \mathrm{lcm}(I_1T, I_2T) = \mathrm{lcm}(I_1, I_2)MT =$$
$$\mathrm{lcm}(I_1M, I_2M)T = \mathrm{lcm}(N, K)T$$

The converse is obvious.

(ii) This proof is similar to that in case (i) and we omit it. ■

Lemma 2.9. *Let N_i ($1 \leq i \leq n$) be a finite collection of submodules in $S(M)$.*

(i) *If* $\gcd(N_1, N_2, ..., N_n)$ *and* $\gcd(N_1, N_2, ..., N_{n-1})$ *exist and*

$$U = \gcd(N_1, N_2, ..., N_{n-1})$$

then $\gcd(N_1, N_2, ..., N_n) = \gcd(U, N_n)$.

(ii) *If* $\mathrm{lcm}(N_1, N_2, ..., N_n)$ *and* $\mathrm{lcm}(N_1, N_2, ..., N_{n-1})$ *exist and*

$$V = \mathrm{lcm}(N_1, N_2, ..., N_{n-1})$$

then $\mathrm{lcm}(N_1, N_2, ..., N_n) = \mathrm{lcm}(V, N_n)$.

Proof. Straightforward ∎

Corollary 2.10. *Let* $N, K, T \in S(M)$, *and let* $\gcd(TN, TK)$ *exists and* $\gcd(N, K) = M$. *Then* $\gcd(N, TK) = \gcd(N, T)$.

Proof. By Corollary 2.8, we have $\gcd(TN, TK) = T\gcd(N, K) = TM = T$. Then $\gcd(N, T) = \gcd(N, \gcd(TN, TK))$, and it follows from Lemma 2.9 that

$$\gcd(N, T) = \gcd(\gcd(N, TN), TK) = \gcd(N, TK).$$ ∎

Lemma 2.11. *Let* $N, K, T \in S(M)$.

(i) *If* $TN = TK$ *then* $N = K$.

(ii) *If* $T_1 = \gcd(N, K)$ *then* $\gcd((N : T_1)M, (K : T_1)M) = M$.

Proof. (i) By Lemma 2.3, there exists ideals I_1, I_2 and I_3 in $S(R)$ such that $N = I_1M$, $K = I_2M$ and $T = I_3M$. Suppose that $TN = I_1I_3M = I_2I_3M = TK$, so $I_1I_3 = I_2I_3$ since M is cancellation, and hence $I_2 = I_1$ since I_3 is cancellation ideal. Thus $N = K$.

(ii) Since T_1 is the greatest common divisor of N, K, we have $N, K \subseteq T_1$ and $T_1 \in S(M)$, so $N = (N : T_1)T_1$ and $K = (K : T_1)T_1$. It follows from Corollary 2.8 that

$$T_1M = T_1 = \gcd((N : T_1)T_1, (K : T_1)T_1) = T_1\gcd((N : T_1)M, (K : T_1)M).$$

By (i), we get $\gcd((N : T_1)M, (k : T_1)M) = M$, as required. ∎

Theorem 2.12. *The greatest common divisor of* N *and* K *exists for all* $N, K \in S(M)$ *if and only if* $\operatorname{lcm}(N, K)$ *exists for all* $N, K \in S(M)$.

Proof. Let $N, K \in S(M)$. If $\gcd(N, K) = G$ then, $G \in S(M)$ and $N = (N : G)G$, $K = (K : G)G$. So by Corollary 2.8, $\operatorname{lcm}(N, K)$ exists if and only if $\operatorname{lcm}((N : G)M, (K : G)M)$ exists, and $\gcd((N : G)M, (K : G)M) = M$ by Lemma 2.11 (Since $\operatorname{lcm}(N, K) = \operatorname{lcm}((N : G)MG, (K : G)MG))$. Thus we may assume that $\gcd(N, K) = M$. Now it is enough to show that $\operatorname{lcm}(N, K) = NK$. Clearly, NK is a common multiple of N and K. Let U be a submodule of M such that $N, K \mid U$. Then $U = I_1N = I_2K$ for some ideals I_1 and I_2 of R. As $N \mid I_1N = I_1MN$, we infer from Corollary 2.10 that $\gcd(N, I_1MN) = \gcd(N, I_1M)$, and hence $N \mid I_1M$, so that $NK \mid I_1MN = U$. The converse follows from Theorem 2.6. ∎

Lemma 2.13. *Let* $N, K \in S(M)$.

(i) $\operatorname{lcm}(N,K)$ *exists in* $S(M)$ *if and only if* $N\cap K\in S(M)$. *In particular*, $\operatorname{lcm}(N,K)=N\cap K=(N:K)K$.

(ii) *If* $\operatorname{lcm}(N,K)=N\cap K$ *then* $\gcd(N,K)=(NK:N\cap K)M$.

Proof. (i) Let $\operatorname{lcm}(N,K)=T\in S(M)$. Then T is a common multiple of N and K, so $T\subseteq N\cap K$. As N and K are multiplication, $N\cap K$ is a common multiple of N and K, so $T\mid N\cap K$, and hence $T=N\cap K\in S(M)$. Similarly, if $N\cap K\in S(M)$ then $N\cap K$ is the least common multiple of N and K. Finally, since $N\cap K=(N\cap K:K)K=\operatorname{lcm}(N,K)$, we have $\operatorname{lcm}(N,K)=(N:K)K$.

(ii) Since $NK=(NK:N\cap K)M(N\cap K)=\gcd(N,K)(N\cap K)$, so the result follows from Lemma 2.11. ■

Theorem 2.14. *Let* $N,K\in S(M)$, *and let* $\operatorname{lcm}(N,K)$ *exists. Then the following statements are true*:

(i) $\operatorname{lcm}(N,K)^m=\operatorname{lcm}(N^m,K^m)$ *for all positive integer* m.

(ii) $\gcd(N,K)^m=\gcd(N^m,K^m)$ *for all positive integer* m.

(iii) $(N:K)^m=(N^m:K^m)$ *for all positive integer* m.

Proof. (i) We can write $N=I_1M$ and $K=I_2M$ for some ideals $I_1,I_2\in S(R)$ by Lemma 2.3. Suppose that $\operatorname{lcm}(N,K)=T$. Then there exists an ideal I_3 of R with $T=I_3M$. By [1, Theorem 2.6] and Lemma 2.7, we have

$$\begin{aligned}\operatorname{lcm}(N,K)^m&=\operatorname{lcm}(I_1M,I_2M)^m\\&=(I_3M)^m=I_3^mM\\&=\operatorname{lcm}(I_1,I_2)^mM\\&=\operatorname{lcm}(I_1^m,I_2^m)M\\&=\operatorname{lcm}(I_1^mM,I_2^mM)=\operatorname{lcm}(N^m,K^m)\end{aligned}$$

(ii) This proof is similar to that in case (i) and we omit it.

(iii) There exists elements I_1,I_2 of $S(R)$ such that $N=I_1M$ and $K=I_2M$. from Lemma 2.7 and [1, Theorem 2.6], we have

$$\begin{aligned}(N:K)^m&=(I_1M:I_2M)^m=((I_1:I_2)M)^m=(I_1:I_2)^mM\\&=(I_1^m:I_2^m)M=(I_1^mM:I_2^mM)=(N^m:K^m).\end{aligned}$$ ■

3. GCD Modules

Definition 3.1. An R-module M is called a GCD module if the intersection of every two finitely generated faithful multiplication submodules is also a finitely generated faithful multiplication module.

Let R be a generalized GCD ring (that is, the intersection of every two finitely generated faithful multiplication ideals of R is also a finitely generated faithful multiplication ideal). Then R is GCD as an R-module. But $R = Z[\sqrt{5}]$ is not GCD as an R-module (see [1, Section 3]).

Theorem 3.2. *Let $S(M)$ be the set of as described in section 2. Then the following conditions are equivalent*:

(i) *M is a GCD module.*
(ii) *For each $N, K \in S(M)$, $\mathrm{lcm}(N, K)$ exists in $S(M)$.*
(iii) *For each $N, K \in S(M)$, $\gcd(N, K)$ exists in $S(M)$.*
(iv) *For each $N, K \in S(M)$, $(N : K)M \in S(M)$.*

Proof. From Theorm 2.12 and Lemma 2.13, it is enough to show that $(ii) \Rightarrow (iv)$ and $(iv) \Rightarrow (i)$.

$(ii) \Rightarrow (iv)$. There exists an ideal I of R such that $K = IM$. By Theorem 2.12 and Lemma 2.13, we have

$$\mathrm{lcm}(N, K) = (N : K)K = I(N : K)M$$

It follows that $(N : K)M \mid \mathrm{lcm}(N, K)$. Now the assertion follows from the Corollary 2.5.

$(iv) \Rightarrow (i)$. Let N, K and $(N : K)M \in S(M)$. we can write $K = IM$ for some ideal of R. As

$$(N : K)M = (N \cap K : K)M \in S(M),$$

we have $N \cap K = (N \cap K : K)K = (N \cap K : K)(RM)(IM) \in S(M)$, as required. ■

Corollary 3.3. *Let M be a GCD module, and let $N, K, T \in S(M)$. Then the following statements are true*:

(i) $(\gcd(N, K) : T)M = \gcd((N : T)M, (K : T)M)$.
(ii) $(T : \mathrm{lcm}(N, K))M = \gcd((T : N)M, (T : K)M)$.

Proof. (i) There exists ideals I_1, I_2 and I_3 of R such that $N = I_1M$, $K = I_2M$ and $T = I_3M$. By Lemma 2.7 and [1, Corollary 3.2], we have

$$\begin{aligned}
G &= ((\gcd(I_1M, I_2M) : I_3M))M \\
&= ((\gcd(I_1, I_2)M : I_3M))M \\
&= ((\gcd(I_1, I_2) : I_3)M \\
&= \gcd((I_1 : I_3), (I_2 : I_3))M \\
&= \gcd((I_1 : I_3)M, (I_2 : I_3)M) \\
&= \gcd((I_1M : I_3M), (I_2M : I_3M))
\end{aligned}$$

So, $G = \gcd((N:T)M, (K:T)M)$, as required.

(ii) This proof is similar to that the case (i) and we omit it. ■

Corollary 3.4. *Let M be a GCD module, and let $N, K, T \in S(M)$. Then the following statements are true*:

(i) $\mathrm{lcm}(\gcd(N,K),T)) = \gcd(\mathrm{lcm}(N,T), \mathrm{lcm}(K,T))$

(ii) $\gcd(\mathrm{lcm}(N,K),T)) = \mathrm{lcm}(\gcd(N,T), \gcd(K,T))$

Proof. (i) We can write $N = I_1M$, $K = I_2M$ and $T = I_3M$ for some ideals I_1, I_2 and I_3 of R. By Lemma 2.7 and [1, Corollary 3.3], we have

$$\begin{aligned}
&\mathrm{lcm}(\gcd(I_1M, I_2M), I_3M)) \\
&= \mathrm{lcm}(\gcd(I_1, I_2)M, I_3M) \\
&= \mathrm{lcm}(\gcd(I_1, I_2), I_3)M \\
&= \gcd(\mathrm{lcm}(I_1, I_3), \mathrm{lcm}(I_2, I_3))M \\
&= \gcd(\mathrm{lcm}(I_1, I_3)M, \mathrm{lcm}(I_2, I_3)M) \\
&= \gcd(\mathrm{lcm}(I_1M, I_3M), \mathrm{lcm}(I_2M, I_3M)) \\
&= \gcd(\mathrm{lcm}(N,T), \mathrm{lcm}(K,T)).
\end{aligned}$$

(ii) This proof is similar to that the case (i) and we omit it. ■

Lemma 3.5. *Let M be a GCD module, and let $N, K, T \in S(M)$. If* $\gcd(N,T) = \gcd(K,T) = M$, *then* $\gcd(\mathrm{lcm}(N,K),T) = M = \gcd(NK,T)$.

Proof. This follows from Corollary 2.10 and Corollary 3.4. ■

Let M be a GCD module, and let $I, J \in S(R)$, $N, K \in S(M)$. Set

$$\Phi_{I,J} = \{A : A \text{ is an ideal of } R,\ A \mid I,\ \gcd(A,J) = R\}$$

$$\Phi_{N,K}M = \{T : T \text{ is a submodule of } M,\ T \mid N,\ \gcd(T,K) = M\}$$

Clearly, $\Phi_{N,K}M \subseteq S(M)$ and $M \in \Phi_{N,K}M$.

Lemma 3.6. *Let $N = IM, K = JM \in S(M)$. Then $T = I_1M \in \Phi_{N,K}M$ if and only if $I_1 \in \Phi_{I,J}$.*

Proof. There exists an ideal J_1 of R such that $IM = J_1I_1M$, so $I = J_1I_1$ since M is cancellation, and hence $I_1 \mid I$. As $T \in \Phi_{N,K}M$, by Lemma 2.7, we have

$$\gcd(T,K) = \gcd(I_1M, JM) = \gcd(I_1, J)M = RM$$

It follows that $\gcd(I_1, J) = R$. Thus $I_1 \in \Phi_{I,J}$. The converse is obvious. ■

Theorem 3.7. *Let M be a GCD module and $N = IM, K = JM \in S(M)$. Then the following statements are true*:

(i) $\Phi_{N,K}M$ *formes a lattice of submodules of* M. *Moreover*, *if* $\Phi_{N,K}M$ *contains a minimal element*, *then it is unique.*

(ii) *If* $A, B \in \Phi_{N,K}M$ *then* $(A : B)M \in \Phi_{N,K}M$.

Proof. Let $A, B \in \Phi_{N,K}M$. Then $A, B \in S(M)$, and $\gcd(A, B)$ and $\operatorname{lcm}(A, B)$ exist. There exists ideals I_1 and I_2 in $\Phi_{I,J}$ such that $A = I_1M$ and $B = I_2M$ by Lemma 3.6. From Lemma 2.7, [1, Theorem 3.5], and Lemma 3.6, we have

$$\gcd(A, B) = \gcd(I_1M, I_2M) = \gcd(I_1, I_2)M \in \Phi_{N,K}M$$

$$\operatorname{lcm}(A, B) = \operatorname{lcm}(I_1M, I_2M) = \operatorname{lcm}(I_1, I_2)M \in \Phi_{N,K}M$$

This show that the first assertion follows and the second assertion is obvious.

(ii) We can write $A = I_1M$ and $B = I_2M$ for some ideals I_1 and I_2 in $\Phi_{I,J}$. From Lemma 2.7, [1, Theorem 3.5], and Lemma 3.6, we have

$$(A : B)M = (I_1M : I_2M)M = (I_1 : I_2)M \in \Phi_{N,K}M \qquad \blacksquare$$

Proposition 3.8. *Let* M *be a* GCD *module and* $N = IM, K = JM \in S(M)$. *Then* $T = I_1M$ *is the smallest element in* $\Phi_{N,K}M$ *if and only if* I_1 *is the smallest element in* $\Phi_{I,J}$.

Proof. Suppose first that T is the smallest element in $\Phi_{N,K}M$. Then $I_1 \in \Phi_{I,J}$ by Lemma 3.6. Let I_2 be an ideal in R such that $I_2 \mid (I : I_1)$ and $\gcd(I_2, J) = R$. By [1, Theorem 3.7], it is enough to show that $I_2 = R$. Since $I_2M \mid (I : I_1)M = (IM : I_1M)M$, we have from Theorem 3.2 and Corollary 2.5 that $I_2M \in S(M)$. Moreover, $I_2T = I_2TM \mid (N : T)TM = NM = N$, $\gcd(I_2M, K) = M$ and $\gcd(T, K) = \gcd(I_1, J)M = RM = M$, so by Lemma 3.5,

$$\gcd(I_2MT, K) = \gcd(I_2T, K) = M.$$

It follows that $I_2T \in \Phi_{N,K}M$, and hence $T \subseteq I_2T \subseteq T$. Thus $I_2T = T = RT$ and therefore $I_2 = R$ since T is cancellation. Conversely, assume that I_1 is the smallest element in $\Phi_{I,J}$. Let $T_1 = J_1M \in \Phi_{N,K}M$. Then $J_1 \in \Phi_{I,J}$ by Lemma 3.6, and hence $I_1 \subseteq J_1$, as required. $\blacksquare$

Theorem 3.9. *Let* M *be a* GCD *module and* $N = IM, K = JM \in S(M)$. *Then* $T = I_1M$ *is the smallest element in* $\Phi_{N,K}M$ *if and only if the only submodule of* M *dividing* $(N : T)M$ *and relatively prime to* K *is* M.

Proof. This follows from [1, Theorm 3.7] and Proposition 3.8.

Theorem 3.10. *Let* M *be a* GCD *module and* $N = IM, K = JM$ *and* $T = I_3M \in S(M)$, *and let* $G = I_4M = \gcd(N, K)$. *Then the following statements are equivalent*:

(*i*) $T \mid N$ *and* $\gcd(T, K) = M$.

(*ii*) $T \mid (N : G)M$ *and* $\gcd(T, G) = M$.

Proof. $(i) \Rightarrow (ii)$. By (i) we get $I_3 \mid I_1$ and $\gcd(I_3, I_2) = R$ since M is cancellation, so by [1, Theorem 3.8], we have $I_3 \mid (I_1 : I_4)$ and $\gcd(I_3, I_4) = R$. Now the assertion follows from Lemma 2.7.

$(ii) \Rightarrow (i)$. Similarly, this follows from [1, Theorem 3.8] and Lemma 2.7. ■

Remark 3. It is clear from Theorem 3.10 that if $G = \gcd(N, K)$, then $\Phi_{N,K}M = \Phi_{(N:G)M,G}M$.

Let M be a GCD module, and let $N, K \in S(M)$. Then $N = IM$ and $K = JM$ for some ideals I and J in $S(R)$. Define two sequences of ideals in R and two sequences of submodules of M recursively as follows: $I_0 = I$, $J_0 = J$, $J_{i+1} = \gcd(I_i, J_i)$ and $I_{i+1} = (I_i : J_{i+1})$ for all $i \geq 0$, and $N_0 = N, K_0 = K, K_{i+1} = \gcd(N_i, K_i)$ and $N_{i+1} = (N_i : K_{i+1})M$ for all $i \geq 0$.

Lemma 3.11. *Let M be a GCD module and $N, K \in S(M)$ with the sequences I_i, J_i, N_i and K_i as above. Then $N_i = I_iM$ and $K_i = J_iM$ for all $i \geq 0$.*

Proof. We shall prove the assertion by induction on i. The result is trivial for $i = 0$. Assume that $i \geq 1$ and that $N_i = I_iM$, $K_i = J_iM$. Thus from Lemma 2.7, we have

$$K_{i+1} = \gcd(N_i, K_i) = \gcd(I_i, J_i)M = J_{i+1}M$$

$$N_{i+1} = (N_i : K_{i+1})M = (I_iM : J_{i+1}M)M = (I_i : J_{i+1})M = I_{i+1}M.$$ ■

Theorem 3.12. *Let M be a GCD module and $N, K \in S(M)$ with the sequences I_i, J_i, N_i and K_i as above. Then the following statements are equivalent*:

(i) $\bigcup_{i=1}^{\infty} N_i$ *is the smallest element in* $\Phi_{N,K}M$.

(ii) $\bigcup_{i=1}^{\infty} N_i \in \Phi_{N,K}M$.

(iii) $\bigcup_{i=1}^{\infty} N_i \in S(M)$.

(iv) $\bigcup_{i=1}^{\infty} N_i = N_n$ *for some positive integer* n.

(v) $N_n = N_{n+1}$ *for some positive integer* n.

(vi) $N_{n+1} = M$ *for some positive integer* n.

Proof. This follows from Lemma 3.6, Proposition 3.8, [1, Theorem 3.9] and Lemma 3.11. ■

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