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MINIMAX INEQUALITIES IN THE SPACES WITHOUT LINEAR STRUCTURE

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Abstract. In this paper, we found a new result by relaxing the condition of [15, Corollary 2]. As its application, we have obtained some new minimax inequalities of Ky Fan and minimax theorems in the spaces without linear structure.

1. Introduction and Preliminaries

Since Ky Fan ($[8]$) generalized KKM theorem, a number of applications have been found. Fan's theorem is now becoming a very versatile tool in nonlinear analysis, such as fixed point, variational inequalities (see [2-7]). Fan's theorem was used by many authors to prove fixed point and minimax theorems in topological vector spaces (see [3], [8]). Ha [1-2] has given the generalization of Fan's theorem and Fan's minimax inequalities. This paper has two purposes. First we obtain a new theorem by relaxing closed condition of sets of [15, Corollary 2] , and next, as its application, we obtain some minimax inequalities and minimax theorems in the spaces without linear structure.

To begin with we explain the notions of an H-space introduced by Horvath $[9-11]$ and related concepts on H-spaces.

Let X be a topological space and let $F(X)$ be the family of all nonempty finite subsets of X. Let $\{\Gamma_A\}$ be a family of nonempty contractible subsets of X indexed by $A \in F(X)$ such that $\Gamma_A \subset \Gamma_{A'}$, whenever $A \subset A'$. The pair $(X, \{\Gamma_A\})$ is called an H-space. Given an H-space $(X, \{\Gamma_A\})$, a nonempty subset D of X is called

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H-convex if $\Gamma_A \subset D$ for each nonempty finite subset A of D. For a nonempty K of X, we define the H-convex hull of K, denote by H-co$K$, as

$$H - coK = \bigcap \{D \subset X : D \text{ is } H\text{-convex and } D \supset K\}.$$

Clearly, $H - coK$ is H-convex and is the smallest H-convex containing K.

Let $(X, \{\Gamma_A\})$ be an H-space, let Y be a topological space, and let $f : X \times Y \to R$ be a function. For each $y \in Y$, $f(x, y)$ is said to be H-quasiconvex (or H-quasiconcave) on X if the set $\{x \in X : f(x, y) < t\}$ (or $\{x \in X : f(x, y) > t\}$) is H-convex for all $t \in R$.

A topological space is called acyclic if all of its reduced $\check{C}ech$ homology groups over rationals vanish. In particular, any contractible space is acyclic, thus, any nonempty convex or star-shaped set is acyclic.

Now let X, Y be two topological spaces. By a set-valued mapping T defined on X with values in Y, we mean that to each point $x \in X$, T assigns an unique nonempty subset $T(x)$ of Y. T is called upper semicontinuous if for each open subset G of Y, the set $\{x \in X : T(x) \subset G\}$ is open in X. It is easy to show (e.g., [14]) that if Y is a compact Hausdorff and if $T(x)$ is closed for each x, then T is upper semicontinuous if and only if the graph $\{(x, y) \in X \times Y : y \in T(x)\}$ of T is closed in $X \times Y$.

2. Main Results

Our main result is the following Theorem 2.1. To prove it we need to cite a lemma in Tarafdar [12].

Lemma 2.1. *Let X be a compact topological space and let $(Y, \{\Gamma_A\})$ be an H-space. Let $T : X \to 2^Y$ be a set-valued mapping and $T^{-1}(y) = \{x \in X : y \in Tx\}$ for each $y \in Y$.*

Suppose the following conditions are fulfilled:

(i) *For each $x \in X$, $T(x)$ is a nonempty H-convex subset of Y,*

(ii) *$\{intT^{-1}(y) : y \in Y\}$ is an open covering of X.*

Then there is a continuous selection $f : X \to Y$ of T such that $f = g\varphi$, where $g : \Delta_n \to Y$ and $\varphi : X \to \Delta_n$ are continuous mappings, and Δ_n is the standard n-dimensional simplex for some positive n.

Theorem 2.1. *Let X be a Hausdorff topological space, let $(Y, \{\Gamma_A\})$ be an H-space, and let M, N be two subsets of $X \times Y$ with $M \subset N$. Suppose the following conditions are fulfilled*:

(i) *For each* $y \in Y$, *there exists a closed subset* $X_y \subset X$ *such that the set* $\{x \in X : (x, y) \in N\} \subset X_y$,

(ii) *For each* $x \in X$, *the set* $\{y \in Y : (x, y) \notin M\}$ *is* H*-convex or empty.*

Suppose also that there exists a subset P *of* M *and a compact subset* K *of* X *such that* P *is closed in* $X \times Y$, *and*

(iii) *For each* $y \in Y$, *the set* $\{x \in K : (x, y) \in P\}$ *is nonempty acyclic.*

Then

$$\bigcap_{y \in Y} X_y \bigcap K \neq \emptyset.$$

Proof. If the conclusion of Theorem 2.1 is false, that is, $\bigcap_{y \in Y} X_y \subset X \backslash K$, we prove that there exists $x_0 \in K$ such that

$$\{x_0\} \times Y \subset N, \tag{2.1}$$

or else, for each $x \in K$, there exists $y_0 \in Y$ such that $(x, y_0) \notin N$. Let

$$S(x) = \{y \in Y : (x, y) \notin N\}, T(x) = \{y \in Y : (x, y) \notin M\}.$$

Then $S, T : K \to 2^Y$ are two set-valued mappings such that for each $x \in K$, there exists a point $y \in S(x)$. By $S^{-1}(y) = \{x \in X : (x, y) \notin N\}$ and (i), $S^{-1}(y) \supset X \backslash X_y = V_y$, V_y is an open set of X, and $\bigcup_{y \in Y} V_y = X \backslash \bigcap_{y \in Y} X_y \supset K$. Therefore, $\{\text{int}S^{-1}(y) : y \in Y\}$ is an open covering of K. Consequently, $\{\text{int}(H - coS)^{-1}(y) : y \in Y\}$ is an open covering of K, where the following mapping $H - coS : K \to 2^Y$ is defined by

$$H - coS(x) = H - co(S(x)) \quad \text{for each } x \in K.$$

By Lemma 2.1, there exists a continuous mapping $f : K \to Y$ such that $f = g\Psi$, and

$$f(x) \in H - coS(x) \subset T(x)$$

for all $x \in K$, where $\Psi : K \to \Delta_n$, $g : \Delta_n \to Y$ are continuous mappings and Δ_n is the standard n-simplex. Hence

$$(x, f(x)) \notin M \quad \text{for all } x \in K. \tag{2.2}$$

On the other hand, we define a set-valued mapping $G : Y \to 2^K$ by

$$G(y) = \{x \in K : (x, y) \in P\} \quad \text{for all } y \in Y.$$

By (iii), $G(y)$ is nonempty and acyclic for all $y \in Y$. Since P is closed in $X \times Y$, each $G(y)$ is closed in K and the graph of G is closed in $Y \times K$; thus, G is an upper semicontinuous set-valued mapping defined on Y. Consequently, so is the mapping $F : \Delta_n \to 2^K$ defined by $F(\mu) = G(g(\mu))$. By virtue of $[13, \text{Lemma } 2.1]$, there exists a point $\overline{\mu} \in \Delta_n$ such that $\overline{\mu} \in \Psi(F(\overline{\mu})) = \Psi(G(g(\overline{\mu})))$, and so there is a point $\overline{x} \in G(g(\overline{\mu})) \subset K$ such that $\overline{\mu} = \Psi(\overline{x})$. Let $\overline{y} = g(\overline{\mu})$, then $\overline{y} = g(\Psi(\overline{x})) = f(\overline{x})$ and $\overline{x} \in G(\overline{y})$, i.e.,

$$(\overline{x}, f(\overline{x})) = (\overline{x}, \overline{y}) \in P \subset M.$$

This contradicts (2.2), hence, (2.1) is true. But this contradicts $\bigcap_{y\in Y} X_y \subset X \backslash K$ again, therefore, we must have $\bigcap_{y\in Y} X_y \bigcap K \neq \emptyset$. This proves the theorem.

Remark 2.1. Theorem 2.1 improve and extend $[16, \text{Theorem } 2]$ to a topological space and an H-space.

As an immediate consequence of Theorem 2.1, we obtain some new minimax theorems and some minimax inequalities in the spaces without linear structure.

Theorem 2.2. *Let X be a Hausdorff topological space. Let $(Y, \{\Gamma_A\})$ be an H-space, and let $f, g, h : X \times Y \to R$ be functions. Let $\beta = \inf_{K\in\overline{K}} \sup_{y\in Y} \min_{x\in K} h(x, y)$, where $\overline{K} = \{K \subset X : K$ is compact subset of $X\}$. Suppose the following conditions are fulfilled*:

(i) $f(x, y) \leq g(x, y) \leq h(x, y)$ *for all* $(x, y) \in X \times Y$,

(ii) $f(x, y)$ *is lower semicontinuous on X for each $y \in Y$*,

(iii) $g(x, y)$ *is H-quasiconcave on Y for each $x \in X$*,

(iv) $h(x, y)$ *is lower semicontinuous on $X \times Y$, and the set $\{x \in K : h(x, y) < t\}$ is acyclic or empty for each $t > \beta$, $K \in \overline{K}$ and $y \in Y$. Then*

$$\inf_{x\in X} \sup_{y\in Y} f(x, y) \leq \inf_{K\in\overline{K}} \sup_{y\in Y} \min_{x\in K} h(x, y). \tag{2.3}$$

If X is compact, then

$$\min_{x\in X} \sup_{y\in Y} f(x, y) \leq \sup_{y\in Y} \min_{x\in X} h(x, y). \tag{2.4}$$

Proof. We can assume that the right-hand side of (2.3) is not $+\infty$. If the conclusion of Theorem 2.2 is false, then there is a real number t such that

$$\inf_{x\in X} \sup_{y\in Y} f(x, y) > t > \inf_{K\in\overline{K}} \sup_{y\in Y} \min_{x\in K} h(x, y).$$

Let $M = \{(x, y) \in X \times Y : g(x, y) \leq t\}$ and $P = \{(x, y) \in X \times Y : h(x, y) \leq t\}$. Then

(a) For each $x \in X$, by (iii), the set $\{y \in Y : (x, y) \notin M\}$ is H-convex and satisfies (ii) of Theorem 2.1,

(b) For each $y \in Y$, by (i), $\{x \in X : (x, y) \in M\} \subset \{x \in X : f(x, y) \leq t\} = X_y$, by (ii), X_y is closed and satisfies (i) of Theorem 2.1. It is easy to verify that P is closed in $X \times Y, P \subset M$,

(c) Let K be a compact subset of X such that

$$t > \sup_{y \in Y} \min_{x \in K} h(x, y).$$

Then for any $y \in Y$, the set $\{x \in K : h(x, y) \leq t\}$ is nonempty and we know the set $\{x \in K : h(x, y) \leq t\} = \bigcap_{\epsilon > 0} \{x \in K : h(x, y) < t + \epsilon\}$ is acyclic (this follows from the continuity of Cech homology. See, e.g., McClendon [17]). Thus, by Theorem 2.1,

$$\bigcap_{y \in Y} X_y \bigcap K \neq \emptyset,$$

that is, there exists $x_0 \in K$ such that

$$f(x_0, y) \leq t \text{ for all } y \in Y.$$

Hence

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) \leq t.$$

This contradicts the choice of t, therefore, (2.3) is proved.

We shall establish the following similarities of the proof of theorem 2.2.

Theorem 2.3. *Let $f, g, h : X \times Y \to R$ be as in Theorem 2.2 and X is compact. Then for each $\lambda \in R$, one of the following situations hold*:

(i) *There exists $x_0 \in X$ such that $f(x_0, y) \leq \lambda$ for all $y \in Y$,*

(ii) *There exists $y_0 \in Y$ such that $f(x, y_0) > \lambda$ for all $x \in X$.*

Remark 2.2. Theorem 2.3 is the generalization of [16,Theorem 4].

The following three minimax theorems are obtained from Theorem 2.2 as special cases by taking $f = g$, $g = h$, $f = g = h$.

Corollary 2.1. *Let $f,\ h : X \times Y \to R$ be two functions. Let $\beta = \inf_{K \in \overline{K}} \sup_{y \in Y} \min_{x \in K} h(x, y)$, where $\overline{K} = \{K \subset X : K$ is compact subset of $X\}$. Suppose the following conditions are fulfilled*:

(i) $f(x, y) \leq h(x, y)$ *for all* $(x, y) \in X \times Y$,

(ii) $f(x,y)$ *is lower semicontinuous on* X *for each* $y \in Y$, *and* $f(x,y)$ *is* H*-quasiconcave on* Y *for each* $x \in X$,

(iii) $h(x,y)$ *is lower semicontinuous on* $X \times Y$ *and the set* $\{x \in K : h(x,y) < t\}$ *is acyclic or empty for each* $t > \beta$, $K \in \overline{K}$ *and* $y \in Y$. *Then*

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) \leq \inf_{K \in \overline{K}} \sup_{y \in Y} \min_{x \in K} h(x,y).$$

If X *is compact, then*

$$\min_{x \in X} \sup_{y \in Y} f(x,y) \leq \sup_{y \in Y} \min_{x \in X} h(x,y).$$

Corollary 2.2. *Let* $f, g : X \times Y \to R$ *be two functions. Let* $\beta = \inf_{K \in \overline{K}} \sup_{y \in Y} \min_{x \in K} g(x,y)$, *where* $\overline{K} = \{K \subset X : K$ *is compact subset of* $X\}$. *Suppose the following conditions are fulfilled*:

(i) $f(x,y) \leq g(x,y)$ *for all* $(x,y) \in X \times Y$,

(ii) $f(x,y)$ *is lower semicontinuous on* X *for each* $y \in Y$,

(iii) $g(x,y)$ *is* H*-quasiconcave on* Y *for each* $x \in X$,

(iv) $g(x,y)$ *is lower semicontinuous on* $X \times Y$ *and the set* $\{x \in K : g(x,y) < t\}$ *is acyclic or empty for each* $t > \beta$, $K \in \overline{K}$ *and* $y \in Y$. *Then*

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) \leq \inf_{K \in \overline{K}} \sup_{y \in Y} \min_{x \in K} g(x,y).$$

If X *is compact*, *then*

$$\min_{x \in X} \sup_{y \in Y} f(x,y) \leq \sup_{y \in Y} \min_{x \in X} g(x,y).$$

Corollary 2.3. *Let* $f : X \times Y \to R$ *be a real-valued function. Let* $\beta = \inf_{K \in \overline{K}} \sup_{y \in Y} \min_{x \in K} f(x,y)$, *where* $\overline{K} = \{K \subset X : K$ *is compact subset of* $X\}$. *Suppose the following conditions are fulfilled*:

(i) $f(x,y)$ *is lower semicontinuous on* $X \times Y$,

(ii) $f(x,y)$ *is* H*-quasiconcave on* Y *for each* $x \in X$, *and the set* $\{x \in K : f(x,y) < t\}$ *is acyclic or empty for each* $t > \beta$, $K \subset \overline{K}$ *and* $y \in Y$. *Then*

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) = \inf_{K \in \overline{K}} \sup_{y \in Y} \min_{x \in K} f(x,y).$$

If X is compact, then

$$\min_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\min_{x\in X} f(x,y).$$

Remark 2.3. Corollary 2.3 is the generalization of [1, Theorem 4], thus, Theorem 2.2, Corollary 2.1 and Corollary 2.2 are all the generalizations of [1, Theorem 4].

Theorem 2.4. *Let Y be a compact Hausdorff topological space and $(X, \{\Gamma_A\})$ be an H-space, let $f, g : X \times Y \to R$ be two real-valued functions such that:*

(i) $f(x,y) \leq g(x,y)$ *for all* $(x,y) \in X \times Y$,

(ii) $f(x,y)$ *is H-quasiconvex on X for each $y \in Y$,*

(iii) $g(x,y)$ *is upper semicontinuous on Y for each $x \in X$.*

If T is an upper semicontinuous set-valued mapping defined on X such that Tx is a nonempty compact acyclic subset of Y for each $x \in X$, then

$$\inf_{y\in Tx} f(x,y) \leq \max_{y\in Y}\inf_{x\in X} g(x,y). \tag{2.5}$$

Proof. If the conclusion of Theorem 2.4 is false, then there is a real number t such that

$$\inf_{y\in Tx} f(x,y) > t > \max_{y\in Y}\inf_{x\in X} g(x,y).$$

Let

$$M = \{(x,y) \in X \times Y : f(x,y) \geq t\},\ P = \{(x,y) \in X \times Y : y \in Tx\},$$

and $Y_x = \{y \in Y : g(x,y) \geq t\}$ for each $x \in X$. It is easy to verify that M and Y_x satisfies (i) and (ii) of Theorem 2.1, and P is closed in $X \times Y$ and satisfies (iii) of Theorem 2.1 by taking $K = Y$. Thus, by theorem 2.1, $\bigcap_{x\in X} Y_x \bigcap Y \neq \emptyset$, that is, there exists $y_0 \in Y$ such that

$$g(x, y_0) \geq t, \text{ for all } x \in X.$$

Hence

$$\max_{y\in Y}\inf_{x\in X} g(x,y) \geq t.$$

This contradicts the choice of t, therefore, (2.5) is proved.

Remark 2.4. Theorem 2.4 is the generalization of [2, Theorem 1].

By Theorem 2.4, we can obtain the following corollary.

Corollary 2.4. *Let f, g, T be as in Theorem 2.4. Assume further, that given $\lambda \in R$, we have*

$$\inf_{y \in Tx} f(x, y) \geq \lambda \text{ for all } x \in X.$$

Then there exists $y_0 \in Y$ such that $g(x, y_0) \geq \lambda$ for all $x \in X$.

Remark 2.5. Corollary 2.4 is similar to the result of [4, Theorem 13.4].

We can obtain the following two theorems whose proofs are similar to the proof of Theorem 2.4.

Theorem 2.5. *Let $(Y, \{\Gamma_A\})$ be a Hausdorff H-space, let X be nonempty compact acyclic H-convex subset of Y. Let $f,\ g : X \times Y \to R$ be two real-valued functions satisfying $(i) - (iii)$ of Theorem 2.4. Then*

$$\inf_{x \in X} f(x, x) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y).$$

Theorem 2.6. *Let $(Y, \{\Gamma_A\})$ be a Hausdorff H-space, let X be nonempty compact acyclic H-convex subset of Y. Let $f : X \times Y \to R$ be a real-valued function such that*

(i) *$f(x, y)$ is a H-quasiconvex on X for each $y \in Y$,*

(ii) *$f(x, y)$ is upper semicontinuous on Y for each $x \in X$.*

Then

$$\inf_{x \in X} f(x, x) \leq \sup_{y \in Y} \inf_{x \in X} f(x, y).$$

Corollary 2.5. *Let $(Y, \{\Gamma_A\})$ be a Hausdorff H-space, let X be nonempty compact acyclic H-convex subset of Y. Let $f,\ g : X \times Y \to R$ be two real-valued functions such that*

(i) *$f(x, y) \leq g(x, y)$ for all $(x, y) \in X \times Y$,*

(ii) *$f(x, y)$ is lower semicontinuous on Y for each $x \in X$,*

(iii) *$g(x, y)$ is H-quasiconcave on X for each $y \in Y$.*

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} g(x, x).$$

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