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ON A SYSTEM OF NONLINEAR WAVE EQUATIONS

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Abstract. We consider the existence and nonexistence of global solutions of the following initial-boundary value problem for a system of nonlinear wave equations in a domain $\Omega \times [0, T)$:

$$\begin{cases} u_{tt} - \Delta u + m_1^2 u = -4\lambda(u + \alpha v)^3 - 2\beta u v^2, \\ v_{tt} - \Delta v + m_2^2 v = -4\alpha\lambda(u + \alpha v)^3 - 2\beta u^2 v, \end{cases}$$

where Ω is a bounded domain in R^3 with a smooth boundary. Some sufficient conditions on the given parameters λ, α and β for the global existence and blow-up are imposed. The estimates for the lifespan of solutions is given.

1. Introduction

In this paper we shall consider the global existence and blow-up of solutions of the following initial-boundary value problem for a system of nonlinear wave equations in a bounded domain $[0, T) \times \Omega$:

$$\begin{cases} u_{tt} - \Delta u + m_1^2 u = -4\lambda(u + \alpha v)^3 - 2\beta u v^2, \\ v_{tt} - \Delta v + m_2^2 v = -4\alpha\lambda(u + \alpha v)^3 - 2\beta u^2 v, \end{cases} \tag{1.1}$$

with initial conditions

$$\begin{cases} u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ x \in \Omega, \end{cases} \tag{1.2}$$

and boundary condition

$$u(x,t) = v(x,t) = 0, \ \text{on } \partial\Omega \times (0,T), \tag{1.3}$$

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where Ω is a bounded domain in R^3 with a smooth boundary $\partial\Omega$, and λ, β and α are real numbers, and $T > 0$.

This problem was proposed by Reed [18]. As a model it describes the interactions of scalar fields u, v of mass m_1, m_2 respectively. The system defines the motion of charged mesons in an electro-magnetic field that was first introduced by Segal [20].

The global existence and uniqueness of the solutions of the Cauchy problem for a single wave equation

$$u_{tt} - \Delta u + f(u) = 0, \text{ in } R^+ \times R^n, n \geq 3$$

has been discussed by several authors during the past thirty years. ([1-2, 6, 8, 19, 23, 24]), and some blow up results are obtained by ([3, 5, 8-12, 21-22]). For the system of hyperbolic equations, only a few existence results are known ([7, 14-17]).

In this paper, we shall discuss the global existence and blow-up properties of solutions in $H1 = C^1(0,T,L^2(\Omega) \cap C^0(0,T,H_0^1(\Omega))$ for (1.1) – (1.3) in a bounded domain in R^3. In section 2, we first derive apriori estimate of any solution of linear wave equation. Then from the uniqueness and existence of solutions of linear wave equation, we obtain the local existence in Theorem 2.4 by using successive approximation method. In section 3, we shall prove the local uniqueness and the global existence of solutions for (1.1) – (1.2). We also show the triviality of the solution when the initial data are zero functions. In section 4, we first define an energy function $E(t)$ in (3.1) and show that it is a constant function of t. It follows immediately from some essential identities that will be used later for estimating the lifespan T. And then we obtain our main result, Theorem 4.4, which shows the blow-up of solutions under some restriction on the parameters λ, β and α. The estimates for the upper bound of blow-up time T are also given.

2. Local Existence Results

In this section we shall prove local existence of solutions for nonlinear wave equation (1.1) with initial and boundary conditions (1.2), (1.3) by using a fixed-point theorem.

We first give some notations below. Denote $C^0(0,T,V)$ and $C^1(0,T,V)$ respectively by

$$C^0(0,T,V) = \{u : [0,T) \to V | u \text{ is continuous}\}$$

and

$$C^1(0,T,V) = \{u : [0,T) \to V | u \text{ and } u_t \text{ is continuous}\}.$$

Let

$$H1 = C^1(0,T,L^2(\Omega) \cap C^0(0,T,H_0^1(\Omega)),$$

and

$$H2 = C^2(0, T, L^2(\Omega) \cap C^1(0, T, H_0^1(\Omega)),$$

with the induced norms

$$\|u\|_{H1} = \sup_{0 \le t < T} (\|u_t\|_{L^2} + \|\nabla u\|_{L^2}),$$

and

$$\|u\|_{H2} = \sup_{0 \le t < T} (\|u_{tt}\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla_t\|_{L^2}).$$

respectively.

We say $f \in W^{1,2}(0, T, L^2(\Omega))$, it means that $f \in L^2(0, T, L^2(\Omega))$ and $f_t \in L^2(0, T, H_0^1(\Omega))$.

Before proving the existence theorem for nonlinear wave equations, we need the existence result for linear wave equation which is given by Lions [13, p. 95] and Haraux [4,p. 31].

Lemma 2.1. *Assume that* $f \in W^{1,2}(0, T, L^2(\Omega))$ *and that* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$. *Then the linear problem*

$$\begin{aligned} &u_{tt} - \Delta u + f(t, x) = 0,\ x \in \Omega,\ t \in [0, T) \\ &u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),\ x \in \Omega \\ &u(x, t) = 0 \text{ on } \partial\Omega \times (0, T) \end{aligned} \tag{2.1}$$

has a unique solution $u \in H2$.

By using Divergence theorem and Hölder inequality, we can get the following lemma.

Lemma 2.2. (Apriori Estimate). *Let* u *be a solution of* (2.1). *Then we have the inequality*

$$\|Du\|_2(t) \le \|Du\|_2(0) + \int_0^t \|f\|_2(r)dr, \tag{2.2}$$

where $Du = (u_t, \nabla u)$ *and* $\|Du\|_2^2(t) = \int_\Omega (u_t^2 + |\nabla u|^2)dx.$

By Sobolev inequality and Hölder inequality, we also obtain the following trivial result.

Lemma 2.3. *If* $u, v \in H2$, *then* $u^3,\ u^2v,\ uv^2$ *and* $v^3 \in W^{1,2}(0, T, L^2(\Omega))$.

Theorem 2.4. (local existence) *If* $u_0, v_0 \in H_0^1(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$, *then there exists a solution* (u, v) *of* (1.1) – (1.3) *in* $H1 \times H1$.

Proof. Due to the fact that $H^2(\Omega) \cap H_0^1(\Omega)$ is dense in $H_0^1(\Omega)$ and $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, it suffices to consider this problem for $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1, v_1 \in H_0^1(\Omega)$. Let $\{(u^m, v^m)\}_{m \geq 1}$ be a sequence of solutions obtained by considering the following linear wave equations:

$$\text{(2.3)} \qquad \begin{cases} (u_{tt}^{m+1}) - \Delta u^{m+1} = -m_1^2 u^m - 4\lambda(u^m + \alpha v^m)^3 - 2\beta u^m (v^m)^2, \\ (v_{tt}^{m+1}) - \Delta v^{m+1} = -m_2^2 v^m - 4\alpha\lambda(u^m + \alpha v^m)^3 - 2\beta (u^m)^2 v^m, \end{cases}$$

$$\text{(2.4)} \qquad \begin{cases} u^{m+1}(x,0) = u_0(x), \ (u^{m+1})_t(x,0) = u_1(x), \\ v^{m+1}(x,0) = v_0(x), \ (v^{m+1})_t(x,0) = v_1(x) \ \text{ in } \Omega, \\ u^{m+1}(x,t) = v^{m+1}(x,t) = 0 \text{ on } \partial\Omega \times (0,T) \end{cases}$$

with $u^1(x,t) = u_0(x), v^1(x,t) = v_0(x)$.

The existence and uniqueness of the solution $(u^m, v^m) \in H2$ of (2.3), (2.4) is guaranteed by Lemma 2.1 since the right hand sides of (2.3) belong to $W^{1,2}(0, T, L^2(\Omega))$ due to Lemma 2.3. By Lemma 2.2, we have

$$\text{(2.5)} \qquad \begin{aligned} \|Du^{m+1}\|_2(t) &\leq \|Du^{m+1}\|_2(0) \\ &\quad + \int_0^t \|m_1^2 u^m + 4\lambda(u^m + \alpha v^m)^3 + 2\beta u^m (v^m)^2\|_2(r)dr, \end{aligned}$$

$$\text{(2.6)} \qquad \begin{aligned} \|Dv^{m+1}\|_2(t) &\leq \|Dv^{m+1}(0)\|_2 \\ &\quad + \int_0^t \|m_2^2 v^m + 4\alpha\lambda(u^m + \alpha v^m)^3 + 2\beta v^m (u^m)^2\|_2(r)dr. \end{aligned}$$

Denoted by

$$\text{(2.7)} \qquad \beta_1 \equiv \|Du^{m+1}\|_2(0) = \left(\|u_1\|_2^2 + \|\nabla u_0\|_2^2\right)^{\frac{1}{2}}.$$

$$\text{(2.8)} \qquad \beta_2 \equiv \|Dv^{m+1}\|_2(0) = \left(\|v_1\|_2^2 + \|\nabla v_0\|_2^2\right)^{\frac{1}{2}}.$$

For $m \geq 1$, define

$$\text{(2.9)} \qquad G_{m1} = m_1^2 \|u^m\|_2 + 2|\beta| \ \|u^m (v^m)^2\|_2 + 4|\lambda| \ \|(u^m + \alpha v^m)^3\|_2,$$

and

$$G_{m2} = m_2^2\|v^m\|_2 + 2|\beta|\ \|v^m(u^m)^2\|_2 + 4|\alpha\lambda|\ \|(u^m + \alpha v^m)^3\|_2. \tag{2.10}$$

Let

$$A^i(t) \equiv \|Dw^i\|_2(t) = \|Du^i\|_2(t) + \|Dv^i\|_2(t),\ i \geq 1, \tag{2.11}$$

where $Dw^i = (Du^i, Dv^i)$ for $w^i = (u^i, v^i)$.

By Sobolev inequality and Hölder inequality, we see that

$$G_{m1} + G_{m2} \leq C\|Dw^m\|_2^3(t). \tag{2.12}$$

From (2.5), we have

$$\begin{aligned} \|Du^2\|_2(t) \leq \beta_1 + &\int_0^t (m_1^2\|u_0\|_2 + 2|\beta|\ \|u_0 v_0^2\|_2 \\ &+4|\lambda|\ \|(u_0 + \alpha v_0)^3\|_2)dr \\ &\leq \beta_1 + G_{11}t. \end{aligned} \tag{2.13}$$

Similarly, we have

$$\begin{aligned} \|Dv^2\|_2(t) \leq \beta_2 + &\int_0^t (m_2^2\|v_0\|_2 + 2|\beta|\ \|v_0 u_0^2\|_2 \\ &+4|\alpha\lambda|\ \|(u_0 + \alpha v_0)^3\|_2)dr \\ &\leq \beta_2 + G_{12}t. \end{aligned} \tag{2.14}$$

Note that

$$A^2(t) \leq \beta_1 + \beta_2 + Ct\|Dw^1\|_2^3(t) \tag{2.15}$$

Define

$$\|w\|_{\infty,\tau} = \sup\{\|Dw\|_2(t) \,|\, 0 \leq t \leq \tau\}.$$

Let M be a positive constant such that $M > \beta_1 + \beta_2$. From (2.16) it follows that $A^1(t) \leq M$, for $0 \leq t \leq \tau$, or $\|w^1\|_{\infty,\tau} \leq M$.

Thus from (2.15), we have

$$A^2(t) \leq \beta_1 + \beta_2 + C\,t\,M^3 \leq M \quad \text{for } 0 \leq t \leq \tau, \tag{2.16}$$

provided that $\tau = \frac{M-\beta_1-\beta_2}{CM^3}$. That is,

$$\|w^2\|_{\infty,\tau} \leq M. \tag{2.17}$$

Suppose that $\|w^m\|_{\infty,\tau} \leq M$. By adding (2.5) and (2.6) and by using (2.12), we have

$$A^{m+1}(t) \leq \beta_1 + \beta_2 + Ct\|Dw^m\|_2^3(t) \leq M, \; 0 \leq t \leq \tau. \tag{2.18}$$

Thus $\|w^{m+1}\|_{\infty,\tau} \leq M$. Therefore we have

$$\|w^m\|_{\infty,\tau} \leq M, \quad \text{for} \quad m \geq 1. \tag{2.19}$$

Next we shall claim that $\{w^m\}_{m\geq 1}$ is a Cauchy sequence in $H1$. From (2.3) and (2.4), we have

$$\begin{aligned} &(u^{m+1} - u^m)_{tt} - \Delta(u^{m+1} - u^m) \\ &= -m_1^2(u^m - u^{m-1}) - 4\lambda[(u^m + \alpha v^m)^3 - (u^{m-1} + \alpha v^{m-1})^3] \\ &\quad -2\beta[u^m(v^m)^2 - u^{m-1}(v^{m-1})^2], \end{aligned} \tag{2.20}$$

$$\begin{aligned} &(v^{m+1} - v^m)_{tt} - \Delta(v^{m+1} - v^m) \\ &= -m_2^2(v^m - v^{m-1}) - 4\alpha\lambda[(u^m + \alpha v^m)^3 - (u^{m-1} + \alpha v^{m-1})^3] \\ &\quad -2\beta[v^m(u^m)^2 - v^{m-1}(u^{m-1})^2], \end{aligned} \tag{2.21}$$

$$\begin{cases} (u^{m+1} - u^m)(x, 0) = 0, \; (u^{m+1} - u^m)_t(x, 0) = 0, \; x \in \Omega, \\ (v^{m+1} - v^m)(x, 0) = 0, \; (v^{m+1} - v^m)_t(x, 0) = 0, \; x \in \Omega, \\ (u^{m+1} - u^m)(x, t) = (v^{m+1} - v^m)(x, t) = 0 \text{ on } \partial\Omega \times (0, T). \end{cases} \tag{2.22}$$

As in the previous argument, we see that

$$\begin{aligned} &\|D(w^{m+1} - w^m)\|_2(t) \\ &\leq \|D(w^{m+1} - w^m)\|_2(0) \\ &+ \int_0^t \{m_1^2\|u^m - u^{m-1}\|_2 + m_2^2\|v^m - v^{m-1}\|_2 \\ &+4|\lambda|(1 + |\alpha|)\|(u^m + \alpha v^m)^3 - (u^{m-1} + \alpha v^{m-1})^3\|_2 \\ &+2|\beta|(\|u^m(v^m)^2 - u^{m-1}(v^{m-1})^2\|_2 \\ &+\|v^m(u^m)^2 - v^{m-1}(u^{m-1})^2\|_2)\}dr. \end{aligned} \tag{2.23}$$

From (2.22), we have $\|D(w^{m+1} - w^m)\|_2(0) = 0$. By (2.19), Hölder inequality and Sobolev inequality, we obtain

$$\|D(w^{m+1} - w^m)\|_2(t) \leq K \int_0^t \|D(w^m - w^{m-1})\|_2(r)dr, \; 0 \leq t \leq \tau, \tag{2.24}$$

where K is a constant depending on $\alpha, \beta, \lambda, m_1, m_2$ and Sobolev constant.

Thus by induction we have

$$\|w^{m+1} - w^m\|_{\infty,\tau} \leq K\tau \|w^m - w^{m-1}\|_{\infty,\tau} \leq (K\tau)^{m-1}\|w^2 - w^1\|_{\infty,\tau} \tag{2.25}$$

for $m \geq 1$. Therefore, for any positive integer p, we get

$$\begin{aligned}
&\|w^{m+p} - w^m\|_{\infty,\tau} \\
&\leq [(K\tau)^{m+p-2} + \ldots + (K\tau)^{m-1}]\|w^2 - w^1\|_{\infty,\tau} \\
&= (K\tau)^{m-1} \cdot \frac{1-(K\tau)^p}{1-K\tau} \cdot \|w^2 - w^1\|_{\infty,\tau}, \ 0 < K\tau < 1. \\
&\leq \frac{(K\tau)^{m-1}}{1-K\tau} \cdot \|w^2 - w^1\|_{\infty,\tau} \to 0 \text{ as } m \to \infty, \text{ if } 0 < K\tau < 1.
\end{aligned}$$

Thus $\{w^m\}_{m\geq 1}$ has a convergent subsequence in $H1$ and the limit function $w = \lim_{m\to\infty} w^m$ in $H1$ is a local solution of (1.1) – (1.3).

Remark. If $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1, v_1 \in H_0^1(\Omega)$, then the solution (u, v) of (1.1) – (1.3) is in $H2 \times H2$.

3. Global Existence

In this section, we shall show local uniqueness and global existence of the solution $w = (u, v)$ for the problem (1.1) – (1.3). Before doing this, we shall prove that $\|Dw\|_2(t)$ is uniformly bounded by a constant (independent of t) for all $0 \leq t < T$, $T \leq \infty$.

We first define an energy function $E(t)$ by

$$\begin{aligned}
E(t) = &\frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 + u_t^2 + v_t^2 + m_1^2 u^2 + m_2^2 v^2 \\
&+ 2\lambda(u + \alpha v)^4 + 2\beta u^2 v^2]dx
\end{aligned} \tag{3.1}$$

Lemma 3.1. *Let* (u, v) *be a solution of* (1.1) – (1.3). *Then*

$$\begin{aligned}
E(t) = &\frac{1}{2}\int_\Omega \{|\nabla u_0|^2 + |\nabla v_0|^2 + u_1^2 + v_1^2 + m_1^2 u_0^2 + m_2^2 v_0^2 \\
&+ 2\lambda(u_0 + \alpha v_0)^4 + 2\beta u_0^2 v_0^2)\}dx.
\end{aligned} \tag{3.2}$$

Proof. we see that $\frac{dE}{dt} = 0$ by using the Divergence theorem, (1.1), (1.2) and (1.3). Thus $E(t) = E(0)$ for all $t > 0$.

Lemma 3.2. *Let* $w = (u, v)$ *be a solution of* (1.1) – (1.3). *Assume that*

$$m_1^2\xi^2 + m_2^2\eta^2 + 2\lambda(\xi + \alpha\eta)^4 + 2\beta\xi^2\eta^2 \geq 0 \text{ for } \xi, \eta \in R. \tag{3.3}$$

Then we have

$$\|Dw\|_2^2(t) \leq 4E(0) \text{ for } t \geq 0. \tag{3.4}$$

Proof. From (3.2) and (3.3), we get

$$\|Du\|_2^2(t) + \|Dv\|_2^2(t) \leq 4E(0) \text{ for } t \geq 0. \tag{3.5}$$

Remark. For the particular case $\lambda \geq 0, \beta \geq 0, \alpha \in R$, (3.3) holds. However, λ or β may be negative in (3.3).

Since the proof of local uniqueness of the solution of (1.1) – (1.3) is standard, we omit it here. And the global solution of (1.1), (1.3) can be obtained in an usual manner due to (3.4). Once we have a local solution in $[0, \tau)$, we set

$$\bar{u}_0(\cdot) = u(\cdot, \tau/2) \in H_0^1(\Omega), \quad \bar{v}_0(\cdot) = v(\cdot, \tau/2) \in H_0^1(\Omega)$$

and

$$\bar{u}_1(\cdot) = u_t(\cdot, \tau/2) \in L^2(\Omega), \quad \bar{v}_1(\cdot) = v_t(\cdot, \tau/2) \in L^2(\Omega).$$

Then we have a local solution $(\bar{u}, \bar{v})$ of (1.1) – (1.3) on $[\tau/2, 3\tau/2)$. By local uniqueness of solution, we have $u = \bar{u}, v = \bar{v}$ on $[\tau/2, \tau)$. Now we have extended the solution (u, v) up to $[0, 3\tau/2)$. Continuing in this way, we then obtain a global solution of (1.1) – (1.3). In conclusion, we have the following result.

Theorem 3.3 (Global Existence). *Assume that* (3.3) *holds*, *then there exists a global solution of* (1.1) – (1.3).

In the following, we shall prove the triviality of the solution provided that the initial data are zero functions. We first derive an essential equality that will be used later.

Lemma 3.4. *Let* $(u, v) \in C^2(R^+, H_0^1(\Omega))$ *be a solution of* (1.1) – (1.3). *Then we have*

$$\begin{aligned} &2\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx \\ &= 2E(0) - \frac{A''(t)}{2} - \int_\Omega [2m_1^2u^2 + 2m_2^2v^2 + 6\lambda(u + \alpha v)^4 + 6\beta u^2v^2]dx, \end{aligned} \tag{3.6}$$

or

$$2\int_\Omega (u_t^2+v_t^2)dx = \frac{A''(t)}{2} + 2E(0) + 2\int_\Omega [\lambda(u+\alpha v)^4 + \beta u^2 v^2]dx, \tag{3.7}$$

where

$$A(t) = \int_\Omega (u^2+v^2)(x,t)\,dx. \tag{3.8}$$

Proof. By differentiating (3.8) once and twice respectively, we obtain

$$A'(t) = 2\int_\Omega (uu_t + vv_t)\,dx \tag{3.9}$$

and

$$\begin{aligned} A''(t) =& 2\int_\Omega (u_t^2+v_t^2)\,dx - 2\int_\Omega \big[|\nabla u|^2 + |\nabla v|^2 \\ &+ m_1^2 u^2 + m_2^2 v^2 + 4\lambda(u+\alpha v)^4 + 4\beta u^2 v^2\big]dx. \end{aligned} \tag{3.10}$$

By (3.2), we then obtain (3.6). (3.7) follwos directly from (3.10) and (3.6).

Theorem 3.5. *Let* $(u,v) \in C^2(R^+, H_0^1(\Omega))$ *be a solution of* (1.1) – (1.3). *Assume that*

$$m_1^2\xi^2 + m_2^2\eta^2 + 3\lambda(\xi+\alpha\eta)^4 + 3\beta\xi^2\eta^2 \geq 0 \quad for \quad \xi,\eta \geq 0 \tag{3.11}$$

holds. If $u_0 = 0 = v_0$, $u_1 = 0 = v_1$, *then the only global solution of* (1.1) – (1.3) *is the trivial solution.*

Proof. From the assumptions and (3.1), we have $E(0) = 0$. By (3.6) and (3.11), we get

$$\begin{aligned} & 2\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx \\ &= -\frac{A''(t)}{2} - \int_\Omega [2m_1^2u^2 + 2m_2^2v^2 + 6\lambda(u+\alpha v)^4 + 6\beta u^2v^2]dx \\ &\leq -\frac{A''(t)}{2}. \end{aligned} \tag{3.12}$$

From (3.11), using Poincaré inequality yields

$$A''(t) + 2C_\Omega A(t) \leq 0, \tag{3.13}$$

where C_Ω is a Poincaré constant. Note that $A(0) = 0,\ A'(0) = 0$, we then obtain $A(t) \leq 0$ for $t \geq 0$. Hence $A(t) \equiv 0$ for $t \geq 0$.

4. Blow-Up of Solutions

In this section, we shall discuss the blow-up property of solutions for a system (1.1) – (1.3). Before doing this, let us give the following two lemmas that will be used later.

Lemma 4.1. *Let $\delta > 0$ and let $b(t) : R^+ \to R^+$ be a C^2-function satisfying*

$$b''(t) - 4(\delta+1)\, b'(t) + 4(\delta+1)b(t) \geq 0 \quad for \quad t \geq 0. \tag{4.1}$$

If

$$b'(0) > r_2 b(0), \tag{4.2}$$

then $b'(t) > 0$ for $t > 0$, where $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$ is the smallest root of the quadratic equation $r^2 - 4(\delta+1)r + 4(\delta+1) = 0$.

Proof. Let r_1 and r_2 be the roots of $r^2 - 4(\delta+1)r + 4(\delta+1) = 0$ with $r_2 \leq r_1$. Then (4.1) is equivalent to

$$(D - r_1)(D - r_2)b(t) \geq 0, \quad \text{where} \quad D = \frac{d}{dt}. \tag{4.3}$$

Integrating (4.3) from 0 to t, yeilds

$$b'(t) \geq r_2 b(t) + [b'(0) - r_2 b(0)]e^{r_1 t}. \tag{4.4}$$

By (4.2), we obtain $b'(t) > 0$ for $t > 0$.

Lemma 4.2. *If $J(t)$ is a non-increasing function on $[t_0, \infty)$, $t_0 \geq 0$ and satisfies the differential inequality*

$$J'(t)^2 \geq a + bJ(t)^{2+\frac{1}{\delta}} \quad for \quad t \geq t_0, \tag{4.5}$$

where $a > 0$ and $b \in R$, then there exists a finite positive number T^* such that $\lim_{t\to T^*} -J(t) = 0$ and an upper bound of T^* can be estimated respectively in the following cases:

(i) *when $b < 0$, and $J(t_0) < \min\left\{1, \sqrt{\frac{a}{-b}}\right\}$,*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)}, \tag{4.6}$$

(ii) *when* $b = 0$,

$$T^* \leq t_0 + \frac{J(t_0)}{\sqrt{a}}, \tag{4.7}$$

(iii) *when* $b > 0$,

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left\{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\right\}, \tag{4.8}$$

where $c = \left(\frac{a}{b}\right)^{2+\frac{1}{\delta}}$.

Proof. (i) Since $\sqrt{c^2 - d^2} \geq c - d$ for $c \geq d > 0$, from (4.5) it follows that,

$$\begin{aligned} J'(t) &\leq -\sqrt{a} + \sqrt{-b}J(t)^{1+\frac{1}{2\delta}} \\ &\leq -\sqrt{a} + \sqrt{-b}J(t) \quad \text{for} \quad t \geq t_0. \end{aligned} \tag{4.9}$$

Thus we get

$$J(t) \leq [J(t_0) - \sqrt{-(a/b)}]\, e^{(t-t_0)\sqrt{-b}} + \sqrt{-(a/b)}, \tag{4.10}$$

and there exists a positive $T^* < \infty$ such that $\lim_{t\to T^*-} J(t) = 0$, and an upper bound of T^* is given by (4.6).

(ii) When $b = 0$, from (4.5) we have

$$J(t) \leq J(t_0) - \sqrt{a}(t - t_0) \quad \text{for} \quad t \geq t_0.$$

Thus there exists $T^* < \infty$ such that $\lim_{t\to T^*-} J(t) = 0$ and an upper bound of T^* is given by (4.7).

(iii) When $b > 0$, from (4.5) we have

$$J'(t) \leq -\sqrt{a\big[1 + [cJ(t)]^{2+\frac{1}{\delta}}\big]}, \tag{4.11}$$

where $c = (a/b)^{2+\frac{1}{\delta}}$.

By using the following inequality

$$m^q + n^q \geq 2^{1-q}(m+n)^q \quad \text{for} \quad m, n > 0 \quad \text{and} \quad q \geq 1 \tag{4.12}$$

with $q = 2 + \frac{1}{\delta}$, we obtain

$$J'(t) \leq -\sqrt{a}\, 2^{\frac{-\delta-1}{2\delta}} [1 + cJ(t)]^{1+\frac{1}{\delta}}. \tag{4.13}$$

By solving the differential inequality (4.13), we obtain

$$J(t) \leq \frac{1}{c}\left\{-1+\left[(1+cJ(t_0))^{-\frac{1}{2\delta}}+\frac{\sqrt{a}}{\delta\, c}2^{-\frac{3\delta+1}{2\delta}}(t-t_0)\right]^{-2\delta}\right\}. \tag{4.14}$$

Hence there exists $T^* < \infty$ such that $\lim_{t\to T^*-} J(t) = 0$, and an upper bound of T^* is given by (4.8).

Hereafter we shall consider the blow-up of the solution under the following assumption: for fixed α, β and λ with $\lambda^2+\beta^2 \neq 0$, we have

(A4) $$m_1^2\xi^2 + m_2^2\eta^2 + \lambda(\xi+\alpha\eta)^4 + \beta\xi^2\eta^2 \leq 0 \quad \text{for} \quad \xi, \eta \in R.$$

Let

$$J(t) = A(t)^{-\delta} \quad \text{for} \quad t \geq 0, 0 < \delta \leq \frac{1}{2}. \tag{4.15}$$

By differentiating (4.16) once and twice respectively, we obtain

$$J'(t) = -\delta\, A(t)^{-\delta-1}A'(t), \tag{4.16}$$

and

$$J''(t) = \delta\, A(t)^{-\delta-2}\left\{(\delta+1)[A'(t)]^2 - A(t)A''(t)\right\}. \tag{4.17}$$

Note that by Schwarz inequality and the triangle inequality, we obtain

$$[A'(t)]^2 \leq 4A(t)\int_\Omega (u_t^2+v_t^2)dx. \tag{4.18}$$

From (4.17) and (4.18), we get

$$J''(t) \leq -\delta\, A(t)^{-\delta-1}K(t) \text{ for } t > 0, \tag{4.19}$$

where

$$K(t) = A''(t) - 4(\delta+1)\int_\Omega (u_t^2+v_t^2)dx. \tag{4.20}$$

By (3.6) and (3.7), we have

(4.21)
$$\begin{aligned} K(t) = & -4(1+2\delta)E(0) + \int_\Omega \left[4\delta(|\nabla u|^2+|\nabla v|^2) + 2(m_1^2u^2+m_2^2v^2)\right]dx \\ & +(4\delta-2)\int_\Omega [m_1^2u^2+m_2^2v^2+2\lambda(u+\alpha v)^4+2\beta u^2v^2]dx. \end{aligned}$$

By (A4), we have

$$A''(t) - 4(\delta+1)\int_\Omega (u_t^2 + v_t^2)dx \geq -4(1+2\delta)\,E(0). \tag{4.22}$$

Now we shall consider three different cases on the sign of the initial energy $E(0)$.

(i) If $E(0) < 0$, by integration of (4.22), we have

$$A'(t) \geq A'(0) - 4(1+2\delta)E(0)t \quad \text{for } t \geq 0.$$

Thus we have $A'(t) > 0$ for $t > t^*$, where

$$t^* = \max\left\{\frac{A'(0)}{4(1+2\delta)E(0)},\ 0\right\}. \tag{4.23}$$

(ii) If $E(0) = 0$, then $A''(t) \geq 0$ for $t \geq 0$. Furthermore, if $A'(0) > 0$, then $A'(t) > 0$ for $t > 0$.

(iii) If $E(0) > 0$, by the triangle inequality, we have

$$A'(t) \leq A(t) + \int_\Omega (u_t^2 + v_t^2)dx. \tag{4.24}$$

From (4.22) and (4.24), we have the following differential inequality

$$A''(t) - 4(\delta+1)\,A'(t) + 4(\delta+1)A(t) + 4(1+2\delta)\,E(0) \geq 0. \tag{4.25}$$

Let

$$b(t) = A(t) + \frac{(1+2\delta)\,E(0)}{1+\delta}, \quad \text{for } \ t > 0.$$

Then $b(t)$ satisfies (4.1). By Lemma 4.1, we obtain $A'(t) > 0$ for $t > 0$, provided that

$$A'(0) > r_2\left\{A(0) + \frac{(1+2\delta)\,E(0)}{1+\delta}\right\}. \tag{4.26}$$

Consequently, we have

Lemma 4.3. *Assume that* (4.15) *holds and that either one of the following statements is satisfied*:

(i) $E(0) < 0$,

(ii) $E(0) = 0$ *and* $A'(0) > 0$,

(iii) $E(0) > 0$ *and* (4.26) *holds*.

Then $A'(t) > 0$ *for* $t > t_0$, *where* $t_0 = t^*$ *is given by* (4.23) *in case* (i) *and* $t_0 = 0$ *in cases* (ii) *and* (iii).

Hereafter, we shall find the estimate for the life span of $A(t)$. From (4.19) and (4.22), we have

$$J''(t) \leq 4\delta(1+2\delta)E(0)A(t)^{-\delta-1} \text{ for } t \geq t_0. \tag{4.27}$$

Note that $J'(t) < 0$ for $t > t_0$ by Lemma 4.3. Hence multiplying (4.27) by $J'(t)$, and then integrating from t_0 to t, we get

$$J'(t)^2 \geq a + bJ(t)^{2+\frac{1}{\delta}} \text{ for } t \geq t_0, \tag{4.28}$$

where

$$a = J'(t_0)^2 - bJ(t_0)^{2+\frac{1}{\delta}} = \delta^2 A(t_0)^{-2\delta-2}\{A'(t_0)^2 - 8E(0)A(t_0)\} \tag{4.29}$$

and

$$b = 8\delta^2 E(0). \tag{4.30}$$

Note that $a > 0 \Leftrightarrow E(0) < \frac{A'(t_0)^2}{8A(t_0)}$.

In the case that $E(0) < 0$, we easily obtain the rough estimate for an upper bound of blow up time $T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}$. For the remaining cases, by Lemma 4.2, we then obtain the following main result.

Theorem 4.4. *Assume that* (4.15) *holds and that either one of the following statements is satisfied*:

(i) $E(0) < 0$,

(ii) $E(0) = 0$ *and* $A'(0) > 0$,

(iii) $\frac{A'(0)^2}{8A(0)} > E(0) > 0$ *and* (4.26) *holds*.

Then any solution (u, v) *of* (1.1) – (1.3) *blows up at time* T^* *in the sense that* $\lim_{t \to T^*-} A(t) = \infty$. *In case* (i),

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}.$$

Furthermore, if $J(t_0) \leq \min\left\{1, \sqrt{\frac{a}{-b}}\right\}$, *then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln\left[\sqrt{\frac{a}{-b}} \Big/ \left[\sqrt{\frac{a}{-b}} - J(t_0)\right]\right].$$

In case (ii),

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}.$$

In case (iii),

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left\{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\right\}.$$

Here $c = (a/b)^{2+\frac{1}{\delta}}$ *with* $a = \delta^2 A(t_0)^{-2\delta-2}\{A'(t_0)^2 - 8E(0)A(t_0)\}$ *and* $b = 8\delta^2 E(0)$. *Note that in case* (i), $t_0 = t^*$ *is given in* (4.27) *and* $t_0 = 0$ *in cases* (ii) *and* (iii).

Remark. In this paper we give some sufficient conditions for global existence and blow up of solutions. The estimates of the lifespan of solutions are given. Some remarks should be mentioned below.

(i) The upper bounds obtained in Theorem 4.4 is not optimal.

(ii) In this paper we use the norm $\sup_{0 \leq t < T}(\|u_t\|_{L^2} + \|\nabla u\|_{L^2})$ that is greater than the norm $\sup_{0 \leq t < T}(\frac{\partial}{\partial t}\|u\|_{L^2} + \|\nabla u\|_{L^2})$. Thus the class we considered is more restricted.

(iii) The classification on the coefficients on λ and β into two categories - global existence and blow up is not completely solved yet. It is more complicated and will be the subject of our future research.

(iv) The domain Ω may be unbounded. In this case, we consider the class of functions with compact support. Analogous results will be obtained.

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