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LOCAL C-COSINE FAMILIES AND N-TIMES INTEGRATED LOCAL COSINE FAMILIES

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Abstract. This paper is concerned with local C-cosine families and local n-times integrated cosine families. The characterization of each cosine families is mentioned through their infinitesimal generators. The relationship among local C-cosine families, local integrated cosine families and the abstract Cauchy problem is also mentioned.

1. Introduction

This paper is concerned with a local C-cosine families and local n-times integrated cosine families in a Banach space X. Here C is a bounded linear operator in X and $\overline{R(C)} = X$.

We first mention the recent development of theories of exponentially bounded *C*-cosine families and *n*-times integrated cosine families. In the first order case in time, they have being studied as C-semigroups and n-times integrated semigroups. Many authors gave characterizations of the complete infinitesimal generator of exponentially bounded C-semigroups (for example [2,3,4,9]). On the other hand the characterizations of exponentially bounded n-times integrated semigroups were given (see [1,4,10]). In Tanaka and Okazawa [16], the theories of local C-semigroups and local n-times integrated semigroups were established. In the second order case in time, C-cosine families and n-times integrated cosine family were also studied and the analogous results were given. In [8] and [12] Y.-C Li and S.-Y Shaw gave characterization of the generator of an exponentially bounded integrated *C*-cosine family. Local integrated cosine family was studied by Tingwen [17]. In his paper, it was shown that there exists integrated cosine family which are not defined on all $R = (-\infty, \infty)$. Thus, our first purpose is to establish the theory

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of C-cosine families which are neither exponentially bounded nor defined on R (local C-cosine families). In [7] F. Huang and T. Huang gave a characterization of the complete infinitesimal generator in terms of the asymptotic C-resolvent. They showed the similar theorem to our theorem (Theorem 3.2). In the mean time their paper appeared in the same time as this paper was written independently. But their theorem had an extra condition, which is not necessary to be mentioned. Such an extra condition is technical and it appears if one used reduction of the second order equation to the system with matrix operator. Also we do not give here our original proof of Theorem 5.2 and just quote [7].

The second purpose is to develop the theory of local n-times integrated cosine families, by which we mean that they are neither exponentially bounded nor defined on R. Here we use the different definition of n-times integrated cosine families from ones defined by other authors (Definition 4.1). We investigate basic properties of the complete infinitesimal generator of a local n-times integrated cosine family (Proposition 5.4). Next, we clarify the relationship between local C-cosine families and local n-times integrated cosine families (Theorem 5.9).

In Section 2 we introduce the notion of local C-cosine families of bounded linear operators on a Banach space and derive some properties of the complete infinitesimal generator of a local C-cosine family. We also deal with a characterization of the complete infinitesimal generator of a local C-cosine family in terms of its asymptotic C-resolvent. In Section 3 we give a generation theorem of local C-cosine families. In Section 4 we introduce the notion of a local n-times integrated cosine family. We also investigate properties of a local n-times integrated cosine family. In Section 5 we shall show that the abstract Cauchy problem, formulated for the complete infinitesimal generator, is given by an n-times integrated cosine family. We shall also state the relationships among a local n-times integrated cosine family, a local C-cosine family and the abstract Cauchy problem.

2. PRELIMINARIES

Let X be a Banach space. We denote by $B(X)$ the set of all bounded linear operators on X and by $N = \{1, 2, 3, ...\}$ the set of natural numbers. Let $C \in B(X)$.

Definition 2.1. A family of bounded operators $\{\mathbf{C}(t) : |t| < T\}$ is called a **local C-cosine family** on X

(i) $[\mathbf{C}(t+s) + \mathbf{C}(t-s)]C = 2\mathbf{C}(t)\mathbf{C}(s)$ for any $t, s, t \pm s \in (-T, T)$,

(ii) $\mathbf{C}(0) = C$,

(iii) $\mathbf{C}(\cdot)$ is strongly continuous on $(-T, T)$.

We recall that a local C-cosine family called **nondegenerate** if condition $\mathbf{C}(t)x = 0$ for any $t \in (0, T/2)$ implies $x = 0$.

The following proposition is well-known for the long time.

Proposition 2.2. $([12, 19])$ *A local C-cosine family is nondegenerate iff C is an injective operator.*

Starting from now we will consider only the case of $C \in B(X)$, which is an injective operator such that $R(C)$ is dense in X, i.e. $\overline{R(C)} = X$.

Remark 2.3. A local C-cosine family on X has the following properties:

$$1^0.\ \ \mathbf{C}(-s) = \mathbf{C}(s) \ \text{ and } \ 2^0.\ \ \mathbf{C}(s)C = C\mathbf{C}(s).$$

Indeed, setting $t = 0$ in condition (i) of Definition 2.1 we have

$$[\mathbf{C}(s) + \mathbf{C}(-s)]C = 2C\mathbf{C}(s),$$

which implies 1^0. In fact, $C\mathbf{C}(-s) = \frac{1}{2}[\mathbf{C}(-s)+\mathbf{C}(s)]C = C\mathbf{C}(s)$, i.e. $C[\mathbf{C}(-s) - \mathbf{C}(s)] = 0$, which implies $\mathbf{C}(-s) = \mathbf{C}(s)$. The assertion 2^0 follows from 1^0 and (i) of Definition 2.1.

Definition 2.4. The **infinitesimal generator** of $\{\mathbf{C}(t) : |t| < T\}$ is defined as the limit

$$G_0 x := \lim_{h \to 0+} \frac{2}{h^2}[C^{-1}\mathbf{C}(h)x - x], x \in D(G_0) \tag{2.1}$$

with a natural domain $D(G_0) := \{x \in R(C) : \exists \lim_{h\to 0+} \frac{2}{h^2}[C^{-1}\mathbf{C}(h)x - x]\}$.

Remark 2.5. There are some other possibilities to define infinitesimal generator. For example, in [19] they define

$$Ax = C^{-1} \lim_{h \to 0+} 2h^{-2}(\mathbf{C}(h)x - Cx).$$

In such a situation A is closed and satisfied $C^{-1}AC = A$.

Proposition 2.6. *The operator G_0 is closable and $D(G_0)$ is dense in X.*

Proof. For any $x \in X$ we define

$$x^h = \frac{2}{h^2}\int_o^h (h-s)\mathbf{C}(s)\, x ds.$$

Then in the same way as in Fattorini [5] one obtains

$$\begin{aligned}\frac{2}{h^2}(C^{-1}\quad \mathbf{C}(h)-I)x^\tau &= \frac{2}{\tau^2h^2}\int_h^{h+\tau}(h+\tau-s)\mathbf{C}(s)x\,ds\\ &+\frac{2}{\tau^2h^2}\int_{-h}^{\tau-h}\tau-h-s\mathbf{C}s\,xds-\frac{4}{\tau^2h^2}\int_0^\tau(\tau-s)\mathbf{C}(s)\,xds\\ &= \frac{2}{\tau^2h^2}\int_\tau^{\tau+h}(\tau+h-s)\mathbf{C}(s)\,xds-\frac{2}{\tau^2h^2}\int_0^h(\tau+h-s)\mathbf{C}(s)\,xds\\ &-\frac{2}{\tau^2h^2}\int_{\tau-h}^{\tau}(\tau-h-s)\mathbf{C}(s)\,xds+\frac{2}{\tau^2h^2}\int_{-h}^{0}(\tau-h-s)\mathbf{C}(s)\,xds.\end{aligned}$$

Thus $x^h \in D(G_0)$ with

$$G_0x^h = \frac{2}{h^2}(\mathbf{C}(h)-C)x.$$

It is clear that $x^h \to Cx$ as $h \to 0$, so it means that $D(G_0)$ is dense in X. To show that G_0 is closable we note that $\mathbf{C}(s)\mathbf{C}(t) = \mathbf{C}(t)\mathbf{C}(s)$ for all s, t and hence

$$\frac{2}{h^2}(C^{-1}\mathbf{C}(h)-I)\mathbf{C}(t)x = \mathbf{C}(t)\frac{2}{h^2}(C^{-1}\mathbf{C}(h)-I)x$$

for $h > 0$. So for any $x \in D(G_0)$ one gets

$$G_0\mathbf{C}(t)x = \mathbf{C}(t)G_0x \tag{2.2}$$

and moreover,

$$\frac{\tau^2}{2}(G_0x)^\tau = \frac{\tau^2}{2}G_0x^\tau = (\mathbf{C}(\tau)-C)x. \tag{2.3}$$

Now, let $\{x_n\}$ be a sequence in $D(G_0)$ such that $x_n \to 0$ and $G_0x_n \to v$. To prove the closability we are going to show that $v = 0$. Since $v^\tau = 0$ for any τ one obtains $v^\tau \to Cv = 0$ for $\tau \to 0$, i.e. $v = 0$. ∎

Definition 2.4. The operator $G = \overline{G_0}$ is said to be the **complete infinitesimal generator** of a local C-cosine family $\mathbf{C}(\cdot)$.

Now we have

$$\begin{aligned}&G\int_0^t(t-s)\mathbf{C}(s)\,xds = \mathbf{C}(t)x - Cx,\\ &x\in X, G\int_0^t(t-s)\mathbf{C}(s)\,xds = \int_0^t(t-s)\mathbf{C}(s)\,G\,xds,\quad x\in D(G).\end{aligned} \tag{2.4}$$

Hence $\mathbf{C}(\cdot)x$ for $x \in D(G)$ is twice continuously differentiable in t with $|t| < T$ and

$$\mathbf{C}''(t)x = \mathbf{C}(t)Gx = G\mathbf{C}(t)x,\ x \in D(G). \tag{2.5}$$

We associate with $\mathbf{C}(\cdot)$ the local C-sine family by the formula

$$\mathbf{S}(t)x = \int_0^t \mathbf{C}(s)x ds, \; x \in X. \tag{2.6}$$

Proposition 2.8. *The following relations hold for* $t, h, t \pm h \in (-T, T)$*:*

(*i*) $\Big(\mathbf{S}(t+h) + \mathbf{S}(t-h)\Big)C = 2\mathbf{S}(t)\mathbf{C}(h)$,

(*ii*) $\mathbf{S}(t+h)C = \mathbf{S}(t)\mathbf{C}(h) + \mathbf{S}(h)\mathbf{C}(t)$,

(*iii*) $\Big(\mathbf{C}(t+h) - \mathbf{C}(t-h)\Big)C = 2G\mathbf{S}(t)\mathbf{S}(h)$,

(*iv*) $\mathbf{C}'(t)x = \mathbf{S}(t)Gx = G\mathbf{S}(t)x$ *for any* $x \in D(G)$,

(*v*) $\mathbf{C}''(t)x = \mathbf{C}(t)Gx = G\mathbf{C}(t)x$ *for any* $x \in D(G)$,

(*vi*) $\mathbf{S}''(t)x = \mathbf{S}(t)Gx = G\mathbf{S}(t)x$ *for any* $x \in D(G)$.

Proof. Integrating (i) of Definition 2.1 in t one gets

$$\Big(\int_0^t \mathbf{C}(s+h)ds + \int_0^t \mathbf{C}(s-h)ds\Big)C = 2\mathbf{S}(t)\mathbf{C}(h).$$

Hence

$$\Big(\int_h^{t+h} \mathbf{C}(\eta)d\eta + \int_{-h}^{t-h} \mathbf{C}(\eta)d\eta + \int_0^h \mathbf{C}(\eta)d\eta - \int_{-h}^0 \mathbf{C}(\eta)d\eta\Big)C = 2\mathbf{S}(t)\mathbf{C}(h).$$

From this (i) follows. Now we can add $\Big(\mathbf{S}(t+h) + \mathbf{S}(h-t)\Big)C = 2\mathbf{S}(h)\mathbf{C}(t)$ to (i) and obtain (ii). To prove (iii) take $z = \int_0^h \int_0^s \mathbf{C}(\mu)\mathbf{C}(\nu)xd\mu d\nu$, where $x \in X$. We are going to show that $z \in D(G)$ and $2Gz = \Big(\mathbf{C}(h+s) - \mathbf{C}(h-s)\Big)Cx$. To do this we can write

$$2C^{-1}\mathbf{C}(t)z = 2C^{-1}\int_0^h \int_0^s \mathbf{C}(t)\mathbf{C}(\mu)\mathbf{C}(\nu)xd\mu d\nu$$

$$= \int_0^h \int_0^s \Big(\mathbf{C}(t+\mu) + \mathbf{C}(t-\mu)\Big)\mathbf{C}(\nu)xd\mu d\nu = \int_0^h \int_{t-s}^{t+s} \mathbf{C}(\eta)\mathbf{C}(\nu)xd\eta d\nu.$$

It is clear that

$$\begin{aligned}2C^{-1}\frac{d}{dt}\mathbf{C}(t)z &= \int_0^h \Big(\mathbf{C}(t+s)+\mathbf{C}(t-s)\Big)\mathbf{C}(\nu)xd\nu \\ &= \frac{1}{2}\int_0^h \Big(\mathbf{C}(t+s+\nu)+\mathbf{C}(t+s-\nu)-\mathbf{C}(t-s+\nu)-\mathbf{C}(t-s-\nu)\Big)Cxd\nu \\ &= \frac{1}{2}\Big(\int_{t+s-h}^{t+s+h}\mathbf{C}(\eta)d\eta - \int_{t-s-h}^{t-s+h}\mathbf{C}(\eta)\Big)Cxd\eta.\end{aligned}$$

Hence

$$4C^{-1}\frac{d^2}{dt^2}\mathbf{C}(t)z = \Big(\mathbf{C}(t+h+s)-\mathbf{C}(t+s-h)-\mathbf{C}(t-s+h)+\mathbf{C}(t-s-h)\Big)Cx,$$

which means, using (2.5) at $t=0$, that

$$2Gz = \Big(\mathbf{C}(h+s)-\mathbf{C}(h-s)\Big)Cx.$$

To obtain (iv) just take derivatives of (2.4). Statement (v) follows from (2.5) and (vi) follows from (2.6) taking two derivatives and using (iv). ■

Next, let $\tau \in (0,T)$. Put

$$L_\tau(\lambda)x := \int_0^\tau e^{-\lambda t}\mathbf{S}(t)xdt, \quad x \in X, \tag{2.7}$$

this is so-called the "**local Laplace transform**" of $\mathbf{S}(\cdot)$.

Proposition 2.9. *Let G be the complete infinitesimal generator of $\mathbf{C}(\cdot)$ and $L_\tau(\lambda)$ be a local Laplace transform of $\mathbf{S}(\cdot)$. Then*

$$\lambda L_\tau(\lambda) = \int_0^\tau e^{-\lambda t}\mathbf{C}(t)xdt - e^{-\lambda\tau}\mathbf{S}(\tau)x \tag{2.8}$$

and $L_\tau(\lambda)x \in D(G)$ with

$$(\lambda^2 - G)L_\tau(\lambda)x = Cx - e^{-\lambda\tau}\Big(\mathbf{C}(\tau)x + \lambda\mathbf{S}(\tau)x\Big) \quad \text{for } \lambda > 0 \text{ and } x \in X. \tag{2.9}$$

Proof. Equality (2.8) follows from (2.7) by using integration by parts . One more integration by parts gives

$$\lambda^2 L_\tau(\lambda)x = \int_0^\tau e^{-\lambda t}\mathbf{C}'(t)xdt - \lambda e^{-\lambda\tau}\mathbf{S}(\tau)x + Cx - e^{-\lambda\tau}\mathbf{C}(\tau)x, x \in D(G). \tag{2.10}$$

Hence from Proposition 2.8 (iv) it follows (2.9). ■

Setting $V_\tau(\lambda)x := -e^{-\lambda\tau}\Big(\mathbf{C}(\tau)x + \lambda\mathbf{S}(\tau)x\Big)$, we have for $\lambda > 0$

$$\frac{d^n}{d\lambda^n}V_\tau(\lambda)x = -(-\tau)^n e^{-\lambda\tau}\mathbf{C}(\tau)x - (\lambda(-\tau)^n + n(-\tau)^{n-1})e^{-\lambda\tau}\mathbf{S}(\tau)x,$$

and

$$\|\frac{d^n}{d\lambda^n}V_\tau(\lambda)x\| \le M_\tau\Big((n+1)\tau^n + \lambda\tau^{n+1}\Big)e^{-\lambda\tau}\|x\|,$$

where $M_\tau := \sup_{0\le t\le\tau}\|\mathbf{C}(t)\|$.

Proposition 2.10. *The following estimates hold for $\lambda > 0$ and $x \in X$:*

$$\|\frac{d^n}{d\lambda^n}L_\tau(\lambda)x\| \le M_\tau\frac{(n+1)!}{\lambda^{n+2}}\|x\|,$$

$$\|\frac{d^n}{d\lambda^n}(\lambda L_\tau(\lambda)x)\| \le M_\tau(\frac{n!}{\lambda^{n+1}} + \tau^{n+1}e^{-\lambda\tau})\|x\|.$$

Proof. In fact, we have from (2.7) and (2.8) for any $\lambda > 0$

$$\frac{d^n}{d\lambda^n}L_\tau(\lambda)x = \int_0^\tau(-t)^n e^{-\lambda t}\mathbf{S}(t)x dt,$$

$$\frac{d^n}{d\lambda^n}[\lambda L_\tau(\lambda)x] = \int_0^\tau(-t)^n e^{-\lambda t}\mathbf{C}(t)x dt - (-\tau)^n e^{-\lambda\tau}\mathbf{S}(\tau)x. \qquad ■$$

The proof of the next Proposition is exactly the same as in Okazawa [11].

Proposition 2.11. *$CD(G)$ is a core for G, i.e. $\overline{G|_{CD(G)}} = G$.*

Remark 2.12. An asymptotic C-resolvent $L_\tau(\lambda)$ is compact for some λ (and then for any λ large enough) iff $\mathbf{S}(\cdot)$ is compact. Indeed, if $\mathbf{S}(\cdot)$ is compact then by (2.7) it follows that $L_\tau(\lambda)$ is compact. Conversely, taking derivative of $L_\tau(\lambda)$ in τ and using the fact that $\mathbf{S}(\cdot)$ is uniformly continuous in t we have that $\mathbf{S}(\cdot)$ is compact as the uniform limit of compact operators.

3. Generation Theorem

Let us denote by $C^\infty((a,\infty);X)$ the set of functions which are infinitely many times differentiable.

Definition 3.1. Let A be a closed linear operator in X. Let $a \in R$ and $\tau \in (0,T)$. A family of operators $\{L_\tau(\lambda) : \lambda > a\} \subset B(X)$ is called **the asymptotic C-resolvent** of A if the following conditions are satisfied:

(i) $L_\tau(\cdot)x \in C^\infty((a,\infty);X)$ for any $x \in X$,

(ii) $L_\tau(\lambda)L_\tau(\mu) = L_\tau(\mu)L_\tau(\lambda)$,

(iii) $L_\tau(\lambda)x \in D(A)$ for any $x \in X$ and

$$(\lambda^2 - A)L_\tau(\lambda)x = Cx + V_\tau(\lambda)x, \tag{3.1}$$

where $V_\tau(\lambda) \in C^\infty((a,\infty);X)$, and

$$\|\frac{d^n}{d\lambda^n}V_\tau(\lambda)x\| \le M_\tau \lambda \tau^{n+1} e^{-\lambda\tau}\|x\|, \tag{3.2}$$

(iv) $AL_\tau(\lambda)x = L_\tau(\lambda)Ax$ for any $x \in D(A)$.

Theorem 3.2. **(Generation Theorem)** *Let A be a closed linear operator in X. Then A is the complete infinitesimal generator of a local C-cosine family $\{\mathbf{C}(t) : |t| < T\}$ on X iff*

(i) *$D(A)$ is dense in X,*

(ii) *for any $\tau \in (0,T)$ there is the asymptotic C-resolvent $L_\tau(\lambda)$ of operator A such that*

$$\|\frac{d^n}{d\lambda^n}[\lambda L_\tau(\lambda)x]\| \le M_\tau \frac{n!}{\lambda^{n+1}}\|x\|, \;\; x \in X, \tag{3.3}$$

with $0 \le n/\lambda \le \tau, \; \lambda > a, \; n \in N \cup \{0\}$,

(iii) *$CD(A)$ is a core for A.*

Before proving this theorem, let us introduce the following notions:

$$F^n_{\lambda,\tau}x = (d/d\lambda)^n L_\tau(\lambda)x, \quad G^n_{\lambda,\tau}x = (d/d\lambda)^n\Big[\lambda L_\tau(\lambda)x\Big].$$

Lamma 3.3. *Let A be a closed linear operator in X satisfying conditions (i) and (ii) in Theorem 3.2. Then the following hold:*

$$\lambda F^n_{\lambda,\tau} + nF^{n-1}_{\lambda,\tau} = G^n_{\lambda,\tau}, \tag{3.4}$$

$$\lambda G^n_{\lambda,\tau} + nG^{n-1}_{\lambda,\tau} = AF^n_{\lambda,\tau} + V^{(n)}_\tau(\lambda), \tag{3.5}$$

$$\|\frac{\lambda^{n+1}}{n!}F^{n-1}_{\lambda,\tau}\| \equiv \|\frac{\lambda^{n+1}}{n!}\frac{d^{n-1}}{d\lambda^{n-1}}L_\tau(\lambda)\| \le M_\tau, \tag{3.6}$$

with $0 \le n/\lambda \le \tau, \lambda > a, n \in N \cup \{0\}$.

Proof. Differentiating n times both sides of (3.1) in λ we have (3.5). We prove (3.6) by induction with respect to n. In the case of $n = 1$ estimate (3.6) is clear from (3.3). Assume that (3.6) is valid for $n \le k-2$. By (3.4) we have

$$\frac{\lambda^{k+1}}{k!}F^{k-1}_{\lambda,\tau}x = \frac{\lambda^k}{k!}G^{k-1}_{\lambda,\tau}x - \frac{k-1}{k}\frac{\lambda^k}{(k-1)!}F^{k-2}_{\lambda,\tau}x. \tag{3.7}$$

Hence we have for $n = k-1$

$$\|\frac{\lambda^{k+1}}{k!}F^{k-1}_{\lambda,\tau}\| \le \Big(\frac{1}{k}+\frac{k-1}{k}\Big)M_\tau = M_\tau.$$

■

Lemma 3.4. *Let A be as in Lemma 3.3. Then the following hold:*

$$Cx \in D(A) \text{ and } ACx = CAx \quad \text{for } x \in D(A); \tag{3.8}$$

$$L_\tau(\lambda)Cx = CL_\tau(\lambda)x \quad \text{for } x \in X \text{ and } \lambda > 0; \tag{3.9}$$

$$\lim_{\lambda\to\infty}\frac{(-1)^n}{n!}\lambda^{n+1}F^n_{\lambda,\tau}x = 0 \quad \text{for any } n \in N; \tag{3.10}$$

$$\lim_{\lambda\to\infty}\frac{(-1)^n}{n!}\lambda^{n+1}G^n_{\lambda,\tau}x = Cx \quad \text{for any } n \in N. \tag{3.11}$$

Proof. Let us prove first (3.8) and (3.9). By (3.1) and (3.2) we obtain that

$$\|\lambda^2L_\tau(\lambda)x - Cx\| \le \|L_\tau(\lambda)\|\|Ax\| + \|V_\tau(\lambda)x\| \le \frac{M_\tau}{\lambda^2}\|Ax\| + M_\tau\lambda\tau e^{-\lambda\tau}\|x\| \tag{3.12}$$

for $x \in D(A)$ and $\lambda > \max(a, \frac{1}{\tau})$. Hence $\lim_{\lambda\to\infty}\lambda^2L_\tau(\lambda)x = Cx$ for $x \in D(A)$. Since $\|\lambda^2L_\tau(\lambda)\| \le M_\tau$ for $\lambda > \max(a, \frac{1}{\tau})$ and $D(A)$ is dense in X, one gets

$$\lim_{\lambda\to\infty}\lambda^2L_\tau(\lambda)x = Cx \text{ for any } x \in X. \tag{3.13}$$

By definition we have $A\Big(\lambda^2L_\tau(\lambda)x\Big) = \lambda^2L_\tau(\lambda)Ax$ for $x \in D(A)$. Hence we see from (3.12) and the closedness of A that (3.8) holds. In the same way (3.9) follows from (ii) of Definition 3.1 and (3.13). We are going to prove (3.10) and (3.11) by induction. Since $D(A)$ is dense in X, we obtain (3.10) and (3.11) in the case of $n = 0$. Assume that (3.10) and (3.11) are valid for $n \le k-1$. By (3.5) we have

$$\begin{aligned} \tfrac{(-1)^k}{k!}\lambda^{k+1}G^k_{\lambda,\tau}x \ &= \tfrac{(-1)^{k-1}}{(k-1)!}\lambda^k G^{k-1}_{\lambda,\tau}x \\ &\quad + \tfrac{(-1)^k}{k!}\lambda^k AF^k_{\lambda,\tau}x + \tfrac{(-1)^k}{k!}\lambda^k V^{(k)}_\tau(\lambda)x. \end{aligned} \tag{3.14}$$

The first term on the right hand side is converges to Cx as $\lambda \to \infty$ by the induction hypotheses. From (3.6) for $x \in D(A)$ we have

$$\|\frac{(-1)^k\lambda^k}{k!}F^k_\lambda Ax\| \le M_\tau \frac{k+1}{\lambda^2}\|Ax\| \le M_\tau(\tau + 1/\lambda)\|Ax\|/\lambda.$$

From (3.2) one obtains

$$\|\frac{(-1)^k\lambda^k}{k!}V^{(k)}_\tau(\lambda)x\| \le M_\tau \frac{\lambda^{k+1}\tau^{k+1}}{k!}e^{-\lambda\tau}\|x\|.$$

Therefore the second and third terms in (3.14) converge to 0 as $\lambda \to \infty$. Since $D(A)$ is dense in X and (3.3) is valid, (3.11) is true for any $x \in X$.

Next, (3.4) yields

$$\frac{(-1)^k}{k!}\lambda^{k+1}F^k_{\lambda,\tau}x = \frac{(-1)^{k-1}}{(k-1)!}\lambda^k F^{k-1}_{\lambda,\tau}x + \frac{(-1)^k}{k!}\lambda^k G^k_{\lambda,\tau}x.$$

The first term on the right-hand side is convergent to 0 by the induction hypotheses and moreover, from (3.3) it follows that

$$\|\frac{(-1)^k}{k!}\lambda^k G^k_{\lambda,\tau}x\| \le M_\tau \frac{1}{\lambda}\|x\|.$$

Hence (3.10) is valid. ∎

Proof of Theorem 3.2. Noting that $n \le \lambda\tau$ the "only if" part follows from Propositions 2.9, 2.10 and 2.11. We are going to prove "if" part. The prove is divided into several steps. The general idea is the same as in Takenaka-Okazawa and Tanaka-Okazawa [13,16].

Step 1. Fix $\tau \in (0, T)$ arbitrarily. Then define $C_{n,\tau}(\cdot)$ by putting

$$C_{n,\tau}(t)x = \frac{(-1)^{n-1}}{(n-1)!}\lambda^n \frac{d^{n-1}}{d\lambda^{n-1}}\Big[\lambda L_\tau(\lambda)x\Big]\Big|_{\lambda=n/t},\ x \in X, \tag{3.15}$$

where $0 < t = n/\lambda \le \tau$. We will prove that A is the complete infinitesimal generator of the local C-cosine family $\mathbf{C}(\cdot)$ on X given by

$$\mathbf{C}(t)x = \lim_{n\to\infty} C_{n,\tau}(t)x \ \text{ for any } \ t \in [0, \tau).$$

Define also

$$S_{n,\tau}(t)x = \frac{(-1)^n}{n!}(n/t)^{n+1}F^n_{n/t,\tau}x \quad \text{for} \quad 0 < t \leq \tau, \tag{3.16}$$

$$W_{n,\tau}(t)x = \frac{(-1)^{n+1}}{(n+1)!}(n/t)^{n+2}G^{n+1}_{n/t,\tau} \quad \text{for} \quad 0 < t \leq \tau. \tag{3.17}$$

For $t = 0$, we set $C_{n,\tau}(0)x = W_{n,\tau}(0)x = Cx$ and $S_{n,\tau}(0)x = 0$. By virtue of Lemma 3.4, we get $\lim_{t\to 0+} C_{n,\tau}(t)x = C_{n,\tau}(0)x$, $\lim_{t\to 0+} S_{n,\tau}(t)x = S_{n,\tau}(0)x$ and $\lim_{t\to 0+} W_{n,\tau}(t)x = W_{n,\tau}(0)x$, respectively. Therefore $C_{n,\tau}(\cdot), S_{n,\tau}(\cdot)$ and $W_{n,\tau}(\cdot)$ are strongly continuous on $[0,\tau]$ and for $0 \leq t \leq \tau$ we have

$$\|C_{n,\tau}(t)\| \leq M_\tau, \tag{3.18}$$

$$\|S_{n,\tau}(t)\| \leq M_\tau \frac{n+1}{n} t, \tag{3.19}$$

$$\|W_{n,\tau}(t)\| \leq M_\tau. \tag{3.20}$$

Differentiating $C_{n,\tau}(\cdot)x$, and $S_{n,\tau}(\cdot)x$ in t we see from (3.5) and (3.4) that for any $x \in D(A)$

$$\begin{aligned} \frac{d}{dt}C_{n,\tau}(t)x &= \frac{(-1)^n}{n!}(n/t)^{n+1}\left[nG^{n-1}_{n/t,\tau}x + (n/t)G^n_{n/t,\tau}\right] \\ &= AS_{n,\tau}(t)x + P_{n,\tau}(t)x = S_{n,\tau}(t)Ax + P_{n,\tau}(t)x, \end{aligned} \tag{3.21}$$

where $P_{n,\tau}(t)x = \frac{(-1)^n}{n!}(n/t)^{n+1}V^{(n)}_\tau(n/t)x$ and

$$\frac{d}{dt}S_{n,\tau}(t)x = \frac{(-1)^{n+1}(n+1)}{n(n+1)!}(n/t)^{n+2}G^{n+1}_{n/t,\tau}x = \frac{n+1}{n}W_{n,\tau}(t)x. \tag{3.22}$$

Moreover, the following equality holds:

$$\begin{aligned} W_{n,\tau}(t)x - C_{n,\tau}(t)x &= \frac{(-1)^{n+1}}{(n+1)!}(n/t)^{n+1}\left[(n/t)G^{n+1}_{n/t,\tau}x + (n+1)G^n_{n/t,\tau}x\right] \\ &\quad + \frac{(-1)^n}{n!}(n/t)^n\left[(n/t)G^n_{n/t,\tau}x + nG^{n-1}_{n/t,\tau}x\right] \\ &= \frac{(-1)^{n+1}}{(n+1)!}(n/t)^{n+1}AF^{n+1}_{n/t,\tau}x + \frac{t}{n}AS_{n,\tau}(t)x + Q_{n,\tau}(t)x, \end{aligned}$$

where $Q_{n,\tau}(t)x = \frac{(-1)^{n+1}}{(n+1)!}(n/t)^{n+1}V^{(n+1)}_\tau(n/t)x + \frac{(-1)^n}{n!}(n/t)^nV^{(n)}_\tau(n/t)x$. We have

$$\|P_{n,\tau}(t)x\| \leq \frac{M_\tau}{\tau n!}(\frac{n\tau}{t})^{n+2}e^{-\frac{n\tau}{t}}\|x\|$$

and

$$\|Q_{n,\tau}(t)x\| \le M_\tau\Big[\frac{1}{(n+1)!}(\frac{n\tau}{t})^{n+2}e^{-\frac{n\tau}{t}} + \frac{1}{n!}(\frac{n\tau}{t})^{n+1}e^{-\frac{n\tau}{t}}\Big]\|x\|$$
$$\le 2M_\tau\frac{1}{(n+1)!}(\frac{n\tau}{t})^{n+2}e^{-\frac{n\tau}{t}}\|x\|,$$

for $n \ge 0$ and $t \in (0, \frac{n}{n+1}\tau]$. Therefore we have by (3.6) and (3.2)

$$\|W_{n,\tau}(t)x - C_{n,\tau}(t)x\| \le M_\tau\Big(\frac{2n+3}{n^2}t^2\Big)\|Ax\|$$
$$+2M_\tau\frac{1}{(n+1)!}(\frac{n\tau}{t})^{n+2}e^{-\frac{n\tau}{t}} \tag{3.23}$$
$$\le \frac{2t^2}{n}(1+\frac{3}{2n})M_\tau\|Ax\| + 2M_\tau\frac{1}{(n+1)!}(\frac{n\tau}{t})^{n+2}e^{-\frac{n\tau}{t}}\|x\|$$

for $0 < t \le \tau$.

Step 2. To prove that $C_{n,\tau}(t)x$ and $S_{n,\tau}(t)x$ converge as $n \to \infty$ we are going to show that $C_{n,\tau}(t)x$ is the Cauchy sequence. Let $\epsilon > 0$ and $x \in D(A)$. It follows from (3.21) and (3.22) that

$$C_{m,\tau}(\epsilon)C_{n,\tau}(t-\epsilon)x - C_{m,\tau}(t-\epsilon)C_{n,\tau}(\epsilon)x$$
$$+S_{m,\tau}(\epsilon)S_{n,\tau}(t-\epsilon)Ax - S_{m,\tau}(t-\epsilon)S_{n,\tau}(\epsilon)Ax$$
$$= \int_\epsilon^{t-\epsilon}\frac{d}{ds}[C_{m,\tau}(t-s)C_{n,\tau}(s)x]ds + \int_\epsilon^{t-\epsilon}\frac{d}{ds}[S_{m,\tau}(t-s)S_{n,\tau}(s)x]Axds$$
$$= \int_\epsilon^{t-\epsilon}\Big[C_{m,\tau}(t-s)AS_{n,\tau}(s)x - S_{m,\tau}(t-s)AC_{n,\tau}(s)x$$
$$+C_{m,\tau}(t-s)P_{n,\tau}(s)x - P_{m,\tau}(t-s)C_{n,\tau}(s)x\Big]ds$$
$$+\int_\epsilon^{t-\epsilon}\Big[\frac{n+1}{n}S_{m,\tau}(t-s)AW_{n,\tau}(s)x - \frac{m+1}{m}W_{m,\tau}(t-s)AS_{n,\tau}(s)x\Big]ds.$$

Letting $\epsilon \to 0$ we obtain that

$$C_{n,\tau}(t)Cx - C_{m,\tau}(t)Cx$$
$$= \int_0^t S_{m,\tau}(t-s)\Big[W_{n,\tau}(s) - C_{n,\tau}(s)\Big]Axds$$
$$+\int_0^t\Big[C_{m,\tau}(t-s) - W_{m,\tau}(t-s)\Big]S_{n,\tau}(s)Axds$$

$$+\int_0^t \Big[\frac{1}{n}S_{m,\tau}(t-s)W_{n,\tau}(s)-\frac{1}{m}W_{m,\tau}(t-s)S_{n,\tau}(s)\Big]Axds$$
$$+\int_0^t \Big[C_{m,\tau}(t-s)P_{n,\tau}(s)x-P_{m,\tau}(t-s)C_{n,\tau}(s)x\Big]ds.$$

We note that

$$\int_0^t (n\tau/s)^{n+2}e^{-n\tau/s}ds=\int_0^t (n\tau)^{n+1}/s^n\frac{d}{ds}(e^{-n\tau/s})ds$$
$$=(n\tau)^{n+1}/t^ne^{-n\tau/t}+n\int_0^t (n\tau/s)^{n+1}e^{-n\tau/s}ds\le\frac{2t^2}{\tau}(n\tau/t)^{n+2}e^{-n\tau/t}$$

because the estimate

$$(n\tau/s)^{n+1}e^{-n\tau/s}\le(n\tau/t)^{n+1}e^{-n\tau/t}$$

for $0<s\le t\le\frac{n}{n+1}\tau$. Let $0<\beta<\tau$. Then for $n,m>|a|\tau$ it follows

(3.24)

$$\sup_{0\le t\le\beta}\|C_{n,\tau}(t)Cx-C_{m,\tau}(t)Cx\|\le M_\tau^2\beta^4(\frac{1}{n}+\frac{1}{m})\|A^2x\|$$
$$+M_\tau^2\beta^2(\frac{1}{n}+\frac{1}{m})\|Ax\|$$
$$+8M_\tau^2\frac{\beta^3}{\tau}\Big(\frac{1}{(n+1)!}(\frac{n\tau}{\beta})^{n+2}e^{-n\tau/\beta}+\frac{1}{(m+1)!}(\frac{m\tau}{\beta})^{m+2}e^{-m\tau/\beta}\Big)\|Ax\|$$
$$+2M_\tau^2\frac{\beta^2}{\tau^2}\Big(\frac{1}{n!}(\frac{n\tau}{\beta})^{n+2}e^{-n\tau/\beta}+\frac{1}{m!}(\frac{m\tau}{\beta})^{m+2}e^{-m\tau/\beta}\Big)\|x\|$$

for $x\in D(A^2)$. We know that

$$a_m=\frac{1}{m!}(\frac{m\tau}{\beta})^{m+2}e^{-m\tau/\beta}\to 0 \tag{3.25}$$

as $m\to\infty$, because

$$\lim_{m\to\infty}\frac{a_{m+1}}{a_m}=\lim_{m\to\infty}\frac{\tau}{\beta}e^{-\tau/\beta}(1+1/m)^{m+2}=\frac{\tau}{\beta}e^{1-\tau/\beta}<1$$

if $\beta<\tau$. So from (3.24) it follows that for $x\in CD(A^2)$ and for $t\in[0,\tau)$, the function $C_{n,\tau}(t)x$ converges as $n\to\infty$ and the convergence is uniform on every bounded subinterval of $[0,\tau)$. Since $CD(A)$ is a core of A one has $CD(A^2)$ is dense in X and it follows from (3.18), (3.24) and the continuity of $C_{n,\tau}(\cdot)$ at $t=0$ that $\lim_{n\to\infty}C_{n,\tau}(t)x=C_\tau(t)x$ for $x\in X$ and $t\in[0,\tau)$. The convergence is uniform on every bounded subinterval of $[0,\tau)$. Therefore the function $C_\tau(\cdot)x$ is continuous on $[0,\tau)$ for any $x \in X$. Moreover, we see from (3.23) that $W_{n,\tau}(\cdot)x$ converges too. Since

$$S_{n,\tau}(t)x = \int_0^t \frac{n+1}{n} W_{n,\tau}(s)xds,$$

then $S_{n,\tau}(t)x$ also converges uniformly on every bounded subinterval of $[0,\tau)$.

Step 3. For $x \in X$ we define

$$C_\tau(t)x = \lim_{n\to\infty} C_{n,\tau}(t)x = \lim_{n\to\infty} W_{n,\tau}(t)x,$$

$$S_\tau(t)x = \lim_{n\to\infty} S_{n,\tau}(t)x = \lim_{n\to\infty} \int_0^t C_{n,\tau}(s)xds$$

for $t \in [0,\tau)$. We can extend $C_\tau(t)x$ and $S_\tau(t)x$ for $-\tau < t < 0$ as follows. For $t \geq 0$ we put $C_\tau(-t) = C_\tau(t)x$ and $S_\tau(-t) = -S_\tau(t)x$. We denote the extensions again by $C_\tau(\cdot)$ and $S_\tau(\cdot)$. Then $C_\tau(\cdot)x$ and $S_\tau(\cdot)x$ satisfy

$$\|C_\tau(t)x\| \leq M_\tau\|x\|, \quad \|S_\tau(t)x\| \leq M_\tau|t|\|x\| \quad \text{for any } |t| < \tau.$$

From (3.21) and (3.22) we have

$$C_{n,\tau}(t) = Cx + \int_0^t \Big[S_{n,\tau}(s)Ax + P_{n,\tau}(s)x\Big]ds$$

$$= Cx + \frac{n+1}{n}\int_0^t\int_0^u W_{n,\tau}(s)Axdsdu + \int_0^t P_{n,\tau}(s)xds$$

$$= Cx + \frac{n+1}{n}\int_0^t (t-s)W_{n,\tau}(s)Axds + \int_0^t P_{n,\tau}(s)xds.$$

Passing to the limit as $n \to \infty$, we have

$$C_\tau(t)x = Cx + \int_0^t (t-s)C_\tau(s)Axds. \tag{3.26}$$

Therefore we infer that $C_\tau(t)x$ is twice continuously differentiable in t for $x \in D(A)$. Noting that $AC_{n,\tau}(s)x = C_{n,\tau}(s)Ax$ for $x \in D(A)$ and A is closed, we see that

$$C_\tau(t)x \in D(A) \text{ and } AC_\tau(t)x = C_\tau(t)Ax \tag{3.27}$$

for $x \in D(A)$ and $|t| < \tau$. Moreover we obtain

$$\frac{d}{dt}C_\tau(t)x = S_\tau(t)Ax, \frac{d^2}{dt^2}C_\tau(t)x = AC_\tau(t)x = C_\tau(t)Ax \tag{3.28}$$

for $x \in D(A)$ and we have

$$S_\tau(t)x \in D(A) \text{ and } A\int_0^t S_\tau(s)xds = C_\tau(t)x - Cx \tag{3.29}$$

for $x \in X$. Hence we have

$$C_\tau(0)x = Cx \text{ for } x \in X \text{ and } C'_\tau(0)x = AS_\tau(0)x = 0 \tag{3.30}$$

for $x \in D(A)$.

Step 4. We are going to show that

$$[C_\tau(t+s) + C_\tau(t-s)]C = 2C_\tau(t)C_\tau(s)$$

for any $t, s, t \pm s \in (-\tau, \tau)$. Noting that $CC_\tau(t)x = C_\tau(t)Cx$, we set

$$v_\tau(t) = [C_\tau(t+s) + C_\tau(t-s)]Cx - 2C_\tau(t)C_\tau(s)x$$

for $x \in D(A)$. We see that $v_\tau(t) \in D(A)$ for $x \in D(A)$ and $v_\tau(0) = 0, v'_\tau(0) = C'_\tau(s)Cx + C'_\tau(-s)Cx = 0$ and $v''_\tau(t) = Av_\tau(t)$. Now, integrating by parts one gets from one side

$$\int_0^t AC_\tau(t-s)v_\tau(s)ds = \int_0^t C''_\tau(t-s)v_\tau(s)ds = \int_0^t C'_\tau(t-s)v'_\tau(s)ds$$

and from other side

$$\int_0^t C_\tau(t-s)Av_\tau(s)ds = \int_0^t C_\tau(t-s)v''_\tau(s)ds = Cv'(t) + \int_0^t C'_\tau(t-s)v'_\tau(s)ds.$$

Therefore, $Cv'_\tau(s) = 0$ for $|s| < \tau$ and $Cv_\tau(\cdot)$ must be constant. But $v_\tau(0) = 0$ and hence $Cv_\tau(t) = 0$ for any $|t| < \tau$. Since C is injective, $v_\tau(t) = 0$ for any $|t| < \tau$. From the density of $D(A)$ we have

$$C_\tau(t+s)Cx + C_\tau(t-s)Cx = 2C_\tau(t)C_\tau(s)x$$

for $x \in X$, and, moreover, $C_\tau(\cdot)$ is strongly continuous in $(-\tau, \tau)$.

Step 5. We shall define a local C-cosine family and prove that A is the complete infinitesimal generator of this local C-cosine family. We define $\mathbf{C}(t)x = C_\tau(t)x$ for $|t| < \tau, \tau \in (0, T)$ and $x \in X$. Then $\mathbf{C}(\cdot)$ is well-defined and it is clearly a local C-cosine family. Indeed, by (3.27) and (3.28) we have that

$$(\frac{d}{ds})\Big[C_{\tau_1}(t-s)C_{\tau_2}(s)x + S_{\tau_1}(t-s)S_{\tau_2}(s)Ax\Big]$$

$$= -S_{\tau_1}(t-s)AC_{\tau_2}(s)x + C_{\tau_1}(t-s)AS_{\tau_2}(s)x$$
$$-C_{\tau_1}(t-s)S_{\tau_2}(s)Ax + S_{\tau_1}(t-s)C_{\tau_2}(s)Ax = 0$$

for $x \in X$ and for any $t, s, t \pm s \in (-min(\tau_1, \tau_2), min(\tau_1, \tau_2))$. Therefore if $\tau_1 < \tau_2$, then $C_{\tau_1}(t)x = C_{\tau_2}(t)x$ for $|t| < \tau_1$ and $x \in X$. This implies that $\mathbf{C}(\cdot)$ is well-defined.

Now, let $\tau < T$ and G_0 be the infinitesimal generator of $\mathbf{C}(\cdot)$. We are going to prove that the operator A is the complete infinitesimal generator of $\mathbf{C}(\cdot)$, that is $G = A$. To this end, let $x \in CD(A)$ and $x = Cy$ for some $y \in D(A)$. Then by (3.26) and (3.27)

$$\frac{2}{h^2}(C^{-1}\mathbf{C}(h) - I)x - Ax = \frac{2}{h^2}(\mathbf{C}(h) - C)y - Ax$$

$$= \frac{2}{h^2}\int_0^h \int_0^s \mathbf{C}(u)Ay\,duds - Ax = \frac{2}{h^2}\int_0^h \int_0^s (\mathbf{C}(u)Ay - Ax)\,duds.$$

Therefore we have

$$\|\frac{2}{h^2}(C^{-1}\mathbf{C}(h) - I)x - Ax\| \leq \sup_{|u|\leq h} \|\mathbf{C}(u)Ay - Ax\|.$$

This implies as $h \to 0$ that $G_0x = CAy = Ax$ from (3.8). Thus $x \in D(G_0)$ and $G_0x = Ax$, which means that $A|_{CD(A)} = G_0|_{CD(A)} \subseteq G$. Therefore $A \subseteq G$ because of Theorem 3.2 (iii). Conversely, to prove that $G \subseteq A$, let $x \in D(G)$. Then there exists a sequence $\{x_n\}$ in $D(G)$ such that $x_n \to x$ and $G_0x_n \to Gx$ as $n \to \infty$. Since $x_n \in D(G_0) \subseteq R(C)$, we have by (3.9) that $L_\tau(\lambda)x_n \in R(C)$ and

$$\frac{2}{h^2}\Big(C^{-1}\mathbf{C}(h)L_\tau(\lambda)x_n - L_\tau(\lambda)x_n\Big) = L_\tau(\lambda)\frac{2}{h^2}(C^{-1}\mathbf{C}(h) - I)x_n \to L_\tau(\lambda)G_0x_n$$

as $h \to 0$. Thus we see that $L_\tau(\lambda)x_n \in D(G_0)$ and $G_0L_\tau(\lambda)x_n = L_\tau(\lambda)G_0x_n$. Letting $n \to \infty$ in the above equality yields that $L_\tau(\lambda)x \in D(G)$ and

$$GL_\tau(\lambda)x = L_\tau(\lambda)Gx \quad \text{for } x \in D(G). \tag{3.31}$$

But since $L_\tau(\lambda)x \in D(A)$ for $x \in X$ (by (3.1)) and $A \subseteq G$ it follows that $AL_\tau(\lambda)x = L_\tau(\lambda)Gx$ for $x \in D(G)$. Combining this with (2.10) we have that

$$\lambda^2 L_\tau(\lambda)Gx = G(\lambda^2 L_\tau(\lambda)x) = A(\lambda^2 L_\tau(\lambda)x)$$

for $x \in D(G)$. Since A is closed, (3.13) implies that $Cx \in D(A)$ and $ACx = CGx = GCx$ for $x \in D(G)$. This means that $G|_{CD(G)} = A|_{CD(G)} \subseteq A$. Since $CD(G)$ is a core for G, we obtain $G \subseteq A$. ■

4. Local Integrated Cosine Families

In this section we introduce integrated cosine family and clarify the relationship of local integrated cosine families with local C-cosine families and the abstract Cauchy problem ACP.

Definition 4.1. Let $n \geq 1$ and $T \in (0, \infty]$. A family of operators $\{U(t) : |t| < T\}$ in $B(X)$ is called a local n**-times integrated (nondegenerate) cosine family** on X if

$$U(\cdot)x : (-T, T) \to X \text{ is continuous for any } x \in X; \tag{4.1}$$

$$\begin{aligned} 2U(t)U(s)x = \frac{1}{(n-1)!}\Big[\int_t^{t+s}(t+s-r)^{n-1}U(r)xdr \\ -\int_0^s(t+s-r)^{n-1}U(r)xdr\Big] + \frac{(-1)^n}{(n-1)!}\Big[\int_t^{t-s}(t-s-r)^{n-1}U(r)xdr \\ -\int_0^{-s}(t-s-r)^{n-1}U(r)xdr\Big] \end{aligned} \tag{4.2}$$

for $|t|, |s|, |t+s|, |t-s| < T$ and $U(0) = 0$;

$$U(t)x = 0 \text{ for } t \in (-T, T) \text{ implies } x = 0. \tag{4.3}$$

The Definition 4.1 is slightly different from the definitions introduced by Tingwen [17] and Li-Shaw [8,12]. By (4.2) we have $U(t)U(-s)x = (-1)^n U(t)U(s)x$. Moreover, we obtain

$$\begin{aligned} &\int_s^{s-t}(s-t-r)^{n-1}U(r)xdr - \int_0^{-t}(s-t-r)^{n-1}U(r)dr \\ &= (-1)^n\int_{-s}^{t-s}(t-s-r)^{n-1}U(-r)xdr - (-1)^n\int_0^t(t-s-r)^{n-1}U(-r)xdr \\ &= \int_t^{t-s}(t-s-r)^{n-1}U(r)xdr - \int_0^{-s}(t-s-r)^{n-1}U(r)xdr. \end{aligned}$$

This means that $U(t)U(s)x = U(s)U(t)x$. The local C-sine family satisfies (4.2) with $n = 1$.

Let us denote $C^n(\delta) = \{x \in X : U(\cdot)x : (-\delta, \delta) \to X$ is n-times continuously differentiable$\}$.

Lemma 4.2. *Let $U(\cdot)$ be a local n-times integrated cosine family, $|t| < T$*

and $x \in C^k(\delta)$ $(1 \le k \le n-1$ *and* $\delta > 0)$. *Then the following hold:*

$$\begin{aligned}
&2U(t)U^{(k)}(s)x \\
&\quad = \frac{1}{(n-(k+1))!}\Big[\int_t^{t+s}(t+s-r)^{n-k-1}U(r)xdr \\
&\qquad -\int_0^{s}(t+s-r)^{n-k-1}U(r)xdr\Big] \\
&\qquad +\frac{(-1)^{n-k}}{(n-(k+1))!}\Big[\int_t^{t-s}(t-s-r)^{n-k-1}U(r)xdr \\
&\qquad -\int_0^{-s}(t-s-r)^{n-k-1}U(r)xdr\Big] \\
&\qquad -\sum_{i=1}^{k}\frac{t^{n-i}}{(n-i)!}\Big(U^{(k-i)}(s)x+(-1)^{n-k}U^{(k-i)}(-s)x\Big)
\end{aligned} \tag{4.4}$$

for $|s| < min(\delta, T-|t|)$;

$$U(t)U^{(k)}(0)x = 0 \text{ and hence } U^{(k)}(0)x = 0. \tag{4.5}$$

Proof. Let $|t| \in (0,T)$ and $x \in C^1(\delta)$. Then taking derivative of (4.2) in s one gets (4.4) with $k=1$. In the same way one can prove that (4.5) holds for $k=1$. Therefore the conclusion can be proved by induction with respect to k. ■

Lemma 4.3. *Let* $U(\cdot)$ *be a local* n*-times integrated cosine family,* $|t| < T$ *and* $x \in C^n(\delta)$ $(\delta > 0)$. *Then the following hold:*

$$\begin{aligned}
2U(t)U^{(n)}(s)x &= U(t+s)x + U(t-s)x \\
&\quad -\sum_{k=1}^{n}\frac{t^{n-k}}{(n-k)!}\Big(U^{(n-k)}(s)+U^{(n-k)}(-s)\Big)x
\end{aligned} \tag{4.6}$$

for $|s| < min(\delta, T-|t|)$;

$$U(t)U^{(n)}(0)x = U(t)x \text{ and hence } U^{(n)}(0)x = x.$$

Proof. For $k = n-1$ we have $n-k-1=0$, so the conclusion follows from (4.4).

Lemma 4.4. *Let* $|t| < T$ *and* $x \in C^k(\delta)$ $(1 \le k \le n$ *and* $\delta > 0)$. *Then* $U(t)x \in C^k(min(\delta, T-|t|))$ *and* $U(t)U^{(k)}(s)x = U^{(k)}(s)U(t)x$ *for* $|s| < min(\delta, T-|t|)$.

The proof of the above lemma is similar to the proof of Lemma 4.4 of Tanaka-Okazawa [16].

Lemma 4.5. (Travis-Webb [18].) *Let $u(\cdot) : R \to X$ be continuous function. Then $u(\cdot)$ is twice continuously differentiable iff*

$$\lim_{h\to 0} \frac{1}{h^2}\Big(u(t+h) - 2u(t) + u(t-h)\Big)$$

exists uniformly on any compact subsets of R. If the limit exists uniformly on compact subsets of R, then it equals to $d^2u(t)/dt^2$.

Definition 4.6. The infinitesimal generator A_0 of a local n-times integrated cosine family $U(\cdot)$ is defined as the limit:

$$A_0 x = \lim_{h\to 0} 2h^{-2}(U^{(n)}(h)x - x) \quad \text{for } x \in D(A_0) \tag{4.7}$$

with domain

$$D(A_0) = \{x \in \bigcup_{0<\delta<T} C^n(\delta) : \lim_{h\to 0} 2h^{-2}(U^{(n)}x - x) \text{ exists } \}.$$

Proposition 4.7. *Let A_0 be the infinitesimal generator of local n-times integrated cosine family $U(\cdot)$ on X. Then the following hold:*

$$U(t)x \in D(A_0) \text{ and } A_0U(t)x = U(t)A_0x \quad \text{for } |t| < T \text{ and } x \in D(A_0); \tag{4.8}$$

$$\frac{d^2}{dt^2}U(t)x = U(t)A_0x + \frac{t^{n-2}}{(n-2)!}x \quad \text{for } |t| < T \text{ and } x \in D(A_0); \tag{4.9}$$

the operator A_0 is closable;

$$\text{if } C^n(T) \text{ is dense in } X, \text{ then } D(A_0) \text{ is dense in } X. \tag{4.10}$$

Proof. By the definition of A_0 there exists $\delta \in (0, T)$ such that $x \in C^n(\delta)$ and

$$A_0 x = \lim_{h\to 0} 2h^{-2}(U^{(n)}(h)x - x).$$

Choose h such that $|h| < min(\delta, T - |t|)$. Then we see by Lemma 4.4 that $U(t)x \in \cup_{0<\delta<T} C^n(\delta)$ and $U^{(n)}(h)U(t)x = U(t)U^{(n)}(h)x$ and hence for $h \to 0$

$$\frac{2}{h^2}\Big(U^{(n)}(h)U(t) - U(t)\Big)x = U(t)\frac{2}{h^2}(U^{(n)}(h)x - x) \to U(t)A_0x.$$

This means that (4.8) holds. Next we will show (4.9). We note that $U(s)x = (-1)^n U(-s)x$ and hence $U^{(n-k)}(s)x = (-1)^k U^{(n-k)}(-s)x$. Therefore, if $k = 2m$, then

$$U^{(n-k)}(s)x + U^{(n-k)}(-s)x = 2U^{(n-k)}(s)x,$$

and if $k = 2m-1$, then

$$U^{(n-k)}(s)x + U^{(n-k)}(-s)x = 0.$$

Moreover, we have by (4.6)

$$\begin{aligned} &\frac{1}{h^2}\Big(U(t+h) - 2U(t) + U(t-h)\Big)x \\ &\quad = \frac{2}{h^2}\Big(U(t)U^{(n)}(h) - U(t)\Big)x \\ &\qquad - \sum_{k=1}^{m} \frac{t^{n-2k}}{(n-2k)!}\frac{2}{h^2}\Big(U^{(n-2k)}(h)x - U^{(n-2k)}(0)x\Big), \end{aligned} \tag{4.11}$$

where we use $U^{(n-2k)}(0)x = 0$ for $k < m$ and $[n/2] = m$. Noting that by Lemma 4.5 with $t = 0$

$$\frac{2}{h^2}\Big(U^{(n-2k)}(h)x - U^{(n-2k)}(0)x\Big) \to U^{(n-2k+2)}(0)x$$

for $1 \le k \le m$ and $x \in C^n(\delta)$. So as $h \to 0$ we have the right hand side of (4.11) tends to

$$U(t)A_0x + \frac{t^{n-2}}{(n-2)!}U^{(n)}(0)x.$$

Hence (4.9) holds. To prove the next statement we twice integrate (4.9)

$$U(t)x = \frac{t^n}{n!}x + \int_0^t (t-s)U(s)A_0x\,ds \tag{4.12}$$

for $x \in D(A_0)$ and $|t| < T$. Let $\{x_k\}$ be a sequence in $D(A_0)$ such that $x_k \to 0$ and $A_0x_k \to y$ as $k \to \infty$. Putting $k \to \infty$ in (4.12) with x replaced by x_k, we see that $\int_0^t (t-s)U(s)y\,ds = 0$ for $|t| < T$ and hence $U(t)y = 0$ for $|t| < T$. By (4.3) we have $y = 0$. Finally, we will show that (4.10) holds. Let $x \in C^n(T)$. Then by (4.5)

$$\int_0^t (t-s)U^{(n)}(s)x\,ds = U^{(n-2)}(t)x - U^{(n-2)}(0)x = U^{(n-2)}(t)x.$$

Using (4.4) with $k = n-2$ we have that for $|s|, |t| < T/2$

$$\begin{aligned} 2U(s)U^{(n-2)}(t)x &= \int_s^{s+t} (t+s-r)U(r)x\,dr - \int_0^t (t+s-r)U(r)x\,dr \\ &+ \int_s^{s-t} (s-t-r)U(r)x\,dr - \int_0^{-t} (s-t-r)U(r)x\,dr - 2\sum_{i=1}^{m-1} \frac{s^{n-2i}}{(n-2i)!}U^{(n-2i-2)}(t)x, \end{aligned}$$

where $[n/2] = m$. Therefore, $\int_0^t (t-s)U^{(n)}(s)xds \in C^n(T/2)$ and
(4.13)

$$U^{(n)}(h)\int_0^t (t-s)U^{(n)}(s)xds = U^{(n-2)}(h+t)x - 2U^{(n-2)}(h)x + U^{(n-2)}(h-t)x$$

for $x \in C^n(T)$ and $|t|, |h| < T/2$. It follows that

$$\frac{2}{h^2}\Big(U^{(n)}(h)\int_0^t (t-s)U^{(n)}(s)xds - \int_0^t (t-s)U^{(n)}(s)xds\Big)$$
$$= \frac{1}{h^2}\Big(U^{(n-2)}(h+t)x - 2U^{(n-2)}(t)x + U^{(n-2)}(t-h)x\Big)$$
$$-\frac{1}{h^2}\Big(U^{(n-2)}(h)x + U^{(n-2)}(-h)x - 2U^{(n-2)}(0)x\Big) \to U^{(n)}(t)x - x$$

as $h \to 0$ because $U^{(n-2)}(t)x = U^{(n-2)}(-t)x$. Thus $\int_0^t (t-s)U^{(n)}(s)xds \in D(A_0)$ for $x \in C^n(T)$ and $|t| < T/2$. Since $x = \lim_{t\to 0} \frac{2}{t^2}\int_0^t (t-s)U^{(n)}(s)xds$, it follows that $C^n(T) \subseteq \overline{D(A_0)}$, which prove (4.10). ∎

5. The Abstract Cauchy Problem

Let A be a linear operator in a Banach space X, C as before, $T \in (0,\infty]$ and $x, y \in X$. We consider the abstract Cauchy problem on $(-T, T)$ in the form (ACP; T,x, y) :

$$(\frac{d^2}{dt^2})u(t) = Au(t),\ |t| < T, u(0) = x, u'(0) = y.$$

A function $u(\cdot)$ is called a **solution** to (ACP; T,x, y) if

(a) $u(\cdot)$ is twice continuously differentiable in $t \in (-T, T)$, for any $|t| < T$,

(b) $u(t) \in D(A)$ for any $|t| < T$ and $u(\cdot)$ satisfies (ACP; T,x, y).

We denote also (ACP; T,x, y) with $x, y \in CD(A)$ as (ACP;T,CD(A)).

Definition 5.1. The Cauchy problem $(ACP; T, CD(A))$ is said to be **well-posed** if for every $x, y \in CD(A)$ there is a unique solution $u(t; x, y)$ to $(ACP; T, x, y)$ such that $\|u(t; x, y)\| \le M(t)(\|C^{-1}x\| + \|C^{-1}y\|)$ for $|t| < T$ and $x, y \in CD(A)$, where the function $M(t)$ is bounded on every compact subinterval of $(-T, T)$ and independent of x and y.

Theorem 5.2. (Huang-Huang [7].)*Let A be a densely defined closed linear operator in X satisfying*

(*i*) $Cx \in D(A)$ *and* $ACx = CAx$ *for* $x \in D(A)$,

(*ii*) $CD(A)$ *is a core for* A.

Then the following are equivalent :

(I) *The operator A is the complete infinitesimal generator of a local C-cosine family* $\mathbf{C}(\cdot)$;

(II) $(ACP; T, CD(A))$ *is well-posed.*

In this case $u(t; x, y) = C^{-1}\mathbf{C}(t)x + C^{-1}\mathbf{S}(t)y$, $t \in (-T, T)$, *is a unique solution for every initial values* $x, y \in CD(A)$.

In this section we investigate some properties of the complete infinitesimal generator of an n-times integrated local cosine family $U(\cdot)$ and consider the abstract Cauchy problem ACP. Let us denote the resolvent set of operator A as $\rho(A)$.

Definition 5.3. The operator $A = \overline{A_0}$ is said to be the **complete infinitesimal generator of a local** n**-times integrated cosine family** $U(\cdot)$.

Proposition 5.4. *Let A be the complete infinitesimal generator of a local n-times integrated cosine family $\{U(t) : |t| < T\}$ on X. Then the following hold:*

$$U(t)x \in D(A) \text{ and } U(t)Ax = AU(t)x \text{ for } |t| < T \text{ and } x \in D(A); \tag{5.1}$$

$$U(t)x = \frac{t^n}{n!}x + \int_0^t (t-s)U(s)Axds \text{ for } x \in D(A) \text{ and } |t| < T; \tag{5.2}$$

If $D(A)$ is dense in X, then $\int_0^t (t-s)U(s)xds \in D(A)$ *and*

$$A\int_0^t (t-s)U(s)xds = U(t)x - \frac{t^n}{n!}x \tag{5.3}$$

for $x \in X$ *and* $|t| < T$;

$$If\ \overline{D(A)} = X, \text{ then there is a real number } \omega \text{ such that } (\omega, \infty) \subset \rho(A); \tag{5.4}$$

$$D(A) \text{ is dense in } X \text{ iff } C^n(T) \text{ is dense in } X. \tag{5.5}$$

Proof. Proposition 4.7 deduces (5.1),(5.2) and (5.3). The assertion (5.4) can be proved in the same way as in Tingwen [17, Proposition 2.2]. We show the "only if" part of (5.5). Since $D(A)$ is dense in X by (5.4) one gets that $\rho(A) \neq \emptyset$. If $\lambda \in \rho(A)$, then $D(A) = R((\lambda - A)^{-1})$ is dense in X, so $D(A^m) = R((\lambda - A)^{-m})$ is dense in X for $m = 1, 2, \cdots$. This implies that $D(A^{[\frac{n+1}{2}]})$ is dense in X. Using this fact and (5.2) we see that $C^n(T)(\supset D(A^{[\frac{n+1}{2}]}))$ is dense in X. The "if" part of (5.5) follows from (4.10). ∎

Proposition 5.5 *If A is complete infinitesimal generator of an n-times integrated local cosine family $U(\cdot)$ and $[(n+1)/2] = m$, then $(ACP; T, D(A^{m+1}))$ is well-posed in the following sense: for every $x, y \in D(A^{m+1})$ there is a unique solution $u(t; x, y)$ of ACP such that*

$$\|u(t; x, y)\| \leq M(t)(\|x\|_m + \|y\|_m)$$

for $|t| < T$ and $x, y \in D(A^{m+1})$, where $M(t)$ is bounded function on every compact subinterval of $(-T, T)$ and $\|z\|_m = \sum_{i=0}^{m} \|A^i z\|$ for $z \in D(A^m)$.

Proof. Let A be the complete infinitesimal generator of an n-times integrated local cosine family $U(\cdot)$. Let $k \in N$ with $1 \leq k \leq [n/2]$. Then we have

$$(5.6)\quad U^{(2k)}(t)x = U(t)A^k x + \sum_{i=1}^{k} \frac{t^{n-2i}}{(n-2i)!} A^{k-i} x \text{ for } |t| < T \text{ and } x \in D(A^k).$$

Indeed, it is easily get (5.6) for $k = 1$. Assume that (5.6) holds for $k = l$. Then by (5.2) and (5.6)

$$\begin{aligned} U^{(2l+2)}(t)x &= (\frac{d^2}{dt^2})\Big(\frac{t^n}{n!}A^l x + \int_0^t (t-s)U(s)A^{l+1}x ds + \sum_{i=1}^{l} \frac{t^{n-2i}}{(n-2i)!} A^{l-i} x\Big) \\ &= \frac{t^{n-2}}{(n-2)!}A^l x + \sum_{i=1}^{l} \frac{t^{n-2i-2}}{(n-2i-2)!} A^{l-i} x + U(t)A^{l+1}x \\ &= \sum_{i=1}^{l+1} \frac{t^{n-2i}}{(n-2i)!} A^{l+1-i} x + U(t)A^{l+1}x. \end{aligned}$$

So (5.6) holds. Moreover, by (5.2) the same way

$$U^{(2k-1)}(t)x = \sum_{i=1}^{k} \frac{t^{n-2i+1}}{(n-2i+1)!} A^{k-i} x + \int_0^t U(s)A^k x ds$$

for $1 \leq k \leq [n/2]$. Now, let $x, y \in D(A^{n+1})$. Then if $n = 2m$

$$U^{(n)}(t)x = \sum_{k=0}^{m-1} \frac{t^{2k}}{(2k)!} A^k x + U(t)A^m x,$$

$$U^{(n-1)}(t)y = \sum_{k=0}^{m-1} \frac{t^{2k+1}}{(2k+1)!} A^k y + \int_0^t U(t)A^m y ds,$$

and if $n = 2m-1$, then (5.6) and (5.7) give

$$U^{(2m-2)}(t)y = \sum_{k=0}^{m-2} \frac{t^{2k+1}}{(2k+1)!}A^k y + U(t)A^{m-1}y,$$
$$U^{(2m-1)}(t)x = \sum_{k=0}^{m-1} \frac{t^{2k}}{(2k)!}A^k x + \int_0^t U(s)A^m x ds.$$

Noting that

$$(5.8) \qquad U'(t)x = \frac{t^{n-1}}{(n-1)!}x + \int_0^t U(s)Axds, \; U''(t)x = \frac{t^{n-2}}{(n-2)!}x + U(t)Ax$$

for $x, y \in D(A^{m+1})$ with $n = 2m$ one obtains

$$\frac{d^2}{dt^2}U^{(2m)}(t)x = \sum_{k=0}^{m-1} \frac{t^{2k}}{(2k)!}A^{k+1}x + U(t)A^{m+1}x,$$
$$\frac{d^2}{dt^2}U^{(2m-1)}(t)y = \frac{d}{dt}U^{(2m)}(t)y = \sum_{k=0}^{m-1} \frac{t^{2k+1}}{(2k+1)!}A^{k+1}y + \int_0^t U(s)A^{m+1}yds.$$

In the case of $n = 2m-1$, we also have

$$\frac{d^2}{dt^2}U^{(2m-1)}(t)x = \frac{d}{dt}U^{(2m)}(t)x = \sum_{k=0}^{m-1} \frac{t^{2k+1}}{(2k+1)!}A^{k+1}x + \int_0^t U(s)A^{m+1}xds,$$
$$\frac{d^2}{dt^2}U^{(2m-2)}(t)y = \frac{d}{dt}U^{(2m-1)}(t)y = \sum_{k=0}^{m-1} \frac{t^{2k}}{(2k)!}A^{k+1}y + U(t)A^m y.$$

These imply that $u(t;x,y)$ is a solution of (ACP;T,x, y), where we denoted $u(t;x,y) = U^{(n)}(t)x + U^{(n-1)}(t)y$. To check the uniqueness of solution to (ACP; T,x, y), let $v(t;x,y)$ be any solution of (ACP; T,x, y). We set $W(t)x = \int_0^t U(s)xds$ for $x, y \in D(A^{m+1})$ and $|t| < T$. Then by (5.2) we have

$$\frac{d}{ds}\Big[U(t-s)v(s;x,y) + W(t-s)v'(s;x,y)\Big]$$

$$= -U'(t-s)v(s;x,y) + U(t-s)v'(s;x,y) - U(t-s)v'(s;x,y) + W(t-s)v''(s;x,y),$$

and using (5.8) one obtains

$$\frac{d}{ds}\Big[U(t-s)v(s;x,y) + W(t-s)v'(s;x,y)\Big] = -\frac{(t-s)^{n-1}}{(n-1)!}v(s;x,y)$$

for $|t|, |s|, |t-s|, |t+s| < T$. Integrating this from 0 to t yields that

$$U(t)x + W(t)y = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} v(s;x,y)ds.$$

Now, taking n derivatives we get $u(t;x,y) = v(t;x,y)$ for $|t| < T$. ∎

Remark 5.6. If $n = 2m-1$, then for $x \in D(A^{m+1})$ and $y \in D(A^m)$ we have a unique solution to ACP. The conclusion follows from the proof of Proposition 5.5.

Now we clarify the relationship between local C-cosine family and an integrated local cosine family. We set $j_n(t) := \frac{t^n}{n!},\ t \geq 0$.

Lemma 5.7. *Let $\mathbf{C}(\cdot)$ be a local C-cosine family on X and let $n \in N$. Define*

$$(j_n * \mathbf{C})(t)x = \int_0^t \int_0^{t_1} ... \int_0^{t_{n-1}} \mathbf{C}(t_n)xdt_{n-1}...dt_1$$

for $x \in X$ and $0 \leq |t| < T$. Then

$$\begin{aligned} 2(j_n * \mathbf{C})(t)(j_n * \mathbf{C})(s) &= \frac{1}{(n-1)!}\Big(\int_t^{s+t} (s+t-r)^{n-1}(j_n * \mathbf{C})(r)Cxdr \\ &- \int_0^s (s+t-r)^{n-1}(j_n * \mathbf{C})(r)Cxdr\Big) \\ &+ \frac{(-1)^n}{(n-1)!}\Big[\int_t^{t-s} (t-s-r)^{n-1}(j_n * \mathbf{C})(r)Cxdr \\ &- \int_0^{-s} (t-s-r)^{n-1}(j_n * \mathbf{C})(r)Cxdr\Big] \end{aligned} \tag{5.9}$$

for $|t|, |s|, |t-s|, |t+s| < T$.

Proof. To prove (5.9) it suffices to show that

$$\begin{aligned} 2(j_n * \mathbf{C})(t)(j_n * \mathbf{C})(s) &= (j_{2n} * \mathbf{C})(t+s)Cx + (-1)^n (j_{2n} * \mathbf{C})(t-s)Cx \\ &- \sum_{i=0}^{n-1} \Big[\frac{s^i + (-1)^n(-s)^i}{i!} (j_{2n-i} * \mathbf{C})(t)Cx \\ &+ \frac{t^i}{i!}\Big((j_{2n-i} * \mathbf{C})(s)Cx + (-1)^n (j_{2n-i} * \mathbf{C})(-s)Cx\Big)\Big] \end{aligned} \tag{5.10}$$

for $|t|, |s|, |t-s|, |t+s| < T$ and $x \in X$. Indeed, in the same way as in Tanaka-Okazawa [16, Lemma 4.8] we have

$$\frac{(-1)^n}{(n-1)!}\Big[\int_0^{t-s} (t-s-r)^{n-1}(j_n * \mathbf{C})(r)Cxdr - \int_0^t (t-s-r)^{n-1}(j_n * \mathbf{C})(r)Cxdr$$

$$-\int_0^{-s}(t-s-r)^{n-1}(j_n*\mathbf{C})(r)Cxdr\Big]$$

$$=\frac{(-1)^n}{(n-2)!}\Big[\int_0^{t-s}(t-s-r)^{n-2}(j_{n+1}*\mathbf{C})(r)Cxdr-\int_0^{t}(t-s-r)^{n-2}(j_{n+1}*\mathbf{C})(r)Cxdr$$

$$-\int_0^{-s}(t-s-r)^{n-2}(j_{n+1}*\mathbf{C})(r)Cxdr\Big]-(-1)^n\frac{(-s)^{n-1}}{(n-1)!}(j_{n+1}*\mathbf{C})(t)Cx$$

$$-(-1)^n\frac{t^{n-1}}{(n-1)!}(j_{n+1}*\mathbf{C})(-s)Cx,$$

and also

$$\frac{1}{(n-1)!}\Big(\int_0^{t+s}(s+t-r)^{n-1}(j_n*\mathbf{C})(r)Cxdr-\int_0^{t}(s+t-r)^{n-1}(j_n*\mathbf{C})(r)Cxdr$$

$$-\int_0^{s}(s+t-r)^{n-1}(j_n*\mathbf{C})(r)Cxdr\Big)$$

$$=\frac{1}{(n-1)!}\Big[\int_0^{t+s}(t+s-r)^{n-2}(j_{n+1}*\mathbf{C})(r)Cxdr$$

$$-\int_0^{t}(t+s-r)^{n-2}(j_{n+1}*\mathbf{C})(r)Cxdr$$

$$-\int_0^{-s}(t+s-r)^{n-2}(j_{n+1}*\mathbf{C})(r)Cxdr\Big]-(-1)^n\frac{(-s)^{n-1}}{(n-1)!}(j_{n+1}*\mathbf{C})(t)Cx$$

$$-(-1)^n\frac{t^{n-1}}{(n-1)!}(j_{n+1}*\mathbf{C})(-s)Cx.$$

Thus repeating the above equalities one more time, we get the right-hand side of (5.10).

Now we will prove (5.10) by induction with respect to n. In the case of $n=1$, changing variables a few times, we have

$$2(j_1*\mathbf{C})(t)(j_1*\mathbf{C})(s)x=\int_0^t\int_0^s\Big(\mathbf{C}(\xi+\eta)Cx+\mathbf{C}(\xi-\eta)Cx\Big)d\eta d\xi$$

$$=\int_0^t\Big((j_1*\mathbf{C})(\xi+s)Cx-(j_1*\mathbf{C})(\xi-s)Cx\Big)d\xi$$

$$=(j_2*\mathbf{C})(t+s)Cx-(j_2*\mathbf{C})(t-s)Cx-(j_2*\mathbf{C})(s)Cx+(j_2*\mathbf{C})(-s)Cx.$$

This means that (5.10) holds for $n=1$. Assume that (5.10) holds for $n=k$. Then by the induction hypothesis and by changing of variables we obtain that

$$
\begin{aligned}
&2(j_{k+1}*\mathbf{C})(t)(j_{k+1}*\mathbf{C})(s)x = \int_0^t\int_0^s (j_k*\mathbf{C})(\xi)(j_k*\mathbf{C})(\eta)d\eta d\xi \\
&= \int_0^t \Big((j_{2k+1}*\mathbf{C})(\xi+s)Cx - (j_{2k+1}*\mathbf{C})(\xi)Cx - (-1)^{k+1}(j_{2k+1}*\mathbf{C})(\xi-s)Cx \\
&-(-1)^{k+1}(j_{2k+1}*\mathbf{C})(\xi)Cx\Big)d\xi - \sum_{i=0}^{k-1}\left[\frac{s^{i+1}+(-1)^{k+1}(-s)^{i+1}}{(i+1)!}(j_{2k+1-i}*\mathbf{C})(t)\right. \\
&\left.+\frac{t^{i+1}}{(i+1)!}\Big((j_{2k+1-i}*\mathbf{C})(s)+(-1)^{k+1}(j_{2k+1-i}*\mathbf{C})(-s)\Big)\right]Cx.
\end{aligned}
$$

Changing variables yields that

$$
\begin{aligned}
&\int_0^t \Big((j_{2k+1-i}*\mathbf{C})(\xi+s)Cx + (-1)^{k+1}(j_{2k+1-i}*\mathbf{C})(\xi-s)Cx \\
&\qquad -(j_{2k+1}*\mathbf{C})(\xi)Cx - (-1)^{k+1}(j_{2k+1}*\mathbf{C})(\xi)Cx\Big)d\xi \\
&\quad = (j_{2k+2}*\mathbf{C})(t+s)Cx + (-1)^{k+1}(j_{2k+2}*\mathbf{C})(t-s)Cx \\
&\qquad -\Big((j_{2k+2}*\mathbf{C})(s)Cx + (-1)^{k+1}(j_{2k+2}*\mathbf{C})(-s)Cx\Big) \\
&\qquad -\Big(1+(-1)^{k+1}\Big)(j_{2k+2}*\mathbf{C})(\xi)Cx.
\end{aligned}
$$

Noting that

$$
\sum_{i=0}^{k-1}\frac{s^{i+1}}{(i+1)!}(j_{2k+1-i}*\mathbf{C})(t)Cx = \sum_{j=1}^{k}\frac{s^j}{j!}(j_{2(k+1)-j}*\mathbf{C})(t)Cx,
$$

we see that (5.10) holds for $n = k+1$. ■

Next lemma was proved in Tanaka-Miyadera [15, Lemma] in the first order case in time, but the same result is valid in the second order case.

Lemma 5.8. *Let A be the complete infinitesimal generator of a local C-cosine family $\mathbf{C}(\cdot)$ with $C = R(c;A)^m := (c-A)^{-m}$, where $c \in \rho(A)$ and m is a positive integer. Define*

$$
V_m(t)x = \int_0^t\int_0^{t_1}\ldots\int_0^{t_{2m-1}} \mathbf{C}(t_{2m})x dt_{2m-1}\ldots dt_1
$$

for $x \in X$ and $|t| < T$. Then we have for $m \in N, x \in X$ and $|t| < T$

(i) $V_m(t)x \in B(X)$ *and* $\int_0^t \int_0^s (c-A)^{m-1} V_{m-1}(r)xdrds \in D(A)$;

(ii) $(c-A)^m V_m(\cdot)x : (-T, T) \to X$ *is continuous;*

(iii) $(c-A)^m V_m(t) \in B(X)$ *and*

$$(c-A)^m V_m(t) = c(c-A)^{m-1} V_m(t) \\ -(c-A)^{m-1} V_{m-1}(t) + (t^{2m-2}/(2m-2)!)(c-A)^{m-1} C.$$

Proof. By (2.4) we have $A\int_0^t (t-s)\mathbf{C}(s)xds = \mathbf{C}(t)x - Cx$ and hence (i)-(iii) hold for $m = 1$. The conclusion follows by induction with respect m. ■

Theorem 5.9. *Let A be a densely defined closed linear operator in X with $\rho(A) \neq \emptyset$ and $m \in N$. Let $c \in \rho(A), T \in (0, \infty]$. Then the following four conditions are equivalent:*

(i) *A is the complete infinitesimal generator of an $2m$-times integrated local cosine family $U(\cdot)$;*

(ii) *A is the complete infinitesimal generator of a local C-cosine family $\mathbf{C}(\cdot)$ with $C = R(c; A)^m$;*

(iii) *$\rho(A)$ contains a half line $\{\lambda \in R : \lambda > \omega\}$ for some $\omega > 0$, and for every $\tau \in (0, T)$ there exists a constant $M_\tau > 0$, depending on τ, such that for $x \in D(A^m)$,*

$$\| \frac{\lambda^k}{(k-1)!} \frac{d^{k-1}}{d\lambda^{k-1}} \Big(\lambda R(\lambda^2; A)x \Big) \| \le M_\tau \|x\|_m,$$

for $0 \le k/\lambda \le \tau, \lambda > \omega, k \in N$;

(iv) *$(ACP; T, D(A^{m+1}))$ is well-posed.*
In this case

$$U(t)x = (c-A)^m \int_0^t \int_0^{t_1} ... \int_0^{t_{2m-1}} \mathbf{C}(t_{2m}) x dt_{2m-1} ... dt_1 \tag{5.11}$$

for $x \in X$ and $|t| < T$.

Proof. In view of Proposition 5.5 we see that (i) implies (iv). The step from (iv) to (iii) was proved by Takenaka-Okazawa [14, Theorem 3.3]. To prove that (iii) implies (ii) we can apply Theorem 3.2 with $C = R(c; A)^m$ and $L_\tau(\lambda)x = R(\lambda^2; A)R(c; A)^m x$ for $x \in X, \lambda > \omega$ and $\tau \in (0, T)$. To show that (ii) implies (i), let A be the complete infinitesimal generator of a local C-cosine family $\mathbf{C}(\cdot)$ with $C = R(c; A)^m$. Define $U(\cdot)$ by

$$U(t)x = (c-A)^m \int_0^t \int_0^{t_1} ... \int_0^{t_{2m-1}} \mathbf{C}(t_{2m}) x dt_{2m-1} ... dt_1$$

for $x \in X$. By Lemma 5.8 for $x \in X$ it follows that $U(t)x$ is well-defined and $U(t) \in B(X)$ for $|t| < T$. Moreover, $U(\cdot)$ satisfies (4.1) and (4.3). By (5.9) we obtain that

$$\begin{aligned}
&2U(t)R(c;A)^m U(s)R(c;A)^m x \\
&= \frac{1}{(2m-1)!}\Big(\int_t^{t+s}(t+s-r)^{2m-1}U(r)R(c;A)^m Cxdr \\
&\quad -\int_0^s (t+s-r)^{2m-1}U(r)R(c;A)^m Cxdr\Big) \\
&\quad +\frac{(-1)^{2m}}{(2m-1)!}\Big[\int_t^{t-s}(t-s-r)^{2m-1}U(r)R(c;A)^m Cxdr \\
&\quad -\int_0^{-s}(t-s-r)^{2m-1}U(r)R(c;A)^m Cxdr\Big].
\end{aligned}$$

This implies that $U(\cdot)$ is $2m$-times integrated local cosine family. We will prove that A is a complete infinitesimal generator of $U(\cdot)$. To this end, let A_0 and G_0 be the operator defined by (4.7) and (2.1), respectively. Then $A = \overline{G_0}$. Let $x \in D(G_0)$ $(\subset R(C))$. Then by (5.11)

$$\frac{2}{h^2}\Big(U^{(2m)}(h)x - x\Big) = \frac{2}{h^2}\Big(C^{-1}\mathbf{C}(h)x - x\Big) \to G_0 x$$

as $h \to 0$, since $x \in C^{2m}(T)$ and $U^{(2m)}(h)x = (c-A)^m\mathbf{C}(h)x = C^{-1}\mathbf{C}(h)x$. It means that $x \in D(A_0)$ and $A_0x = G_0x$. Hence $A = G \subseteq \overline{A_0}$. Next , let $x \in D(A_0)$. Then by the definition of A_0 there exists $\delta \in (0,T)$ such that $x \in C^{2m}(\delta)$ and $A_0x = \lim_{h\to 0}\frac{2}{h^2}\Big(U^{(2m)}(h)x - x\Big)$. Since by (5.11) $\mathbf{C}(h)x$=$R(c;A)^m U^{(2m)}(h)x$ for $x \in C^{2m}(\delta)$ and we see that

$$\frac{2}{h^2}\left(\mathbf{C}(h)x - Cx\right) = R(c;A)^m\frac{2}{h^2}\Big(U^{(2m)}(h)x - x\Big)$$

for $x \in C^{2m}(\delta)$ and the right-hand side tends to $R(c;A)^m A_0 x \in R(C)$ as $h \to 0$. To prove $\overline{A_0} \subseteq A$, let $x \in D(A_0)$. Then $Cx \in R(C)$ and $\frac{2}{h^2}\Big(C^{-1}\mathbf{C}(h)Cx - Cx\Big) = \frac{2}{h^2}\left(\mathbf{C}(h)x - Cx\right)$. Hence we have $G_0Cx = CA_0x$. Let $x \in D(\overline{A_0})$, then $ACx = C\overline{A_0}x$ and $(c-A)Cx = cCx - C\overline{A_0}x = C(c-\overline{A_0})x$. This implies that $Cx = (c-A)^{-1}C(c-\overline{A_0})x$. Hence $x = C^{-1}(c-A)^{-1}C(c-\overline{A_0})x = (c-A)^{-1}C^{-1}C(c-\overline{A_0})x = (c-A)^{-1}(c-\overline{A_0})x$ (because $C(c-\overline{A_0})x \in D(A^m)$ and (2.5)). This implies that $Ax = \overline{A}_0x$ and $D(\overline{A}_0) \subset D(A)$. ∎

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