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FULL ELEMENTS IN REGULAR RINGS

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Abstract. In this paper we introduce full elements in regular rings, and show that every matrix over regular rings admits a diagonal reduction by full matrices. Also we get that a regular ring R is unit-regular if and only if $aR \cong bR$ with full elements $a, b \in R$ implies $a = ubv$ for some units $u, v \in R$.

1. Introduction

A ring R is a regular ring provided that for every $x \in R$ there exists $y \in R$ such that $x = xyx$. We say that a ring R is unit-regular if for every $x \in R$, there exists a unit $u \in R$ such that $x = xux$. We refer the reader to [6] for the general theory of regular rings. It is well known that every matrix over unit-regular rings admits a diagonal reduction by invertible matrices (cf. [12, Theorem 3]). P. Ara et. al. have extended this result to separative regular rings (cf. [2, Theorem 2.5]). In this paper we introduce full elements in regular rings and observe that these result can be generalized to general regular rings by virtue of full elements. In [10, Theorem 2], D. Handelman proved that a regular ring R is unit-regular if and only if $eR \cong fR$ with idempotents $e, f \in R$ implies $e = ufu^{-1}$ for a unit $u \in R$. We also characterize unit-regularity by full elements, and show that a regular ring R is unit-regular if and only if $aR \cong bR$ with full elements $a, b \in R$ implies $a = ubv$ for some units $u, v \in R$.

Throughout, all rings are associative with identity and all modules are right modules. The notation $I \leq R$ means that I is a two-sided ideal of R, and we use $U(R)$ to denote the set of all units of R.

Definition 1. An element $x \in R$ is said to be full in case $RxR = R$. We denote the set of all full elements of R by $Q(R)$.

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Clearly, $U(R) \subseteq Q(R) \subseteq R$ and $Q(R) = R - \bigcup\{I \mid I \subsetneq R\}$. If R is either a local ring or a commutative ring, then $Q(R) = U(R)$. Let D be a division ring and V a countably infinite dimensional vector space over D. We claim that $Q(\mathrm{End}_D(V)) \neq U(\mathrm{End}_D(V))$. Set $I = \{x \in \mathrm{End}_D(V) \mid \dim_D(xV) < \infty\}$. Then I is the only proper ideal of $\mathrm{End}_D(V)$ and so $Q\big(\mathrm{End}_D(V)\big) = \mathrm{End}_D(V) - I$. Since $\mathrm{End}_D(V)$ is not a local ring, $Q\big(\mathrm{End}_D(V)\big) \neq U\big(\mathrm{End}_D(V)\big)$.

Theorem 2. *Let R be a regular ring. Then $aR + bR = R$ implies that there exists $y \in R$ such that $a + by$ is a full element.*

Proof. Suppose that $aR + bR = R$. Since R is regular, $aR \cap bR$ is a principal right ideal of R and so it is a direct summand of R. Therefore, by Modular Law, $bR = (aR \cap bR) \oplus cR$ and so $R = aR + bR = aR \oplus cR$. Hence there exist two orthogonal idempotents $u, v \in R$ such that $u + v = 1, aR = uR$ and $cR = vR$. Obviously, $c = by$ for $y \in R$. We claim that $a + c = a + by$ is a full element. Indeed, $u(a + c) = a, v(a + c) = c$ and so $a, c \in R(a + c)R$. Therefore $R = aR + cR = R(a + c)R$, as required. ■

Corollary 3. *A ring R is regular if and only if for every $x \in R$, there exists a $w \in Q(R)$ such that $x = xwx$.*

Proof. One direction is obvious. Conversely, let $x \in R$. Then there exists $y \in R$ such that $x = xyx$ and $y = yxy$. From $yx + (1 - yx) = 1$, we have a $z \in R$ such that $y + (1 - yx)z = w \in Q(R)$ by Theorem 2. Hence $x = xyx = x\big(y + (1 - yx)z\big)x = xwx$. ■

Corollary 4. *Let R be a regular ring. Then for every $x \in R$, there exist an idempotent $e \in R$ and a $w \in Q(R)$ such that $x = ew$.*

Proof. Given any $x \in R$, we have a $y \in R$ such that $x = xyx$ and $y = yxy$. Since $xy + (1 - xy) = 1$, by virtue of Theorem 2, we can find a $z \in R$ such that $x + (1 - xy)z = w \in Q(R)$. So $x = xyx = xy\big(x + (1 - xy)z\big) = xyw$. Set $e = xy$. Then $e \in R$ is an idempotent and $x = ew$, as asserted. ■

Corollary 5. *Let R be a regular ring and n a positive integer. If $1/2 \in R$, then for every $A \in M_n(R)$, there exist $U, V \in Q\big(M_n(R)\big)$ such that $A = U + V$.*

Proof. Since R is a regular ring, so is $M_n(R)$ by [6, Theorem 1.7]. Given any $A \in M_n(R)$, from Corollary 4, we can find an idempotent matrix E and a full matrix U such that $A = EU$. It is easy to verify that $E = \mathrm{diag}(\frac{1}{2}, \cdots, \frac{1}{2}) + \frac{1}{2}\big(2E - \mathrm{diag}(1, \cdots, 1)\big)$. Clearly, $\frac{1}{2}\big(2E - \mathrm{diag}(1, \cdots, 1)\big)\big(4E - \mathrm{diag}(2, \cdots, 2)\big) = \frac{1}{2}\big(4E - \mathrm{diag}(2, \cdots, 2)\big)\big(2E - \mathrm{diag}(1, \cdots, 1)\big) = \mathrm{diag}(1, \cdots, 1)$, so we see that

$$\mathrm{diag}(\frac{1}{2}, \cdots, \frac{1}{2}), \frac{1}{2}\bigl(2E - \mathrm{diag}(1, \cdots, 1)\bigr) \in GL_n(R).$$

Therefore $A = \mathrm{diag}(\frac{1}{2}, \cdots, \frac{1}{2})U + \frac{1}{2}\bigl(2E - \mathrm{diag}(1, \cdots, 1)\bigr)U$, as desired. ■

Lemma 6. *Let* R *be a regular ring. Then the following hold:*

(1) *Whenever* $aR = bR$*, there exists a* $w \in Q(R)$ *such that* $a = bw$.

(2) *Whenever* $Ra = Rb$*, there exists a* $w \in Q(R)$ *such that* $a = wb$.

Proof. Suppose that $aR = bR$ with $a, b \in R$. Then we have $x, y \in R$ such that $ax = b$ and $a = by$. Obviously, $b = ax = byx$. Since $yx + (1-yx) = 1$, we have a $z \in R$ such that $y + (1-yx)z = w \in Q(R)$. Therefore $a = by = b\bigl(y+(1-yx)z\bigr) = bw$, as required. The second statement is proved analogously. ■

Lemma 7. *Let* R *be a regular ring. Whenever* $aR \cong bR$*, there exist* $w_1, w_2 \in Q(R)$ *such that* $a = w_1bw_2$.

Proof. Suppose that $\varphi : aR \cong bR$. Since R is a regular ring, by [11, Lemma 1], we know that $Ra = R\varphi(a)$ and $\varphi(a)R = bR$. In view of Lemma 6, there exist $w_1, w_2 \in Q(R)$ such that $\varphi(a) = w_1a$ and $b = \varphi(a)w_2$. Therefore we conclude that $b = w_1aw_2$ with $w_1, w_2 \in Q(R)$. ■

Theorem 8. *Let* R *be a regular ring and* n *be a positive integer. For any* $A \in M_n(R)$*, there exist full matrices* $P, Q \in M_n(R)$ *such that* $PAQ = \mathrm{diag}(e_1, \cdots, e_n)$ *for some idempotents* $e_1, \cdots, e_n \in R$.

Proof. Because R is a regular ring, so is $M_n(R)$ by [6, Theorem 1.7]. Given any $A \in M_n(R)$, there exists $E = E^2 \in M_n(R)$ such that $AM_n(R) = EM_n(R)$. Clearly, ER^n is a finitely generated projective right R-module. By virtue of [6, Proposition 2.6], we can find idempotents $e_1, \ldots, e_n \in R$ such that $ER^n \cong e_1R \oplus \cdots \oplus e_nR \cong \mathrm{diag}(e_1, \ldots, e_n)R^n$ as right R-modules. Hence $ER^{n\times 1} \cong \mathrm{diag}(e_1, \ldots, e_n)R^{n\times 1}$, where $R^{n\times 1}$ consisting of all n-column vectors over R is a right R-module and a left $M_n(R)$-module. Let $R^{1\times n} = \{(x_1, \ldots, x_n) \mid x_i \in R\}$. Then $R^{1\times n}$ is a left R-module and a right $M_n(R)$-module. One easily checks that

$$(ER^{n\times 1}) \bigotimes_R R^{1\times n} \cong \bigl(\mathrm{diag}(e_1, \ldots, e_n)R^{n\times 1}\bigr) \bigotimes_R R^{1\times n}.$$

In addition, $R^{n\times 1} \bigotimes R^{1\times n} \cong M_n(R)$ as right $M_n(R)$-modules. Hence $AM_n(R) = EM_n(R) \cong \mathrm{diag}(e_1, \ldots, e_n)M_n(R)$. According to Lemma 7, we can find $P, Q \in Q\bigl(M_n(R)\bigr)$ such that $PAQ = \mathrm{diag}(e_1, \ldots, e_n)$, as asserted. ■

Let R be a regular ring. Analogously to Theorem 8, we prove that if $A \in M_n(R)$, there exist $P, Q \in Q\big(M_n(R)\big)$ such that $A = P\,\mathrm{diag}(e_1, \cdots, e_n)Q$ for some idempotents $e_1, \cdots, e_n \in R$. Thus, by [13, Lemma 1], we conclude that if $A \in M_n(R)(n \geq 2)$ then there exist $P, Q, W \in Q\big(M_n(R)\big)$ such that $A = (P + Q)W$.

Recall that a is pseudo-similar to b in R provided that there exist $x, y, z \in R$ such that $xay = b, zbx = a$ and $xyx = xzx = x$. We denote it by $a \widetilde{\sim} b$. [5, Theorem] showed that a regular ring R is unit-regular if and only if whenever $a \widetilde{\sim} b$, there exists a unit $u \in R$ such that $a = ubu^{-1}$. In [H. Chen, On Exchange QB-rings, Comm. Algebra, to appear], the author observed the following simple fact.

Lemma 9. *Let R be an associative ring with $a, b \in R$. Then the following are equivalent:*

(1) $a \widetilde{\sim} b$.

(2) *There exist some $x, y \in R$ such that $a = xby, b = yax, x = xyx$ and $y = yxy$.*

Proof. (1)⇒(2) is trivial.

(2)⇒(1) Inasmuch as $a \widetilde{\sim} b$, there are $x, y, z \in R$ such that $b = xay, zbx = a$ and $x = xyx = xzx$. Indeed, $xa(yxy) = xzbx(yxy) = xzb(xyx)y = xzbxy = xay = b$. Analogously, $(zxz)by = a$. By replacing y with yxy and z with zxz, we can assume $y = yxy$ and $z = zxz$. Further, we directly check that $xazxy = xzbxzxy = xzbxy = xay = b, zxybx = zxyxayx = zxayx = zbx = a, zxy = zxyxzxy$ and $x = xzxyx$, thus yielding the result. ■

Theorem 10. *Let R be a regular ring. Whenever $a \widetilde{\sim} b$, there exist full elements $w_1, w_2 \in R$ such that $a = w_1bw_2$.*

Proof. Suppose that a and b are pseudo-similar in R. According to Lemma 9, there exist $x, y \in R$ such that $a = xby, b = yax, x = xyx$ and $y = yxy$. By Corollary 3, we can find a $v \in Q(R)$ such that $y = yvy$. Let $w_1 = (1 - xy - vy)v(1 - yx - yv)$. It is easy to verify that $(1 - xy - vy)^2 = 1 = (1 - yx - yv)^2$. hence $w_1 \in Q(R)$. In addition, we have $aw_1 = a(1 - xy - vy)v(1 - yx - yv) = -av(1 - yx - yv) = -av + ax + av = ax$. Similarly, we see that $xb = w_1b$. From $yx + (1 - yx) = 1$, there is a $z \in R$ such that $y + (1 - yx)z = w_2 \in Q(R)$, whence $y = yxy = yx\big(y + (1 - yx)z\big) = yxw_2$. Therefore $a = xby = xbyxw_2 = axw_2 = w_1bw_2$, as desired. ■

Recall that $a \in R$ is said to be strongly π-regular if there exist $n \geq 1$ and $x \in R$ such that $a^n = a^{n+1}x, ax = xa$ and $x = xax$. Clearly, the solution $x \in R$ is unique, and we say that x is the Drazin inverse a^d of a.

Corollary 11. *Let R be a regular ring. Whenever $ab, ba \in R$ are strongly π-regular, there exist $w_1, w_2 \in Q(R)$ such that $(ab)^d = w_1(ba)^dw_2$.*

Proof. As ab and $ba \in R$ are strongly ¼-regular, we have $k \geq 1$ such that $(ab)^k = (ab)^{k+1}(ab)^d; (ab)(ab)^d = (ab)^d(ab)$ and $(ab)^d = (ab)^d(ab)(ab)^d$. It is easy to verify that

$$\begin{aligned}(ab)^{k+2}a(ba)^d(ba)^d b &= (ab)a(ba)^{k+1}(ba)^d(ba)^d b = (ab)a(ba)^k(ba)^d b\\ &= a(ba)^{k+1}(ba)^d b = a(ba)^k b = (ab)^{k+1};\\ (ab)(a(ba)^d(ba)^d b) &= a(ba)^d(ba)^d(ba)b\\ &= a(ba)^d b = (a(ba)^d(ba)^d b)(ab);\end{aligned}$$

$(a(ba)^d(ba)^d b)(ab)(a(ba)^d(ba)^d b) = a(ba)^d(ba)^d b$: So $(ab)^d = a(ba)^d(ba)^d b$; so $(ab)^d = a(ba)^d(ba)^d b$. In addition, we check that $(ba)^d = (ba)^d(ba)(ba)^d = (ba)^d babab)^d(ab)^d a = (ba)^d b(ab)^d a$: Clearly, $(ba)^d ba(ba)^d b = (ba)^d b$; hence, $(ab)^d \sim (ba)^d$. We see from Theorem 10 that $(ab)^d = w_1(ba)^d w_2$ for $w_1; w_2 \in Q(R)$. ■

Lemma 12. *Let R be a regular ring. Then the following are equivalent:*

(1) R *is unit-regular.*

(2) *Every full element of* R *is unit-regular.*

Proof. $(1) \Rightarrow (2)$ is trivial.

$(2) \Rightarrow (1)$ Given $aR + bR = R$, then $ax + by = 1$ for $x; y \in R$. By Theorem 2, we have $z \in R$ such that $a + bz \in Q(R)$. Since every full element in R is unit-regular, there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a + bz = eu$. Hence $(a + bz)x + b(y - zx) = 1$, and then $eux + b(y - zx) = 1$. Thus, we must have $eux(1-e) + b(y-zx)(1-e) = 1-e$, whence $a + b\big(z + (y - zx)(1-e)u\big) = eu + b(y - zx)(1 - e)u = \big(1 - eux(1 - e)\big)u = \big(1 - eux(1 - e)\big)^{-1}u \in U(R)$. It follows by [6, Proposition 4.12] that R is unit-regular. ■

By [14, Proposition 3.3], if R is a regular ring then an idempotent $e \in Q(R)$ if and only if there exist nilpotent $n_1; n_2 \in R$ such that $e = 1 + n_1n_2$. Now we investigate unit-regularity by virtue of full idempotents.

Lemma 13. *Let R be a regular ring. Then the following are equivalent:*

(1) R *is unit-regular.*

(2) *Whenever* $eR \cong fR$ *with full idempotents* $e; f \in R$, *there exists* $u \in U(R)$ *such that* $e = ufu^{-1}$.

(3) *Whenever* $eR \cong fR$ *with full idempotents* $e; f \in R$, $(1 - e)R \cong (1 - f)R$.

Proof. $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are clear by [10, Theorem 2].

$(3) \Rightarrow (1)$ Given any $x \in Q(R)$, then there exists a $y \in R$ such that $x = xyx$. Since xy and yx are both idempotents of R, we have right R-module decompositions $R = yxR \oplus (1 - yx)R = xyR \oplus (1 - xy)R$. Clearly, Ã : $yxR = yR \cong xyR$ given by $yr \to x(yr)$ for any $r \in R$. If $yx \notin Q(R)$, then there exists an ideal $I \trianglelefteq R$ such that $yx \in I$; hence, $x = xyx \in I$. So $x \notin Q(R)$, a contradiction. Thus, $yx \in Q(R)$. Likewise, $xy \in Q(R)$. By the hypothesis, we have $\psi : (1 - yx)R \cong (1 - xy)R$. Define $u \in End_R(R)$ so that u restricts to $\varphi^{-1} : xR = xyR \to yxR$ and u restricts to $\psi^{-1} : (1 - xy)R \to (1 - yx)R$. One easily checks that $x = xu(1)x$ and $u(1) \in U(R)$. That is, every full elements in R is unit-regular. Therefore we get the result by Lemma 12. ■

As a result, we deduce that a regular ring R is unit-regular if and only if whenever $aR \cong bR$ with full elements $a, b \in R$, then $R/aR \cong R/bR$.

Theorem 14. *Let R be a regular ring. Then the following hold:*

(1) R *is unit-regular.*

(2) *Whenever* $aR \cong bR$ *with full elements* $a, b \in R$*, there exist* $u, v \in U(R)$ *such that* $a = ubv$.

Proof. $(1) \Rightarrow (2)$ Assume that $\varphi : aR \cong bR$ with $a, b \in R$. Since R is unit-regular, a and b are both unit-regular; hence, $a = eu_1$ and $b = fv_1$ with idempotents $e, f \in R$ and units $u_1, v_1 \in R$. Clearly, $eR = aR \cong bR = fR$. By [10, Theorem 2], we have $u \in U(R)$ such that $e = ufu^{-1}$. So $a = eu_1 = ufu^{-1}u_1 = ubv_1^{-1}u^{-1}u_1$. Set $v = v_1^{-1}u^{-1}u_1$. Then there are $u, v \in U(R)$ such that $a = ubv$.

$(2) \Rightarrow (1)$ Given $\varphi : eR \cong fR$ with idempotents $e, f \in Q(R)$, then we have $u, v \in U(R)$ such that $e = ufv$. Set $e' = ufu^{-1}$. Obviously, $eR = e'R$ and so $(e - e')^2 = 0$. Therefore $w = 1 - e + e' \in U(R)$ and $w^{-1} = 1 + e - e'$. Clearly, $we' = e'$ and $e'w^{-1} = e$. Therefore $e = we'w^{-1} = (wu)f(wu)^{-1}$ and so the result follows from Lemma 13. ■

Corollary 15. *Let R be a regular ring, I an ideal of R. Then the following hold:*

(1) R *is unit-regular.*

(2) *Whenever* $aR \cong bR$ *with* $a, b \notin I$*, there exist* $u, v \in U(R)$ *such that* $a = ubv$.

Proof. $(1) \Rightarrow (2)$ is analogous to Theorem 14.

$(2) \Rightarrow (1)$ Given $aR \cong bR$ with full elements $a, b \in R$, then we have $a, b \notin I$; hence, there exist $u, v \in U(R)$ such that $a = ubv$. Therefore we complete the proof by Theorem 14. ■

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