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WEAK TYPE ESTIMATES FOR COMMUTATORS ON HERZ-TYPE SPACES

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Abstract. We consider the commutator of Calderón-Zygmund operator and multiplication operator by b, that is, the commutator of Coifman, Rochberg and Weiss. We study the boundedness of this operator on the Hardy spaces associated with Herz spaces. We show this commutator is bounded from $H\dot{K}_q^{\alpha,p}$ (Hardy pace) to $\dot{K}_q^{\alpha,p,\infty}$ (weak Herz space).

1. Introduction

Let b be a locally integrable function on R^n and let T be a Calderón-Zygmund operator (see section 2). Consider the commutator operator $[b, T]$ defined by

$$[b, T]f = b \cdot Tf - T(bf).$$

Coifman, Rochberg and Weiss [1] proved that $[b, T]$ is a bounded operator on $L^q(R^n), 1 < q < \infty$, when b is a BMO function.

Li and Yang [4] obtained the boundedness of singular integrals on Herz space $\dot{K}_q^{\alpha,p}(R^n)$. Lu and Yang [6] obtained the boundedness on the Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(R^n)$. Lu and Yang [7] proved the that $[b, T]$ is a bounded operator on $\dot{K}_q^{\alpha,p}(R^n)$, where $-n/q < \alpha < n(1 - 1/q)$.

This commutator is not bounded on $\dot{K}_q^{\alpha,p}(R^n)$, where $\alpha = n(1 - 1/q)$. Lu and Yang [7] defined the space $H\dot{K}_{q,b}^{\alpha,p}(R^n)$ (a variant of ordinary Herz-type Hardy space) and show the boundedness of $[b, T]$ when $\alpha = n(1 - 1/q)$.

But the space $H\dot{K}_{q,b}^{\alpha,p}(R^n)$ depends on a function b. In this paper we define weak Herz space $\dot{K}_q^{\alpha,p,\infty}(R^n)$ and prove $[b, T]$ is bounded from $H\dot{K}_q^{\alpha,p}(R^n)$ to $\dot{K}_q^{\alpha,p,\infty}(R^n)$.

In section 4 we shall consider the maximal operator associated with the commutator of the Bochner–Riesz operator.

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2. Definitions and Notations

The following notation is used: For a set $E \subset R^n$ we denote the Lebesgue measure of E by $|E|$. We denote a characteristic function of E by χ_E. We write $B_k = \{x \in R^n; |x| \le 2^k\}$, $A_k = B_k \setminus B_{k-1} = \{x \in R^n; 2^{k-1} < |x| \le 2^k\}$ and $\chi_k = \chi_{A_k}$ where $k \in \mathbb{Z}$.

First we define homogeneous Herz spaces (see [2], [4]).

Let $0 < p \le \infty$, $1 < q < \infty$ and $\alpha \in R$.

Definition 1. $(\dot{K}_q^{\alpha,p})$

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L^q_{\text{loc}}(R^n \setminus \{0\}); \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\int_{A_k} |f(x)|^q dx \right)^{p/q} \right\}^{1/p}.$$

(Here the usual modification are made when $p = \infty$.)

We define weak L^q spaces and weak Herz spaces.

Definition 2. (weak L^q)

$$L^{q,\infty}(R^n) = \{f; \|f\|_{L^{q,\infty}} < \infty\},$$

where

$$\|f\|_{L^{q,\infty}} = \sup_{\lambda > 0} \lambda |\{x \in R^n; |f(x)| > \lambda\}|^{1/q}.$$

Definition 3. (weak $\dot{K}_q^{\alpha,p}$)

$$\dot{K}_q^{\alpha,p,\infty}(R^n) = \left\{f \in L^q_{\text{loc}}(R^n \setminus \{0\});\ \|f\|_{\dot{K}_q^{\alpha,p,\infty}} < \infty\right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p,\infty}} = \sup_{\lambda > 0} \lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \big|\{x \in A_k;\ |f(x)| > \lambda\}\big|^{p/q} \right\}^{1/p}.$$

Remark. $\dot{K}_q^{0,q} = L^q$ and $\dot{K}_q^{0,q,\infty} = L^{q,\infty}$.

Following [6] and [7], we define atoms and the Hardy spaces associated with Herz spaces.

Definition 4. (central atom) A function $a(x)$ is a central (α, q)–atom if

(i) supp $a \subset B(0,r)$ for some $r > 0$,

(ii) $\|a\|_{L^q} \cdot \; r^{-\alpha}$,

(iii) $\int_{R^n} a(x)dx = 0$.

Definition 5. $(H\dot{K}_q^{\alpha,p})$ We define the Hardy space $H\dot{K}_q^{\alpha,p}(R^n)$ by

$$H\dot{K}_q^{\alpha,p}(R^n) = \Bigg\{ f \in \mathcal{S}';\ f = \sum_{j=1}^{\infty} c_j a_j,\ a_j \text{ are central } (\alpha,q)\text{–atoms supported on } B(0,2^j),\ \|a_j\|_{L^q} \cdot \; 2^{-j\alpha} \text{ and } \sum_{j=-\infty}^{\infty} |c_j|^p < \infty \Bigg\},$$

and let the quasi-norm $\|f\|_{H\dot{K}_q^{\alpha,p}}$ be the infimum of $\left(\sum_{j=-\infty}^{\infty} |c_j|^p\right)^{1/p}$ over all representations of f.

$BMO(R^n)$ is the John-Nirenberg space (see [10], chapter 8).

Definition 6. (BMO).

$$BMO(R^n) = \{f \in L^1_{\text{loc}}(R^n); \|f\|_{BMO} < \infty\},$$

where

$$\|f\|_{BMO} = \sup_{Q;\text{ball}} \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx,$$

and $f_Q = \frac{1}{|Q|} \int_Q f(y)dy$.

Finally we define Calderón–Zygmund operators.

Definition 7. (Calderón–Zygmund operator). Let T be a bounded linear operato from $\mathcal{D}$ to $\mathcal{D}'$ and we assume that there exists a function $K(x,y)$ defined on $\{(x,y) \in R^n \times R^n;\ x \neq y\}$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

if f is in $\mathcal{D}$ and x does not in the support of f.

We say T is a Calderón–Zygmund operator if there exists $0 < \varepsilon \cdot \; 1$ and T satisfies the following conditions.

$$|K(x,y)| \cdot \; \frac{C}{|x-y|^n}$$

$$|K(x,y) - K(x,y+z)| \cdot \; \frac{C|z|^\varepsilon}{|x-y|^{n+\varepsilon}} \quad \text{if} \quad 2|z| < |x-y|.$$

$$|K(x+z,y) - K(x,y)| \cdot \; \frac{C|z|^\varepsilon}{|x-y|^{n+\varepsilon}} \quad \text{if} \quad 2|z| < |x-y|.$$

T extends to a continuous operator on $L^2(\mathbb{R}^n)$.

Remark. A Calderón–Zygmund operator T is bounded on L^q where $1 < q < \infty$ (see [3] or [10]), and bounded on Herz spaces (for the details, see [4]).

3. Commutator

Definition 8. (The commutator of Coifman, Rochbeg and Weiss) We define the commutator operator $[b, T]$ by

$$[b, T]f = b \cdot Tf - T(bf).$$

Remark. If T is a Calderón–Zygmund operator and $b \in BMO(R^n)$, then $[b, T]$ is a bounded operator on $L^q(R^n)$ when $1 < q < \infty$ (see [1]).

Lu and Yang [7] proved the following:

Theorem A. *Let T be a Calderón-Zygmund operator and $b \in BMO(R^n)$. If $1 < q < \infty, -n/q < \alpha < n(1 - 1/q)$ and $0 < p \leq \infty$, then $[b, T]$ is bounded on $\dot{K}_q^{\alpha,p}(R^n)$.*

This theorem is not true when $\alpha = n(1 - 1/q)$. As a substitute for this they obtained the next theorem.

They defined the space $H\dot{K}_{q,b}^{\alpha,p}$ as follows.

Definition 9. A function $a(x)$ is a central (α, q, b)–atom if a satisfies (i), (ii), (iii) and (iv) $\int_{R^n} a(x)b(x)dx = 0$.

Definition 10. $(H\dot{K}_{q,b}^{\alpha,p})$ The Hardy space $H\dot{K}_{q,b}^{\alpha,p}(R^n)$ is defined by

$$H\dot{K}_{q,b}^{\alpha,p}(R^n) = \Bigg\{ f \in \mathcal{S}'; f = \sum_{j=-\infty}^{\infty} c_j a_j,\ a_j \text{ are central } (\alpha, q, b)\text{–atoms supported on } B(0, 2^j),\ \|a_j\|_{L^q} \leq 2^{-j\alpha} \text{ and } \sum_{j=-\infty}^{\infty} |c_j|^p < \infty \Bigg\}.$$

Theorem B. *Let T be a Calderón-Zygmund operator and $b \in BMO(R^n)$. If $1 < q < \infty$ and $0 < p \leq \infty$, then $[b, T]$ is bounded from $H\dot{K}_{q,b}^{n(1-1/q),p}(R^n)$ to $\dot{K}_q^{n(1-1/q),p}(R^n)$.*

Our result is the following:

Theorem 1. *Let T be a Calderón-Zygmund operator and $b \in BMO(R^n)$. If $1 < q < \infty$ and $0 < p \leq 1$, then $[b, T]$ is bounded from $H\dot{K}_q^{n(1-1/q),p}(R^n)$ to $\dot{K}_q^{n(1-1/q),p,\infty}(R^n)$;*

$$\|[b,T]f\|_{\dot{K}_q^{n(1-1/q),p,\infty}} \leq C\|f\|_{\dot{K}_q^{n(1-1/q),p}},$$

where C is a positive constant depending only on n, p, q and T.

Remark. In section 7 we shall show the counterexample that Theorem 1 is not true for $p > 1$.

Remark. Lu and Yang [7] showed the boundedness of fractional integral operators on $H\dot{K}_{q,b}^{\alpha,p}(R^n)$. We can also prove the weak type boundedness of fractional integral operators on $H\dot{K}_q^{\alpha,p}(R^n)$.

4. The Bochner–Riesz Operator

Definition 11. We define the Bochner–Riesz operator B_r^δ by using Fourier transform.

$$(B_r^\delta f)\hat{}(\xi) = (1 - r^2|\xi|^2)_+^\delta \hat{f}(\xi).$$

Let $b \in BMO(R^n)$ and we define the maximal operator $B_{*,b}^\delta$ associated with the commutator of the Bochner–Riesz operator by

$$B_{*,b}^\delta f(x) = \sup_{r>0} |[b, B_r^\delta]f(x)|.$$

We also set $B_*^\delta f(x) = \sup_{r>0} |B_r^\delta f(x)|$.

Lu and Yang [7] proved the next theorem.

Theorem C. *Let $1 < q < \infty$, $0 < p \leq \infty$, $\delta > (n-1)/2$ and $b \in BMO(R^n)$. Then $B_{*,b}^\delta$ is bounded from $H\dot{K}_{q,b}^{n(1-1/q),p}(R^n)$ to $\dot{K}_q^{n(1-1/q),p}(R^n)$.*

Our result is the following:

Theorem 2. *Let $1 < q < \infty, 0 < p \leq 1, \delta > (n-1)/2$ and $b \in BMO(R^n)$. Then $B_{*,b}^\delta$ is bounded from $H\dot{K}_q^{n(1-1/q),p}(R^n)$ to $\dot{K}_q^{n(1-1/q),p,\infty}(R^n)$.*

5. Proof of Theorem 1

We need the following elementary lemmas.

Lemma 1. *If T is a Calderón–Zygmund operator then*

$$|Tf(x)| \cdot \; C\int_{R^n} \frac{|f(y)|}{|x-y|^n}\,dy \quad if \quad x \notin \text{supp} \;\; f.$$

Lemma 2. *Let T be a Calderón-Zygmund operator. We assume that a function $a(x)$ satisfies*

$$\text{supp } a \subset B_j \quad and \quad \int_{R^n} a(x)dx = 0.$$

Then we have the following:

$$If \;\; x \in A_k \;\; and \;\; j \cdot \;\; k-2, \;\; then \;\; |T(a)(x)| \cdot \;\; C2^{j\varepsilon}2^{-k(n+\varepsilon)}\|a\|_{L^1}.$$

Lemma 3. ([10, p. 201]). *Let $b \in BMO(R^n)$. Then we have*

$$\Big(\frac{1}{|B_k|}\int_{B_k}|b(y)-\tilde{b}_j|^q dy\Big)^{1/q} \cdot \;\; C\|b\|_{BMO}|k-j|,$$

where $\tilde{b}_j = \frac{1}{|B_j|}\int_{B_j} b(y)dy$.

Now we begin to prove the theorem. Let $1/q + 1/q' = 1$ and $\alpha = n(1-1/q)$. Suppose $f = \sum_{j=-\infty}^{\infty} c_j a_j$ (see Def. 4) and $\lambda > 0$ be fixed. We write

$$\begin{aligned}
&\lambda^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\big|\{x \in A_k; |[b,T]f(x)| > 3\lambda\}\big|^{p/q} \\
&\cdot \;\; C\Big\{\lambda^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\big|\{x \in A_k; |\sum_{j=-\infty}^{k-2} c_j \;[b,T]a_j(x)| > 2\lambda\}\big|^{p/q} \\
&\qquad + \lambda^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\big|\{x \in A_k; |[b,T]\big(\textstyle\sum_{j=k-1}^{\infty} c_j a_j\big)(x)| > \lambda\}\big|^{p/q}\Big\} \\
&= C(J_1 + J_2).
\end{aligned}$$

The estimate of J_2 is easy. By the L^q boundedness of the commutator $[b,T]$ ([1]), we have

$$\begin{aligned}
J_2 \cdot \;\; &C\lambda^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\|b\|_{BMO}\lambda^{-1}\big\|\sum_{j=k-1}^{\infty} c_j a_j\big\|_{L^q}\Big)^p \\
\cdot \;\; &C\|b\|_{BMO}^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{\infty} 2^{-j\alpha}\;|c_j|\Big)^p.
\end{aligned}$$

Because $0 < p \cdot \; 1$, we have

$$\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{\infty} 2^{-j\alpha}\ |c_j|\Big)^p \cdot \quad \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=k-1}^{\infty} 2^{-j\alpha p}\ |c_j|^p \cdot \quad C\sum_{j=-\infty}^{\infty} |c_j|^p.$$

To estimate J_1 we write

$$[b,T]a_j(x) = (b(x)-\tilde{b}_j)T(a_j)(x) - T\big((b-\tilde{b}_j)a_j\big)(x),$$

where $\tilde{b}_j = \frac{1}{|B_j|}\int_{B_j} b(y)dy$, and write

$$\begin{aligned} J_1 \cdot \quad & C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\big|\{x\in A_k; |\sum_{j=-\infty}^{k-2} c_j\ (b(x)-\tilde{b}_j)T(a_j)(x)| > \lambda\}\big|^{p/q} \\ & \quad + \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\big|\{x\in A_k; |\sum_{j=-\infty}^{k-2} c_j\ T\big((b-\tilde{b}_j)a_j\big)(x)| > \lambda\}\big|^{p/q}\Big\} \\ & = C(J_{11}+J_{12}). \end{aligned}$$

First we estimate J_{11}.

By Lemma 2, if $x \in A_k$ we have

$$\begin{aligned} \Big|\sum_{j=-\infty}^{k-2} c_j\,(b(x)-\tilde{b}_j)T(a_j)(x)\Big| \cdot \quad & C\sum_{j=-\infty}^{k-2} |c_j|\ 2^{j\varepsilon}2^{-k(n+\varepsilon)}\|a_j\|_{L^1}|b(x)-\tilde{b}_j| \\ & \cdot \quad C\ 2^{-k(n+\varepsilon)}\sum_{j=-\infty}^{k-2} 2^{j\varepsilon}|c_j||b(x)-\tilde{b}_j|. \end{aligned}$$

So we have

$$\begin{aligned} & \big|\{x\in A_k; |\sum_{j=-\infty}^{k-2} c_j\ (b(x)-\tilde{b}_j)T(a_j)(x)| > \lambda\}\big|^{1/q} \\ & \cdot \quad C\lambda^{-1}2^{-k(n+\varepsilon)}\Big\|\sum_{j=-\infty}^{k-2} 2^{j\varepsilon}|c_j|(b-\tilde{b}_j)\chi_k\Big\|_{L^q} \\ & \cdot \quad C\|b\|_{BMO}\,\lambda^{-1}2^{-k(n+\varepsilon-n/q)}\sum_{j=-\infty}^{k-2} 2^{j\varepsilon}|c_j||k-j|, \end{aligned}$$

by lemma 3.

Therefore we obtain

$$\begin{aligned} J_{11} \cdot \quad & C\|b\|_{BMO}^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p}2^{-k(n+\varepsilon-n/q)p}\Big(\sum_{j=-\infty}^{k-2} 2^{j\varepsilon}|c_j||k-j|\Big)^p \\ \cdot \quad & C\|b\|_{BMO}^p \sum_{k=-\infty}^{\infty} 2^{-k\varepsilon p}\sum_{j=-\infty}^{k-2} 2^{j\varepsilon p}|c_j|^p|k-j|^p \end{aligned}$$

$$\cdot \quad C\|b\|_{BMO}^p \sum_{j=-\infty}^{\infty} 2^{j\varepsilon p}|c_j|^p \sum_{k=j+2}^{\infty} 2^{-k\varepsilon p}|k-j|^p \cdot \quad C\|b\|_{BMO}^p \sum_{j=-\infty}^{\infty} |c_j|^p.$$

Next we estimate J_{12}.
By Lemma 1 and Lemma 3, if $x \in A_k$ we have

$$\begin{aligned} |T\big((b-\tilde{b}_j)a_j\big)(x)| \cdot \quad & C2^{-kn}\int |(b(y)-\tilde{b}_j)a_j(y)|dy \\ \cdot \quad & C2^{-kn}\Big(\int_{B_j} |b(y)-\tilde{b}_j|^{q'}dy\Big)^{1/q'}\|a_j\|_{L^q} \\ \cdot \quad & C2^{-kn}\|b\|_{BMO}. \end{aligned}$$

So we have

$$\begin{aligned} \Big|\sum_{j=-\infty}^{k-2} c_j\, T\big((b-\tilde{b}_j)a_j\big)(x)\Big| \cdot \quad & C2^{-kn}\|b\|_{BMO}\sum_{j=-\infty}^{k-2}|c_j| \\ \cdot \quad & C2^{-kn}\|b\|_{BMO}\Big(\sum_{j=-\infty}^{\infty}|c_j|^p\Big)^{1/p}. \end{aligned}$$

Let $k_0 \in \mathbb{Z}$ be such that

$$C2^{-k_0 n}\|b\|_{BMO}\Big(\sum_{j=-\infty}^{\infty}|c_j|^p\Big)^{1/p} > \lambda \geq C2^{-(k_0+1)n}\|b\|_{BMO}\Big(\sum_{j=-\infty}^{\infty}|c_j|^p\Big)^{1/p}.$$

Then we have

$$\begin{aligned} J_{12} \cdot \quad & \lambda^p \sum_{k=-\infty}^{k_0} 2^{k\alpha p}|A_k|^{p/q} \\ \cdot \quad & C\lambda^p 2^{k_0 np} \cdot \quad C\|b\|_{BMO}^p \sum_{j=-\infty}^{\infty}|c_j|^p. \end{aligned}$$

We obtain the desired result.

6. Proof of Theorem 2

We need the following elementary lemmas.

Lemma 4. ([5, p. 121]) *We can write*

$$B_r^\delta f(x) = r^{-n}\int_{R^n} K^\delta\left(\frac{x-y}{r}\right) f(y)dy$$

and K^δ *satisfies*

$$|K^\delta(x)| + |\nabla K^\delta(x)| \cdot \ \frac{C}{(1+|x|)^{\delta+(n+1)/2}}.$$

By using this lemma we obtain the following:

Lemma 5. *We assume* $\operatorname{supp} f \subset B(0,R)$, *then*

$$|B_r^\delta f(x)| \cdot \ \frac{C}{r^n}\left(1+\frac{|x|}{r}\right)^{-(\delta+(n+1)/2)} \int |f(y)|dy \quad where \quad |x| \geq 2R.$$

Furthermore if $\int f(y)dy = 0$, *then we have*

$$|B_r^\delta f(x)| \cdot \ \frac{C}{r^{n+1}}\left(1+\frac{|x|}{r}\right)^{-(\delta+(n+1)/2)} \int |y|\ |f(y)|\, dy \quad where \quad |x| \geq 2R.$$

Now we begin to prove the theorem. The proof of Theorem 2 is similar to that of Theorem 1. So we shall show only the outline of the proof.

Let $\alpha = n(1-1/q)$ and $f = \sum_{j=-\infty}^{\infty} c_j a_j$ and we write

$$\begin{aligned}
&\lambda^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\big|\{x \in A_k; B_{*,b}^\delta f(x) > 3\lambda\}\big|^{p/q} \\
&\cdot \ C\Big\{\lambda^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\big|\{x \in A_k; \sum_{j=-\infty}^{k-2} |c_j|\ B_{*,b}^\delta(a_j)(x) > 2\lambda\}\big|^{p/q} \\
&\qquad + \lambda^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p}|x \in A_k; |B_{*,b}^\delta\big(\textstyle\sum_{j=k-1}^{\infty} c_j a_j\big)(x)| > \lambda\}\big|^{p/q}\Big\} \\
&= C(J_1 + J_2).
\end{aligned}$$

By the weighted L^q boundedness of B_*^δ (see [9]) and the argument in [8, p. 962], (note that $\delta + (n+1)/2 > n$), we have $B_{*,b}^\delta$ is bounded on L^q. Therefore we obtain

$$J_2 \cdot \ C\|b\|_{BMO}^p \sum_{j=-\infty}^{\infty} |c_j|^p.$$

To estimate J_1 we write

$$[b, B_r^\delta](a_j)(x) = (b(x) - \tilde{b}_j)B_r^\delta(a_j)(x) - B_r^\delta\big((b-\tilde{b}_j)a_j\big)(x),$$

and

$$
\begin{aligned}
J_1 \cdot\ & C\Big\{\lambda^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\big|\{x \in A_k; \sum_{j=-\infty}^{k-2} |c_j|\ |b(x) - \tilde{b}_j| B_*^{\delta}(a_j)(x) > \lambda\}\big|^{p/q} \\
& \qquad + \lambda^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\big|\{x \in A_k; \sum_{j=-\infty}^{k-2} |c_j|\ B_*^{\delta}\big((b - \tilde{b}_j)a_j\big)(x) > \lambda\}\big|^{p/q}\Big\} \\
& = C(J_{11} + J_{12}),
\end{aligned}
$$

where $\tilde{b}_j = \frac{1}{|B_j|}\int_{B_j} b(y)dy$.

First we estimate J_{11}.

By using Lemma 5, if $x \in A_k$ we have

$$|B_*^{\delta}(a_j)(x)| \cdot\ C2^{-k(n+1)}\ 2^j.$$

Therefore by lemma 3, we obtain

$$
\begin{aligned}
J_{11} \cdot\ & \lambda^p \sum_{k=\infty}^{\infty} 2^{k\alpha p}\big|\{x \in A_k; \sum_{j=-\infty}^{k-2} 2^j\ |c_j|\ |b(x) - \tilde{b}_j| > C2^{k(n+1)}\lambda\}\big|^{p/q} \\
\cdot\ & C \sum_{k=\infty}^{\infty} 2^{k(\alpha-n-1)p}\Big\| \sum_{j=-\infty}^{k-2} 2^j\ |c_j|\ |b - \tilde{b}_j|\chi_k\Big\|_{L^q}^p \\
\cdot\ & C\|b\|_{BMO}^p \sum_{k=\infty}^{\infty} 2^{k(\alpha-n-1)p}\Big(\sum_{j=-\infty}^{k-2} 2^j\ |c_j|\ 2^{kn/q}|k-j|\Big)^p \\
\cdot\ & C\|b\|_{BMO}^p \sum_{j=-\infty}^{\infty} |c_j|^p.
\end{aligned}
$$

Next we estimate J_{12}.

By Lemma 5, if $x \in A_k$ we have

$$B_*^{\delta}\big((b - \tilde{b}_j)a_j\big)(x) \cdot\ C\|b\|_{BMO}^p\ 2^{-kn}.$$

So we have

$$J_{22} \cdot\ C\|b\|_{BMO}^p \sum_{j=-\infty}^{\infty} |c_j|^p.$$

We obtain the desired result.

7. Counterexamples

We shall show Theorem 1 is not true for $p > 1$. For the simplicity, we consider on R^1. Let H be the Hilbert transform $Hf(x) = \text{p.v.} \int_{R^1} f(y)/(x-y)dy$.

Counterexample 1. *There exists a function* $b \in BMO(R^1)$ *such that* $[b, H]$ *is not bounded from* $H\dot{K}_q^{1-1/q,p}(R^1)$ *to* $\dot{K}_q^{1-1/q,p,\infty}(R^1)$ *when* $p > 1$.

Proof. For positive integers k and N, we define a function a_k by

$$a_k(x) = \begin{cases} 2^k, & \text{if } \ 2^{-k-1} < x < 2^{-k}, \\ -2^k, & \text{if } \ -2^{-k} < x < -2^{-k-1}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_N(x) = \sum_{k=1}^{N} \frac{a_k(x)}{k}.$$

Then a_k is a central $(1-1/q, q)$–atom and $\|f_N\|_{H\dot{K}_q^{1-1/q,p}} \cdot \ C$ because $p > 1$.

Let $b(x) = \chi_{\{x<0\}}(x) \in BMO(R^1)$. Note that

$$[b, H]f_N(x) = -H(bf_N)(x), \quad \text{where} \quad x > 2$$

and

$$|H(bf_N)(x)| \geq C\Big(\sum_{k=1}^{N} \frac{1}{k}\Big)\frac{1}{x} \quad \text{where} \quad x > 2.$$

Therefore we have $\limsup_{N\to\infty} \|[b, H]f_N\|_{\dot{K}_q^{1-1/q,p,\infty}} = \infty$. ∎

Counterexample 1. *There exists a function* $b \in BMO(R^1)$ *such that* $[b, H]$ *is not bounded from* $H\dot{K}_q^{-1/q,p}(R^1)$ *to* $\dot{K}_q^{-1/q,p,\infty}(R^1)$.

Proof. Let $f(x) = \chi_{[2,3]}(x) - \chi_{[-3,-2]}(x) \in H\dot{K}_q^{-1/q,p}(R^1)$ and we define $b(x) = \chi_{\{x>2\}}(x) \in BMO(R^1)$.

Note that $[b, H]f(x) = -H(bf)(x)$ when $|x| < 1$, and $|H(bf)(x)| > 1/4$ when $|x| < 1$.

So we have

$$\sum_{k=-\infty}^{0} 2^{-kp/q} \Big|\{x \in A_k; |[b, H]f(x)| > 1/4\}\Big|^{p/q} = \sum_{k=-\infty}^{0} 2^{-kp/q}|A_k|^{p/q} = \infty. \quad ∎$$

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