TAIWANESE JOURNAL OF MATHEMATICS
Vol. 8, No. 3, pp. 453-465, September 2004
This paper is available online at http://www.math.nthu.edu.tw/tjm/

A CHARACTERIZATION OF ABSOLUTE SUMMABILITY FACTORS

B. E. Rhoades and Ekrem Savaş

Abstract. Let A and B be two summability methods. We shall use the notation $\lambda \in (A, B)$ to denote the set of all sequences λ such that $\sum a_n \lambda_n$ is summable B, whenever $\sum a_n$ is summable A. In the present paper we characterize the sets $\lambda \in (|\overline{N}, p_n|, |T|_k)$ and $\lambda \in (|\overline{N}, p_n|_k, |T|)$, where T is a lower triangular matrix with positive entries and row sums 1. As special cases we obtain summability factor theorems and inclusion theorems for pairs of weighted mean matrices.

1. Introduction

In a recent paper, Sarigöl and Bor [9] obtained necessary and sufficient conditions for $(|\overline{N}, p_n|, |\overline{N}, q_n|_k)$ and $(|\overline{N}, p_n|_k, |\overline{N}, q_n|)$. The concept of absolute summability of order k was coined by Flett [3] as follows. A series $\sum a_n$ is summable $|C, \delta|_k, k \geq 1, \delta > -1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta \sigma_{n-1}^{\delta}|^k < \infty, \tag{1.1}$$

where σ_n^{δ} denotes the nth term of the (C, δ) transform of the partial sums, s_n, of the series $\sum a_n$.

In extending (1.1) to weighted mean methods, for example, Bor [1], Sarigöl [8], and others, have used the definition

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\Delta u_{n-1}|^k < \infty \tag{1.2}$$

Received August 30, 2002; Accepted January 21, 2003.
Communicated by H. M. Srivastava.
2000 *Mathematics Subject Classification*: Primary 40F05, 40D25, 40G99.
Key words and phrases: Absolute summability, Weighted mean matrices, Summability factors.
This research was supported by the Scientific and Technical Research Council of Turkey.

where u_n is the nth term of the weighted mean transform of $\{s_n\}$.

Let T denote a lower triangular matrix with positive entries and row sums 1. Define

$$\bar{t}_{n\nu} = \sum_{i=\nu}^{n} a_{ni}, \quad n, \nu = 0, 1, \cdots$$

and $\hat{t}_{n\nu} = \bar{t}_{n\nu} - \bar{t}_{n-1,\nu}, n = 1, 2, \cdots$.

Before stating our main results we shall note the following lemma.

Lemma 1.1. [5] *Let $1 \leq k < \infty$. Then an infinite matrix $T : \ell \to \ell^k$ if and only if*

$$\sup_{\nu} \sum_{n=1}^{\infty} |t_{n\nu}|^k < \infty.$$

2. The Main Results

We shall prove the following.

Theorem 2.1. *Let $1 \leq k < \infty$. Then $\lambda \in (|\overline{N}, p_n|, |T|_k)$, i.e., $\sum a_n$ is summable $|\overline{N}, p_n|$, then $\sum a_n \lambda_n$ is summable $|T|_k$, if and only if*

(i) $|t_{\nu\nu}\lambda_\nu| \dfrac{P_\nu}{p_\nu} = O(\nu^{1/k-1})$

(ii) $\Big(\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta(\hat{t}_{n\nu}\lambda_\nu|^k \Big)^{1/k} = O\Big(\dfrac{p_\nu}{P_\nu}\Big)$

(iii) $(\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{t}_{n,\nu+1}\lambda_{\nu+1}|^k)^{1/k} = O(1).$

Remark 1. The theorem of [6] is a corollary of Theorem 2.1.

Theorem 2.2. *Let $1 < k < \infty$. Suppose that T also satisfies*

$$\sum_{n=\nu+1}^{\infty} |\hat{t}_{n\nu}| \quad \textit{converges for each} \quad \nu = 1, 2, \ldots \tag{2.1}$$

and

$$\sum_{n=\nu+1}^{\infty} |\hat{t}_{n,\nu+1}| \quad \textit{converges for each} \quad \nu = 1, 2, \ldots \tag{2.2}$$

Then $(|\overline{N}, p_n|_k, |T|)$, i.e., $\sum a_n$ summable $|\overline{N}, p_n|_k$ implies that $\sum a_n \lambda_n$ is summable $|T|$, if and only if

(i) $\displaystyle\sum_{\nu=1}^{\infty} \frac{1}{\nu}\Big| \sum_{n=\nu+1}^{\infty} \frac{P_\nu}{p_\nu}(\Delta \hat{t}_{n\nu}\lambda_\nu) + \hat{t}_{n,\nu+1}\lambda_{\nu+1}\Big|^{k'} < \infty$

(ii) $\displaystyle\sum_{\nu=1}^{\infty} \frac{1}{\nu}\Big| \frac{t_{\nu\nu}P_\nu\lambda_\nu}{p_\nu}\Big|^{k'} < \infty,$

where k' is the conjugate index of k.

Proof of Theorem 2.1. Let $\{x_n\}$ denote the sequence of $(\overline{N}, p_n)$ means of the series $\sum a_n$. By definition,

$$x_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu = \frac{1}{P_n}\sum_{\nu=0}^{n}(P_n - P_{\nu-1})a_\nu.$$

Thus

$$X_n := x_n - x_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1}a_\nu, \quad n \geq 1.$$

Define

$$y_n = \sum_{\nu=0}^{n}\sum_{i=\nu}^{n} t_{n\nu}\lambda_\nu a_\nu = \sum_{\nu=0}^{n} \bar{t}_{n\nu}\lambda_\nu a_\nu \tag{3.1}$$

and

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n}(\bar{t}_{n\nu} - \bar{t}_{n-1,\nu})\lambda_\nu a_\nu = \sum_{\nu=0}^{n} \hat{t}_{n\nu}\lambda_\nu a_\nu. \tag{3.2}$$

By the hypothesis of the theorem, and applying (1.1) with σ_{n-1}^{δ} replaced by Y_n,

$$\sum_{n=1}^{\infty} n^{k-1}|Y_n|^k < \infty \tag{3.3}$$

whenever

$$\sum_{n=1}^{\infty} |X_n| < \infty. \tag{3.4}$$

For $k \geq 1$ we define

$$B = \{\{a_i\} : \sum a_i \text{ is summable}\,|\bar{N}, p_n|\}$$

$$C = \{\{a_i\} : \sum a_i\lambda_i \text{ is summable}|T|_k\}.$$

These are Banach spaces, if normed by

$$\|X\| = \sum |X_n|, \|Y\| = (|Y_0|^k + \sum n^{k-1}|Y_n|^k)^{1/k}, \tag{3.5}$$

respectively.

Since $\sum a_n$ is summable by $|\overline{N}, p_n|$ implies that $\sum a_n\lambda_n$ is summable by $|T|_k$,by the Banach-Steinhaus theorem, there exists a constant $M > 0$ such that

$$\|Y\| \leq M\|X\| \tag{3.6}$$

for all sequences satisfying (3.4).

Applying (3.1) and (3.2) to the sequence $a_\nu = e_\nu, a_{\nu+1} = -e_{\nu+1}, a_n = 0$, otherwise, where e_ν is the $\nu-$th coordinate sequence, we obtain

$$X_n = \begin{cases} 0, & n < \nu, \\ \dfrac{p_\nu}{P_\nu}, & n = \nu, \\ -\dfrac{p_\nu p_n}{P_n P_{n-1}}, & n > \nu, \end{cases}$$

$$Y_n = \begin{cases} 0, & n < \nu, \\ \hat{t}_{\nu\nu}\lambda_\nu, & n = \nu, \\ \Delta(\hat{t}_{n\nu}\lambda_\nu), & n > \nu. \end{cases}$$

From (3.5),

$$\|X\| = \frac{2p_\nu}{P_\nu}$$

and

$$\|Y\| = (\nu^{k-1}|\hat{t}_{\nu\nu}\lambda_\nu|^k + \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta(\hat{t}_{n\nu}\lambda_\nu|^k)^{1/k}.$$

Hence it follows from (3.6) that

$$\nu^{k-1}|\hat{t}_{\nu\nu}\lambda_\nu|^k + \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta(\hat{t}_{n\nu}\lambda_\nu)|^k \leq (2M)^k(\frac{p_\nu}{P_\nu})^k.$$

Since this inequality holds for every $\nu \geq 1$, we obtain

$$\nu^{k-1}|\hat{t}_{\nu\nu}\lambda_\nu|^k + \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta(\hat{t}_{n\nu}\lambda_\nu)|^k = O((\frac{p_\nu}{P_\nu})^k).$$

The above equality is true if and only if each term on the left side is $O((p_\nu/P_\nu)^k)$. Taking the first term gives

$$\frac{P_\nu}{p_\nu}|\hat{t}_{\nu\nu}\lambda_\nu| = O(\nu^{1/k-1});$$

i.e., (i) is necessary. Taking the second term, we obtain

$$(\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta(\hat{t}_{n\nu}\lambda_\nu)|^k)^{1/k} = O(\frac{p_\nu}{P_\nu}),$$

which is condition (ii).

To prove the necessity of (iii) we again apply (3.1) and (3.2), this time to the sequence with $a_\nu = e_{\nu+1}$. We then obtain

$$X_n = \begin{cases} 0, & \text{if } n < \nu+1, \\ \dfrac{P_\nu p_n}{P_n P_{n-1}}, & \text{if } n \geq \nu+1, \end{cases}$$

and

$$Y_n = \begin{cases} 0, & \text{if } n < \nu+1, \\ t_{n,\nu+1}\lambda_{\nu+1}, & \text{if } n \geq \nu+1. \end{cases}$$

Using (3.5) we obtain

$$\|X\| = 1,$$

and

$$\|Y\| = (\sum_{n=\nu+1}^{\infty} n^{k-1}|t_{n,\nu+1}\lambda_{\nu+1}|^k)^{1/k}.$$

From (3.6) it follows that

$$(\sum_{n=\nu+1}^{\infty} n^{k-1}|t_{n,\nu+1}\lambda_{\nu+1}|^k)^{1/k} = O(1),$$

which gives the necessity of (iii).

To prove the conditions sufficient, from (3.1) we have

$$X_n = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu.$$

so,

$$\frac{P_n P_{n-1} X_n}{p_n} = \sum_{\nu=1}^{n} P_{\nu-1} a_\nu$$

$$\frac{P_{n-1}P_{n-2}X_{n-1}}{p_n} = \sum_{\nu=1}^{n-1} P_{\nu-1}a_\nu$$

$$\frac{P_nP_{n-1}X_n}{p_n} - \frac{P_{n-1}P_{n-2}X_{n-1}}{p_n} = P_{n-1}a_n.$$

Thus

$$a_n = \frac{P_nX_n}{p_n} - \frac{P_{n-2}X_{n-1}}{p_{n-1}}. \tag{3.7}$$

Substituting (3.7) into (3.2) gives

$$\begin{aligned} Y_n &= \sum_{\nu=0}^{n} \hat{t}_{n\nu}\lambda_\nu a_\nu = \hat{t}_{n0}\lambda_0 X_0 + \sum_{\nu=1}^{n} \hat{t}_{n\nu}\lambda_\nu(\frac{P_\nu}{p_\nu}X_\nu - \frac{P_{\nu-2}}{p_{\nu-1}}X_{\nu-1}) \\ &= \hat{t}_{n0}X_0\lambda_0 + \hat{t}_{nn}\lambda_n\frac{P_n}{p_n}X_n + \sum_{\nu=1}^{n-1}(\hat{t}_{n\nu}\lambda_\nu P_\nu - \hat{t}_{n,\nu+1}\lambda_{\nu+1}P_{\nu-1})\frac{X_\nu}{p_\nu} \\ &= \sum_{\nu=1}^{n-1}(\hat{t}_{n\nu}\lambda_\nu P_\nu - \hat{t}_{n,\nu+1}\lambda_{\nu+1}P_{\nu-1})\frac{X_\nu}{p_\nu} + t_{nn}\lambda_n\frac{P_n}{p_n}X_n \\ &= \sum_{\nu=1}^{n-1}(\frac{P_\nu}{p_\nu}\Delta(\hat{t}_{n\nu}\lambda_\nu) + \hat{t}_{n,\nu+1}\lambda_{\nu+1})X_\nu + t_{nn}\frac{P_n\lambda_nX_n}{p_n}. \end{aligned}$$

Set $Y_n^* = n^{1-1/k}Y_n$. Then

$$Y_n^* = n^{1-1/k}\sum_{\nu=1}^{n-1}\Big[\frac{P_\nu}{p_\nu}\Delta(\hat{t}_{n\nu}\lambda_\nu) + \hat{t}_{n,\nu+1}\lambda_{\nu+1}\Big]X_\nu + t_{nn}\frac{P_n}{p_n}\lambda_nX_n.$$

We may therefore write $Y_n^* = \sum_{\nu=1}^{n} a_{n\nu}X_\nu$, where

$$a_{n\nu} = \begin{cases} n^{1-1/k}(\frac{P_\nu}{p_\nu})\Delta(\hat{t}_{n\nu}\lambda_\nu) + \hat{t}_{n,\nu+1}\lambda_{\nu+1}, & \text{if} \quad 1 \le \nu \le n-1, \\ n^{1-1/k}\frac{P_n}{p_n}t_{nn}\lambda_n, & \text{if} \quad n = \nu, \\ 0 & \text{if} \quad n > \nu. \end{cases}$$

Then the statement that $\sum a_\nu\lambda_\nu$ is summable $|T|_k, k \ge 1$ whenever $\sum a_n$ is summable $|\overline{N}, p_n|$, is equivalent to $\sum |Y_n^*|^k < \infty$ whenever $\sum |X_n| < \infty$, or, equivalently,

$$\sup_\nu \sum_n |a_{n\nu}|^k < \infty \tag{3.8}$$

by Lemma 1.1. From the definition of T it follows that

$$\sum_{n=\nu}^{\infty} |a_{n\nu}|^k = \nu^{k-1}\Big(\frac{P_n}{p_n}|t_{nn}\lambda_n|\Big)^k + \sum_{n=\nu+1}^{\infty} n^{k-1}\Big|\frac{P_\nu}{p_\nu}\Delta(\hat{t}_{n\nu}\lambda_\nu) + \hat{t}_{n,\nu+1}\lambda_{\nu+1}\Big|^k.$$

Therefore the conditions (i)–(iii), and Minkowski's inequality imply that

$$\sum_{n=\nu}^{\infty} |a_{n\nu}|^k = O(1)$$

as $\nu \to \infty$. This completes the proof.

Proof of Theorem 2.2. Solving (3.1) for a_n and substituting into (3.2) gives

$$Y_n = \sum_{\nu=1}^{n-1}\Big[\frac{P_\nu}{p_\nu}\Delta(\hat{t}_{n\nu}\lambda_\nu) + \hat{t}_{n,\nu+1}\lambda_{\nu+1}\Big]X_\nu + \frac{t_{nn}\lambda_n P_n X_n}{p_n}.$$

With $X_n^* = n^{1-1/k}X_n$,

$$Y_n = \sum_{\nu=1}^{n} a_{n\nu}X_\nu^*,$$

where

$$a_{n\nu} = \begin{cases} \Big(\frac{P_\nu}{p_\nu}\Delta(\hat{t}_{n\nu}\lambda_\nu) + \hat{t}_{n,\nu+1}\lambda_{\nu+1}\Big)\nu^{1/k-1}, & \text{if} \quad 1 \le \nu \le n-1, \\ t_{nn}\lambda_n \frac{P_n}{p_n} n^{1/k-1}, & \text{if} \quad \nu = n, \\ 0, & \text{if} \quad \nu > n. \end{cases}$$

The condition that $\sum a_n\lambda_n$ is summable $|T|$ whenever $\sum a_n$ is summable $|\overline{N}, p_n|_k$ is equivalent to $\sum |Y_n| < \infty$ whenever $\sum |X_n^*|^k < \infty$. Necessary and sufficient conditions for this are that

$$\sum_{n=\nu}^{\infty} a_{n\nu}z_\nu < \infty \text{ for each bounded sequence } z, \nu = 1, 2, \ldots \tag{3.9}$$

and

$$\sum_{\nu=1}^{\infty}\Big|\sum_{n=\nu}^{\infty} a_{n\nu}z_\nu\Big|^{k'} < \infty \quad \text{for each bounded sequence } z. \tag{3.10}$$

To verify (3.9),

$$\sum_{n=\nu}^{\infty} a_{n\nu} z_\nu = t_{\nu\nu}\lambda_\nu \frac{P_\nu}{p_\nu}\nu^{1/k-1} z_\nu + \sum_{n=\nu+1}^{\infty}\Big[\frac{P_\nu}{p_\nu}\Delta(\hat{t}_{n\nu}\lambda_\nu) + \hat{t}_{n,\nu+1}\lambda_{\nu+1}\Big]\nu^{1/k-1} z_n.$$

$$\begin{aligned}\sum_{n=\nu+1}^{\infty}\Big|\frac{P_\nu}{p_\nu}\Delta(\hat{t}_{n\nu}\lambda_\nu)\nu^{1/k-1} z_n\Big| &\le \frac{MP_\nu \nu^{1/k-1}}{p_\nu}\sum_{n=\nu+1}^{\infty}|\hat{t}_{n\nu}\lambda_\nu - \hat{t}_{n,\nu+1}\lambda_{\nu+1}| \\ &\le \frac{MP_\nu \nu^{1/k-1}}{p_\nu}\Big[|\lambda_\nu|\sum_{n=\nu+1}^{\infty}|\hat{t}_{n\nu}| \\ &+ |\lambda_{\nu+1}|\sum_{n=\nu+1}^{\infty}|\hat{t}_{n,\nu+1}|\Big] \\ &= O(1),\end{aligned}$$

by using (2.1) and (2.2), where M is a bound for z. Therefore the series is convergent.

Also

$$\begin{aligned}\sum_{n=\nu+1}^{\infty}\Big|\hat{t}_{n,\nu+1}\lambda_{\nu+1}\nu^{1/k-1} z_n\Big| &\le M\lambda_{\nu+1}\nu^{1/k-1}\sum_{n=\nu+1}^{\infty}|\hat{t}_{n,\nu+1}| \\ &= O(1),\end{aligned}$$

and (3.9) is satisfied.

Therefore, from (3.10), the necessary and sufficient condition for the conclusion of the theorem is

$$\sum_{\nu=1}^{\infty}\Big|t_{\nu\nu}\lambda_\nu\frac{P_\nu}{p_\nu}\nu^{1/k-1} z_\nu + \sum_{n=\nu+1}^{\infty}\Big[\frac{P_\nu}{p_\nu}\Delta(\hat{t}_{n\nu}\lambda_\nu) + \hat{t}_{n,\nu+1}\lambda_{\nu+1}\Big]\nu^{1/k-1} z_\nu\Big|^{k'} < \infty, \tag{3.11}$$

for each bounded sequence z. It follows from (3.11), by choosing $z_n = 1$ for each n, that

$$\sum_{\nu=1}^{\infty}\Big|t_{\nu\nu}\lambda_\nu\frac{P_\nu}{p_\nu}\nu^{1/k-1}\Big|^{k'} = O(1) \tag{3.12}$$

and

$$\sum_{n=\nu+1}^{\infty}\Big|\frac{P_\nu}{p_\nu}\Delta(\hat{t}_{n\nu}\lambda_\nu)+\hat{t}_{n,\nu+1}\lambda_{\nu+1}\big]\nu^{1/k-1}\Big|^{k'}=O(1), \tag{3.13}$$

which are conditions (i) and (ii).

To show that (i) and (ii) are sufficient, one needs only to use the inequality

$$(a+b)^{k'}\le 2^{k'}(a^{k'}+b^{k'}),\quad a,b\ge 0,$$

along with (3.12) and (3.13), since (3.11) holds for every bounded sequence $z_n = O(1)$.

4. Additional Results

For any sequences $\{a_n\},\{b_n\}$, the statement $a_n \asymp b_n$ means $a_n = O(b_n)$ and $b_n = O(a_n)$.

Theorem 4.1. *Let* $1\le k<\infty, \{q_n\}$ *a positive sequence satisfying*

$$\Big(\sum_{n=\nu+1}^{\infty}n^{k-1}\Big(\frac{q_n}{Q_nQ_{n-1}}\Big)^k\Big)^{1/k}\asymp O\Big(\frac{1}{Q_\nu}\Big). \tag{4.1}$$

Then $\lambda\in(|\overline{N},p_n|,|\overline{N},q_n|_k)$ *if and only if*

(i) $\lambda_n = O(1)$

(ii) $\Delta\lambda_n = O\Big(\dfrac{p_n}{P_n}\Big)$

(iii) $\lambda_n = O\Big(\dfrac{p_nQ_n}{q_nP_n}\dfrac{1}{n^{1/k'}}\Big)$.

Proof. With $t_{nk}=q_k/Q_n$, condition (i) of Theorem 2.1 becomes

$$\Big|\frac{q_\nu\lambda_\nu}{Q_\nu}\Big|\frac{P_\nu}{p_\nu}=O(\nu^{1/k-1}),$$

which is equivalent to condition (iii) of Theorem 4.1.

Then

$$\begin{aligned}\hat{t}_{n\nu} &= \bar{t}_{n\nu}-\bar{t}_{n-1,\nu}=\frac{1}{Q_n}\sum_{i=\nu}^{n}q_i-\frac{1}{Q_{n-1}}\sum_{i=\nu}^{n-1}q_i\\ &=\frac{1}{Q_n}(Q_n-Q_{\nu-1})-\frac{1}{Q_{n-1}}(Q_{n-1}-Q_{\nu-1})\\ &=\frac{-Q_{\nu-1}q_n}{Q_nQ_{n-1}}.\end{aligned} \tag{4.2}$$

Substituting into condition (iii) of Theorem 2.1 we have

$$\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}\Big|\frac{Q_\nu q_n \lambda_{\nu+1}}{Q_n Q_{n-1}}\Big|^k\Big)^{1/k} = O(1)$$

or

$$|\lambda_{\nu+1}|Q_\nu\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{q_n}{Q_n Q_{n-1}}\Big)^k\Big)^{1/k} = O(1),$$

which, using (4.1), implies condition (i) of Theorem 4.1.

$$\begin{aligned} \Delta(\hat{t}_{n\nu}\lambda_\nu) &= -\frac{Q_{\nu-1}q_n}{Q_n Q_{n-1}}\lambda_\nu + \frac{Q_\nu q_n}{Q_n Q_{n-1}}\lambda_\nu + 1 \\ &= -\frac{q_n}{Q_n Q_{n-1}}\Delta(Q_{\nu-1}\lambda_\nu). \end{aligned} \tag{4.3}$$

Substituting into condition (ii) of Theorem 2.1 yields

$$\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}\Big|\frac{q_n}{Q_n Q_{n-1}}\Delta(Q_{\nu-1}\lambda_\nu)\Big|^k\Big)^{1/k} = O\Big(\frac{p_\nu}{P_\nu}\Big),$$

or

$$|\Delta(Q_{\nu-1}\lambda_\nu)|\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{q_n}{Q_n Q_{n-1}}\Big)^k\Big)^{1/k} = O\Big(\frac{p_\nu}{P_\nu}\Big),$$

which, using (4.1) implies that

$$|\Delta(Q_{\nu-1}\lambda_\nu)|\frac{1}{Q_\nu} = O\Big(\frac{p_\nu}{P_\nu}\Big).$$

Thus, since λ_ν is bounded,

$$\begin{aligned} \Delta(Q_{\nu-1}\lambda_\nu) &= Q_{\nu-1}\lambda_\nu - Q_\nu\lambda_{\nu+1} \\ &= Q_\nu\Delta\lambda_\nu - q_\nu\lambda_\nu \\ &= O\Big(\tfrac{Q_\nu p_\nu}{P_\nu}\Big) \end{aligned}$$

or

$$\Delta\lambda_\nu = \frac{q_\nu}{Q_\nu}\lambda_\nu + O\Big(\frac{p_\nu}{P_\nu}\Big) = O\Big(\frac{p_\nu}{P_\nu}\Big),$$

which is condition (ii) of Theorem 4.1.

Remark 2. The theorem of [7] is a special case of Theorem 4.1.

Theorem 4.2. *Let* $1 < k < \infty$. *Then* $\lambda \in (|\overline{N}, p_n|_k, |\overline{N}, q_n|)$ *if and only if*

$(i)\ \sum_{\nu=1}^{\infty} \frac{1}{\nu} \Big| \frac{P_\nu q_\nu}{p_\nu Q_\nu} \lambda_\nu \Big|^{k'} < \infty$

$(ii)\ \sum_{\nu=1}^{\infty} \frac{1}{\nu} \Big| \frac{P_\nu \Delta(Q_{\nu-1}\lambda_\nu)}{p_\nu} + Q_\nu \lambda_{\nu+1} \Big|^{k'} < \infty.$

Proof. Since again $t_{nk} = q_k/Q_n$, using (4.2),

$$\begin{aligned}
\sum_{n=\nu+1}^{\infty} |\hat{t}_{n\nu}| &= \sum_{n=\nu+1}^{\infty} \frac{Q_{\nu-1} q_n}{Q_n Q_{n-1}} \\
&= Q_{\nu-1} \sum_{n=\nu+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \\
&= Q_{\nu-1} \sum_{n=\nu+1}^{\infty} \Big(\frac{1}{Q_{n-1}} - \frac{1}{Q_n} \Big) \\
&= \frac{Q_{\nu-1}}{Q_\nu} \le 1,
\end{aligned}$$

and (2.1) is satisfied. So also is (2.2).

Substituting the value of $t_{\nu\nu}$ into condition (ii) of Theorem 2.2 yields condition (ii) of Theorem 4.2, and substituting (4.3) into condition (i) of Theorem 2.2 gives condition (i) of Theorem 4.2.

We now establish some summability factor theorems for the case $k = 1$.

Corollary 4.1. $\lambda \in (|\overline{N}, p_n|, |T|)$ *if and only if*

$(i)\ \lambda_\nu = O\Big(\frac{p_\nu}{P_\nu t_{\nu\nu}}\Big)$

$(ii)\ \sum_{n=\nu+1}^{\infty} |\Delta(\hat{t}_{n\nu}\lambda_\nu| = O\Big(\frac{p_\nu}{P_\nu}\Big)$

$(iii)\ \sum_{n=\nu+1}^{\infty} |\hat{t}_{n,\nu+1}\lambda_{\nu+1}| = O(1).$

To prove the corollary, simply substitute $k = 1$ in Theorem 2.1.

Corollary 4.2. $\lambda \in (|\overline{N}, p_n|, \overline{N}, q_n|)$ *if and only if*

$(i)\ \lambda_n = O(1),$

$(ii)\ \Delta\lambda_n = O\Big(\dfrac{p_n}{P_n}\Big)$

$(iii)\ \lambda_n = O\Big(\dfrac{p_n Q_n}{q_n P_n}\Big).$

To prove this corollary, use Theorem 4.1 with $k = 1$, recognizing that $1/k' = 0$.

Corollary 4.3. $\{p_n Q_n / q_n P_n\} \in (|\overline{N}, p_n|, |\overline{N}, q_n|)$ *if and only if*

$$\frac{p_n Q_n}{q_n P_n} \asymp O(1)$$

Corollary 4.3 is proved by combining parts (i) and (iii) of Corollary 4.2.

Remark 3. Corollary 4.3 is an improvement of a result of Kishore and Hotta [4]. Summability factor theorems also lead to inclusion theorems.

Corollary 4.4. $|\overline{N}, p_n|$ *and* $|\overline{N}, q_n|$ *are equivalent if and only if*

$$\frac{p_n Q_n}{q_n P_n} \asymp O(1)$$

Proof. Suppose that $\sum a_n$ summable $|\overline{N}, p_n|$ implies that $\sum a_n$ is summable $|\overline{N}, q_n|$. Then, from Corollary 4.2, with $\lambda = 1$ we obtain $q_n P_n / p_n Q_n = O(1)$. Interchanging the roles of $\{p_n\}$ and $\{q_n\}$ gives $p_n Q_n / q_n P_n = O(1)$. ■

Remark 4. Corollary 4.4 was first proved by Sunouchi [8] and Bosanquet [2]

Remark 5. Corollaries 4.2– 4.4 are identical to Corollaries 4.1–4.3, of [7] since (1.1) and (1.2) are the same for $k = 1$.

References

1. H. Bor, On two summability methods, *Math. Proc. Cambrige Philos. Soc.* **98** (1985), 147-149
2. L. S. Bosanquet, *Math. Reviews* **11** (1954), 654
3. T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, *Proc. London Math. Soc.* (Ser. 3) **7** (1957), 113-141.
4. N. Kishore and G. C. Hotta, On $|\overline{N}, p_n|$ summability factors, *Acta Sci. Math.* (*Szeged*) **31** (1970), 9-12.
5. I. J. Maddox, *Elements of Functional Analysis*, Cambridge Univ. Press, Cambridge, (1970).

6. C. Orhan and Ö. Cakar, Some inclusion theorems for absolute summability, *Czech. Math. J.* **46** (1996), 599-605.
7. C. Orhan and M. A. Sarigöl, On absolute weighted mean summability, *Rocky Mountain J. Math.* **23** (1993), 1091-1098.
8. M. Ali Sarigöl, Necessary and sufficient conditions for the equivalence of the summability methods $|\bar{N}, p_n|_k$ and $|C, 1|_k$, *Indian J. Pure Appl. Math.* **22** (1991), 483-489.
9. M. Ali Sarigöl and H. Bor, Characterization of absolute summability factors, *J. Math. Anal. Appl.* **195** (1995), 537-545.
10. G. Sunouchi, Notes on Fourier analysis $(XVIII)$: Absolute summability of a series with constant terms, *Tohoku Math. J.* **1** (1949), 57-65.

B. E. Rhoades
Department of Mathematics
Indiana University
Bloomington, IN 47405-7106
U.S.A.
E-mail: rhoades@indiana.edu

Ekrem Savaş
Department of Mathematics
Yüzüncü Yil University
Van, Turkey
E-mail: ekremsavas@yahoo.com