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THE HÁJECK-RÈNYI INEQUALITY FOR THE AANA RANDOM VARIABLES AND ITS APPLICATIONS

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Abstract. In this paper we study the Hájeck-Rènyi type inequality of asymptotically almost negatively associated (AANA) random variables and derive strong laws of large numbers for weighted sums of AANA sequences.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $\{X_1, \cdots, X_n\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathcal{P})$. A finite family $\{X_1, \cdots, X_n\}$ is said to be negatively associated(NA) if for any disjoint subsets $A, B \subset \{1, \cdots, n\}$ and any real coordinatewise nondecreasing functions $f : R^A \to R$ and $g : R^B \to R$,

$$Cov(f(X_i; i \in A), g(X_j; j \in B)) \leq 0.$$

An infinite family of random variables is negatively associated (NA) if every finite subfamily is negatively associated (NA). This concept was introduced by Joag-Dev and Proschan(1983). A sequence $\{X_n, n \geq 1\}$ of random variables is called asymptotically almost negatively associated (AANA) if there is a nonnegative sequence $q(m) \to 0$ such that

$$\begin{aligned} &Cov(f(X_m), g(X_{m+1}, \cdots, X_{m+k})) \\ &\quad \leq q(m)(Var(f(X_m))Var(g(X_{m+1}, \cdots, X_{m+k})))^{\frac{1}{2}} \end{aligned} \tag{1}$$

for all $m, k \geq 1$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1) is finite. This definition was introduced by Chandra and Ghosal(1996 a,b).

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The family of AANA sequences contains negatively associated(in particular, independent) sequences(with $q(m) = 0, \forall m \geq 1$) and some more sequences of random variables which are not much deviated from being negatively associated. Condition (1) is clearly satisfied if the $R_{2,2}$-measure of dependence(see Bradley et al.(1987)) between $\sigma(X_m)$ and $\sigma(X_{m+1}, X_{m+2}, \cdots)$ converges to zero. The following is a non-trivial example of an AANA sequence constructed by Chandra and Ghosal(1996a): Let $\{Y_n\}$ be i.i.d. N(0,1) variables and define $X_n = (1+a_n^2)^{-\frac{1}{2}}(Y_n + a_n Y_{n+1})$ where $a_n > 0$ and $a_n \to 0$.
Chandra and Ghosal(1996 a) derived the Kolmogorov type maximal inequality for AANA random variables and obtained the strong law of large numbers for AANA random variables by using this inequality. Chandra and Ghosal(1996 b) also derived the almost sure convergence of weighted averages on the AANA random variables. In this paper we derive the H$\acute{a}$jeck-R$\grave{e}$nyi type inequality of asymptotically almost negatively associated (AANA) random variables and apply this inequality to obtain the strong laws of large numbers for weighted sums of AANA sequences. A Marcinkiewicz strong law of large numbers for identically distributed AANA sequence is also obtained as a special case.

2. Results

We start this section with the property of asymptotically almost negatively associated (AANA) random variables which can be obtained easily from the definition of AANA random variables.

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of asymptotically almost negatively associated (AANA). Then $\{f_n(X_n), n \geq 1\}$ is still a sequence of AANA random variables, where $f_n(\cdot), n = 1, 2, \cdots$, are nondecreasing functions.*

From the idea of the proof of Theorem 1 in Chandra and Ghosal(1996 a) we have the following lemma:

Lemma 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_k = 0$ and $EX_k^2 < \infty,\ k \geq 1$. Suppose that there exist $M > 1$ and $D > 0$ such that for all $n \geq 1$*

$$(\sum_{k=1}^{n} \sigma_k^{M/(M-1)})^{1-1/M} \leq D(\sum_{k=1}^{n} \sigma_k^2)^{1/2}, \tag{2}$$

where $\sigma_k^2 = EX_k^2,\ k \geq 1$. Let $A = D(\sum_{m=1}^{n-1} q^M(m))^{1/M}$. Then we have

$$E(\max_{1\leq k\leq n} \sum_{i=1}^{k} X_i)^2 \leq (A + (1+A^2)^{1/2})^2 \sum_{k=1}^{n} \sigma_k^2. \tag{3}$$

$$E(\sum_{k=1}^{n} X_k)^2 \le (A+(1+A^2)^{1/2})^2 \sum_{k=1}^{n} \sigma_k^2, \tag{4}$$

Proof. For fixed $n > 1$, we set

$$U_k = \max(X_k, X_k + X_{k+1}, \cdots, X_k + \cdots + X_n), 1 \le k \le n. \tag{5}$$

Note that U_k's are AANA by Lemma 2.1 and $U_k = \max\{X_k, X_k + U_{k+1}\}$. Consequently it follows from (1) that

$$\begin{aligned} EU_k^2 &\le EX_k^2 I(U_{k+1} \le 0) + E(X_k + U_{k+1})^2 I(U_{k+1} > 0) \\ &\le \sigma_k^2 + EU_{k+1}^2 I(U_{k+1} > 0) + 2EX_k U_{k+1} I(U_{k+1} > 0) \\ &\le \sigma_k^2 + EU_{k+1}^2 + 2q(k)\sigma_k (EU_{k+1}^2)^{1/2}, \ 1 \le k \le n-1. \end{aligned}$$

Define a sequence $\{\xi_k, 1 \le k \le n\}$ by

$$\begin{cases} \xi_k^2 = \sigma_k^2 + \xi_{k+1}^2 + 2q(k)\sigma_k \xi_{k+1}, \ 1 \le k \le n-1, \\ \xi_n^2 = \sigma_n^2 \ . \end{cases}$$

From the definition of ξ_k we have

$$EU_k^2 \le \xi_k^2, \ 1 \le k \le n. \tag{7}$$

Note that $\{\xi_k\}$ is decreasing. Thus

$$\xi_k^2 \le \sigma_k^2 + \xi_{k+1}^2 + 2q(k)\sigma_k \xi_1, \ 1 \le k \le n-1.$$

Substituting sequentially and using the Hölder inequality, we get

$$\begin{aligned} \xi_1^2 &\le \tau^2 + 2\xi_1 \sum_{k=1}^{n-1} q(k)\sigma_k \\ &\le \tau^2 + 2\xi_1 (\sum_{k=1}^{n-1} q^M(k))^{1/M} (\sum_{k=1}^{n-1} \sigma_k^{M/(M-1)})^{(M-1)/M} \\ &\le \tau^2 + 2\xi_1 A\tau, \ where \ \tau^2 = \sum_{k=1}^{n} \sigma_k^2. \end{aligned}$$

Hence

$$(\xi_1 - A\tau)^2 \le (1+A^2)\tau^2 \tag{8}$$

Combining (5), (7) and (8) we have $EU_1^2 \le \xi_1^2 \le (A+(1+A^2)^{1/2})^2\tau^2$ and so (3) follows.

To prove (4), for fixed $n \geq 1$, set

$$T_k = X_k + X_{k+1} + \cdots + X_n,\ 1 \leq k \leq n. \tag{9}$$

Then we have

$$T_k = X_k + T_{k+1}$$

and consequently it follows from (1) that

$$ET_k^2 \leq \sigma_k^2 + ET_{k+1}^2 + 2q(k)\sigma_k(ET_{k+1}^2)^{1/2},\ 1 \leq k \leq n-1.$$

Proceeding as in the last part of the proof of (3), we get (4).

Remark 2.1. (3) and (4) are always true for $M = 2$ (in this case (2) becomes an equality with $D = 1$).

We obtain the following theorem by Lemma 2.2 :

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_n = 0$ and $\sigma_n^2 = EX_n^2 < \infty$, $n \geq 1$. Suppose that condition (2) is satisfied and A is defined as in Lemma 2.2. Then*

$$P\{\max_{1 \leq k \leq n} |S_k| \geq \epsilon\} \leq 2\epsilon^{-2}(A + (1 + A^2)^{\frac{1}{2}})^2 \sum_{k=1}^{n} \sigma_k^2. \tag{10}$$

Remark 2.2. By putting $M = 2$ Theorem 2.3 yields Theorem 1 of Chandra and Ghosal(1996a).

As a consequence of Theorem 2.3 we also have the following result:

Theorem 2.4. *Let $\{a_n, n \geq 1\}$ be a positive sequence of real numbers and $\{b_n, n \geq 1\}$ a positive sequence of nondecreasing real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_n = 0$ and $EX_n^2 < \infty$. Suppose that condition (2) is satisfied and A is defined as in Lemma 2.2. Then*

$$P\left\{\max_{1 \leq k \leq n}\left|\frac{\sum_{i=1}^{k} a_i X_i}{b_k}\right| \geq \epsilon\right\} \leq 8\epsilon^{-2}(A + (1 + A^2)^{\frac{1}{2}})^2 \sum_{k=1}^{n} (\frac{a_k^2 \sigma_k^2}{b_k^2}). \tag{11}$$

Proof. Without loss of generality, setting $b_0 = 0$, we have

$$\begin{aligned} S_k &= \sum_{j=1}^{k} b_j \frac{a_j X_j}{b_j} = \sum_{j=1}^{k} (\sum_{i=1}^{j} (b_i - b_{i-1}) \frac{a_j X_j}{b_j}) \\ &= \sum_{i=1}^{k} (b_i - b_{i-1}) \sum_{i \leq j \leq k} \frac{a_j X_j}{b_j}. \end{aligned}$$

Note that $(1/b_k)\sum_{j=1}^{k}(b_j - b_{j-1}) = 1$. So

$$\{|\frac{S_k}{b_k}| \geq \epsilon\} \subset \{\max_{1\leq i\leq k} |\sum_{i\leq j\leq k} \frac{a_j X_j}{b_j}| \geq \epsilon\}.$$

Therefore

$$\begin{aligned}\{\max_{1\leq k\leq n} |\frac{S_k}{b_k}| \geq \epsilon\} &\subset \{\max_{1\leq k\leq n}\max_{1\leq i\leq k} |\sum_{i\leq j\leq k} \frac{a_j X_j}{b_j}| \geq \epsilon\} \\ &= \{\max_{1\leq i\leq k\leq n} |\sum_{j\leq k} \frac{a_j X_j}{b_j} - \sum_{j<i} \frac{a_j X_j}{b_j}| \geq \epsilon\} \\ &\subset \{\max_{1\leq i\leq n} |\sum_{j=1}^{i} \frac{a_j X_j}{b_j}| \geq \frac{\epsilon}{2}\}.\end{aligned}$$

By Theorem 2.3, we obtain (11).

From Theorem 2.4 we can get the following more generalized Hájeck-Rènyi type inequality.

Theorem 2.5. *Let $\{a_n, n \geq 1\}$ be a positive sequence of real numbers and $\{b_n, n \geq 1\}$ a positive sequence of nondecreasing real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_k = 0$ and $EX_k^2 < \infty$. Suppose that condition (2) is satisfied and A is defined as in Lemma 2.2. Then for any $\epsilon > 0$ and any positive integer $m < n$,*

$$\begin{aligned}&P\left\{\max_{m\leq k\leq n}\left|\frac{\sum_{i=1}^{k} a_i X_i}{b_k}\right| \geq \epsilon\right\} \\ &\leq 8\epsilon^{-2}(A + (1+A^2)^{\frac{1}{2}})^2\left(\sum_{j=m+1}^{n}\frac{a_j^2\sigma_j^2}{b_j^2} + \sum_{j=1}^{m}\frac{a_j^2\sigma_j^2}{b_m^2}\right).\end{aligned} \tag{12}$$

Theorem 2.6. *Let $\{a_n, n \geq 1\}$ be a positive sequence of real numbers and $\{b_n, n \geq 1\}$ a positive sequence of nondecreasing real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_k = 0$ and $EX_k^2 < \infty$. Suppose that condition (2) is satisfied and that*

$$\sum_{n=1}^{\infty}(\frac{a_n}{b_n})^2\sigma_n^2 < \infty \tag{13}$$

and

$$(\sum_{k=1}^{\infty} q^M(k))^{1/M} < \infty \; for \; M \geq 2 \tag{14}$$

hold. Then,
(A) for any $0 < r < 2$, E $\sup_n(|S_n|/b_n)^r < \infty$,
(B) $0 < b_n \uparrow \infty$ implies $S_n/b_n \to 0$ a.s. as $n \to \infty$,

where $S_n = \sum_{i=1}^n a_i X_i,\ n \geq 1$.

Proof. Let $B = D(\sum_{k=1}^\infty q^M(k))^{1/M}$.
(A): Note that, for any $0 < r < 2$

$$E \sup_n \left(\frac{|S_n|}{b_n}\right)^r < \infty \Longleftrightarrow \int_1^\infty P\left\{\sup_n \frac{|S_n|}{b_n} > t^{\frac{1}{r}}\right\} dt < \infty.$$

By Theorem 2.4 it follows from (13) and (14) that

$$\begin{aligned} &\int_1^\infty P\left\{\sup_n \frac{|S_n|}{b_n} > t^{\frac{1}{r}}\right\} dt \\ &\leq 2\int_1^\infty t^{-\frac{2}{r}}(B + (1+B^2)^{\frac{1}{2}})^2 \sum_{n=1}^\infty \left(\frac{a_n}{b_n}\right)^2 \sigma_n^2 dt \\ &= 2(B + (1+B^2)^{\frac{1}{2}})^2 \sum_{n=1}^\infty \left(\frac{a_n}{b_n}\right)^2 \sigma_n^2 \int_1^\infty t^{-\frac{2}{r}} dt < \infty. \end{aligned}$$

Hence, the proof of (A) is complete.
(B): By Theorem 2.5, we have

$$\begin{aligned} &P\{\max_{m \leq k \leq n} \left|\frac{\sum_{i=1}^k a_i X_i}{b_k}\right| \geq \epsilon\} \\ &\leq 8\epsilon^{-2}(B + (1+B^2)^{\frac{1}{2}})^2 \left(\sum_{j=m+1}^n \frac{a_j^2 \sigma_j^2}{b_j^2} + \sum_{j=1}^m \frac{a_j^2 \sigma_j^2}{b_m^2}\right). \end{aligned}$$

But

$$\begin{aligned} &P\left\{\sup_{k \geq m} \left|\frac{\sum_{i=1}^k a_i X_i}{b_k}\right| \geq \epsilon\right\} \\ &= \lim_{n \to \infty} P\left\{\max_{m \leq j \leq n} \left|\frac{\sum_{i=1}^j a_i X_i}{b_j}\right| \geq \epsilon\right\} \\ &\leq 8\epsilon^{-2}(B + (1+B^2)^{\frac{1}{2}})^2 \left(\sum_{j=m+1}^\infty \frac{a_j^2 \sigma_j^2}{b_j^2} + \sum_{j=1}^m \frac{a_j^2 \sigma_j^2}{b_m^2}\right) \\ &< \infty \end{aligned} \tag{15}$$

by (13) and (14). By the Kronecker lemma, it follows from (13) that

$$(16) \qquad \sum_{j=1}^{m} \frac{a_j^2 \sigma_j^2}{b_m^2} \to 0 \ as \ m \to \infty.$$

Hence, combining (13)-(16) yields

$$\lim_{n\to\infty} P\left\{ \sup_{k\geq n} \left| \frac{\sum_{i=1}^{k} a_i X_i}{b_k} \right| \geq \epsilon \right\} = 0,$$

which completes the proof of (B).

Remark. 2.3. Theorem 2.6 (B) shows that Theorem 2 of Matula (1996) remains true if the assumption of negatively associated random variables is relaxed to AANA random variables satisfying (2) and (14).

Corollary 2.7. *Let* $\{a_n, n \geq 1\}$ *be a positive sequence of real numbers satisfying* $\sup_n a_n^2 < \infty$ *and let* $\{X_n, n \geq 1\}$ *be a sequence of AANA random variables with* $EX_k = 0$ *and* $EX_k^2 < \infty$. *Suppose that conditions (2) and (14) are satisfied and* $\sup_n \sigma_n^2 < \infty$. *Then, for* $0 < t < 2$, *and for all* $\epsilon > 0$, $m \geq 1$,

$$\begin{aligned} &P\{\sup_{n\geq m} \left| \frac{\sum_{i=1}^{n} a_i X_i}{n^{\frac{1}{t}}} \right| \geq \epsilon\} \\ &\leq 8\epsilon^{-2}(B' + (1+B'^2)^{\frac{1}{2}})^2 \tfrac{2}{2-t} (\sup_n \sigma_n^2)(\sup_n a_n^2) m^{(t-2)/t} \end{aligned}$$

where $B^{'} = BD$.

Corollary 2.8. *Let* $\{a_n, n \geq 1\}$ *be a positive sequence of real numbers satisfying* $\sup_n a_n^2 < \infty$ *and let* $\{X_n, n \geq 1\}$ *be a sequence of AANA random variables with* $EX_k = 0$ *and* $EX_k^2 < \infty$ *and* $\sup_n \sigma_n^2 < \infty$. *Assume that conditions (2) and (14) hold. Then, for* $0 < t < 2$,

(A) $\sum_{i=1}^{n} a_i X_i / n^{\frac{1}{t}} \longrightarrow 0$ a.s. as $n \to \infty$.

(B) $E \sup_n \left(|\sum_{i=1}^{n} a_i X_i| / n^{\frac{1}{t}} \right)^r < \infty$ for any $0 < r < 2$.

Example 2.9. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with $EX_k = 0$ and $EX_k^2 < \infty$. Assume that

$$\sum_{n=1}^{\infty} \frac{Var(X_n)}{n^4} < \infty$$

and (2) and (14) hold. Then, as $n \to \infty$

$$n^{-1} \sum_{k=1}^{n} X_k / k \longrightarrow 0 \ a.s.$$

Proof. By taking $a_k = 1/k$ and $b_n = n$ from Theorem 2.6 the desired result follows.

Example 2.10. Let $\{X_n, n \geq 1\}$ be a sequence of mean zero, square integrable AANA random variables. Assume that

$$\sum_{n=1}^{\infty} (\log\ n)^{-2} \frac{Var(X_n)}{n^2} < \infty$$

and (2) and (14) hold. Then, as $n \to \infty$

$$(\log\ n)^{-1} \sum_{k=1}^{n} X_k/k \longrightarrow 0\ a.s.$$

Proof. By taking $a_k = \frac{1}{k}$ and $b_n = log\ n$ from Theorem 2.6 the result follows.

3. Marcinkiewicz SLLN for AANA Random Variables

Now we prove the Marcinkiewicz strong law of large numbers for the identically distributed AANA random variables.

Theorem 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed AANA random variables. Assume that conditions (2) and (14) are satisfied.*
(1). *If $E|X_1|^t < \infty$ for some $0 < t < 1$, then*

$$\frac{\sum_{i=1}^{n} X_i}{n^{1/t}} \to 0.\ a.s.$$

(2). *If $E|X_1|^t < \infty$ for some $1 \leq t < 2$, then*

$$\frac{\sum_{i=1}^{n} (X_i - EX_i)}{n^{1/t}} \to 0.\ a.s.$$

Proof. The method of proof is the same as that used in the classical Marcinkiewicz strong law of large numbers for the independent and identically distributed random variables (see Stout, 1972, Theorem 3.2.3). We give the main step of the proof, omitting some details.

Assume that $E|X_1|^t < \infty$ for some $0 < t < 2$. To prove (1), it suffices to show that

$$\frac{\sum_{i=1}^{n} X_i^+}{n^{1/t}} \to 0.\ a.s. \tag{17}$$

$$\frac{\sum_{i=1}^{n} X_i^-}{n^{1/t}} \to 0.\ a.s. \tag{18}$$

To prove (2), it suffices to show that

$$\frac{\sum_{i=1}^{n} (X_i^+ - EX_i^+)}{n^{1/t}} \to 0.\ a.s. \tag{19}$$

$$\frac{\sum_{i=1}^{n} (X_i^- EX_i^-)}{n^{1/t}} \to 0.\ a.s. \tag{20}$$

where $X_i^+ = \max(X_i, 0),\ X_i^- = \max(-X_i, 0)$.
Note that $\{X_i^+, i \geq 1\},\ \{X_i^-, i \geq 1\}$ are AANA random variables by Lemma 2.1, we only prove (17) and (19). (18) and (20) can be proved similarly.
Set $Y_i = X_i^+ \wedge n^{1/t},\ i = 1, \cdots, n$. By Lemma 2.1 $\{Y_i, 1 \leq i \leq n\}$ are identically distributed AANA random variables. Notice that $E|X_1|^t < \infty$ implies $\sum_{n=1}^{\infty} P(|X_1| > n^{1/t}) < \infty$ and on the other hand,

$$P(Y_i \neq X_i^+) = P(X_1^+ \wedge n^{1/t} \neq X_i^+) \leq P(X_1^+ > n^{1/t}) \leq P(|X_1| > n^{1/t}).$$

So

$$P(Y_i \neq X_1^+\ i.o.) = 0. \tag{21}$$

We will prove first

$$\frac{\sum_{i=1}^{n} EY_i}{n^{1/t}} \to 0\ for\ 0 < t < 1, \tag{22}$$

$$\frac{\sum_{i=1}^{n} (EX_i^+ - EY_i)}{n^{1/t}} \to 0\ for\ 1 \leq t < 2 \tag{23}$$

Proof of (22). Notice that

$$\sum_{n=1}^{\infty} \frac{EY_n}{n^{1/t}} = \sum_{n=1}^{\infty} n^{-1/t}\{EX_1^+ I(X_1^+ \leq n^{1/t}) + n^{1/t} P(X_1^+ > n^{1/t})\}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{-1/t} E X_1^+ I(X_1^+ \le n^{1/t}) + \sum_{n=1}^{\infty} P(X_1^+ > n^{1/t}) \\
&\le \sum_{n=1}^{\infty} n^{-1/t} \sum_{k=1}^{n} E X_1^+ I((k-1)^{1/t} < X_1^+ \le k^{1/t}) \\
&\qquad + \sum_{n=1}^{\infty} P(|X_1| > n^{1/t}) \\
&\le \sum_{k=1}^{\infty} E X_1^+ I((k-1)^{1/t} < X_1^+ \le k^{1/t}) \sum_{n=k}^{\infty} n^{-1/t} + E|X_1|^t \\
&\le C \sum_{k=1}^{\infty} k^{-(1/t)+1} E X_1^+ I((k-1)^{1/t} < X_1^+ \le k^{1/t}) + E|X_1|^t \\
&\le C \sum_{k=1}^{\infty} E(X_1^+)^t I((k-1)^{1/t} < X_1^+ \le k^{1/t}) + E|X_1|^t \\
&\le CE|X_1|^t < \infty.
\end{aligned}
$$

By Kronecker's Lemma (22) is true.

Proof of (23). Since $|EX_n^+ - EY_n| \le EX_n^+ I(X_n^+ > n^{1/t}) + n^{1/t} P(X_n^+ > n^{1/t})$, it follows in a similar way that

$$\sum_{n=1}^{\infty} n^{-1/t} |EX_n^+ - EY_n| < \infty.$$

Hence, (23) is proved.
From (21), (22) and (23) it suffices to show that

$$\frac{\sum_{i=1}^{n} (Y_i - EY_i)}{n^{1/t}} \to 0 \ a.s. \tag{24}$$

By Theorem 2.6 taking $b_n = n^{1/t}$, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-2/t} E(Y_n - EY_n)^2 \\
&\le \sum_{n=1}^{\infty} n^{-2/t} EY_n^2 \\
&= \sum_{n=1}^{\infty} n^{-2/t} E(X_1^+ \wedge n^{1/t})^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{-2/t} E(X_1^+)^2 I(X_1^+ \le n^{1/t}) + \sum_{n=1}^{\infty} P(X_1^+ > n^{1/t}) \\
&\le \sum_{n=1}^{\infty} n^{-2/t} \sum_{k=1}^{n} E(X_1^+)^2 I((k-1)^{1/t} < X_1^+ \le k^{1/t}) + E|X_1|^t \\
&= \sum_{n=1}^{\infty} n^{-2/t} \sum_{k=1}^{n} E(X_1^+)^2 I((k-1)^{1/t} < X_1^+ \le k^{1/t}) \sum_{n=k}^{\infty} n^{-2/t} + E|X_1|^t \\
&\le C \sum_{k=1}^{\infty} k^{-(2/t)+1} E(X_1^+)^2 I((k-1)^{1/t} < X_1^+ \le k^{1/t}) + E|X_1|^t \\
&\le C \sum_{k=1}^{\infty} k^{-(2/t)+1} k^{(2/t)-1} E(X_1^+)^t I((k-1)^{1/t} < X_1^+ \le k^{1/t}) + E|X_1|^t \\
&\le CE|X_1|^t < \infty.
\end{aligned}
$$

The proof is complete.

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