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ON PERTURBATION OF K-REGULARIZED RESOLVENT FAMILIES

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Abstract. In this paper we study additive perturbations of a linear Volterra integral equation defined in a Banach space X by means of k-regularized resolvent families. We give also a representation formula for the generator of such family, under certain conditions on the scalar kernel $k(t)$.

1. Introduction

Consider the following Volterra equation of convolution type

$$u(t) = \int_0^t a(t-s)Au(s)ds + f(t), \quad t \geq 0 \tag{1.1}$$

where A is a closed and linear operator defined on a Banach space X.

Let $k \in C(\mathbb{R}_+)$ be a scalar kernel. We recall that a family $\{R(t)\}_{t\geq 0} \subseteq \mathcal{B}(X)$ is called a k-regularized resolvent for (1.1) if the following conditions are satisfied

(R1) $R(t)$ is strongly continuous on $\mathbb{R}_+$ and $R(0) = k(0)I$.

(R2) $R(t)x \in D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.

(R3) The k-regularized resolvent equation holds

$$R(t)x = k(t)x + \int_0^t a(t-s)AR(s)xds,$$

for all $x \in D(A), t \geq 0$.

The notion of k-regularized resolvent has been recently introduced in [7] as well as some properties investigated (see [8]). In this paper we mainly study additive perturbations of (1.1), which generalize a theorem of A. Rhandi [11].

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In the first part, under the assumption that $|k(t)|$ is increasing and satisfies the condition $\limsup_{t\to 0^+} \frac{\|R(t)\|}{|k(t)|} < \infty$ we also give a characterization of the domain of the given operator A in terms of the k-regularized resolvent family. In particular, we obtain the representation of A as the generator of an α-times integrated semigroup.

2. The Domain of A

Using the resolvent method in order to study (1.1), that is, assuming the existence of a family of bounded and linear operators $\{S(t)\}_{t\geq 0}$ which satisfy conditions (R1)-(R3) with $k(t) \equiv 1$, it is natural to ask how to characterize the domain $D(A)$ of the given operator A in terms of the resolvent family. This is important, for instance, in order to study the Favard class in perturbation theory (see [4]).

For very special cases the answer to the above question is well known. For instance, when $a(t) = 1$ or $a(t) = t$, A is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ or a cosine family $\{C(t)\}_{t\in\mathbb{R}}$ and we have:

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \quad \text{exists} \right\}$$

and

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{C(t)x - x}{t^2} \quad \text{exists} \right\}$$

respectively (see [10]).

Recently, a reasonable formula for the generator of resolvent families has been established by assuming very mild conditions on the kernel $a(t)$. See [4] Theorem 2.5 and assumption 2.3.

On the other hand, a new type of operator family has been applied to the study of (1.1). The so called k-regularized resolvent introduced in [7] (see also [5]) generalizes the concept of resolvent family as well as many others. For instance, integrated semigroups, integrated resolvent families and convoluted semigroups falls into the framework of a k-regularized resolvent family.

The main objective in this section is to give a characterization for the domain of the operator A in (1.1) in terms of the k-regularized resolvent $\{R(t)\}_{t\geq 0}$ and the kernel $k(t)$, in the case where $|k(t)|$ is increasing and $\limsup_{t\to 0^+} \|R(t)\|/|k(t)| < \infty$. As a remarkable consequence, for $k(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ and $a(t) \equiv 1$ we obtain the representation of A as the generator of an α-times integrated semigroup which is of the growth of t^α.

In what follows, we will consider the following assumption on $a \in L^1_{loc}(\mathbb{R}_+)$, and $k \in C(\mathbb{R}_+)$.

$(\mathbf{H_a})$ There exists $\epsilon_{a,k} > 0$ and $t_{a,k} > 0$ such that for all $0 < t \leq t_{a,k}$

$$\left| \int_0^t a(t-s)k(s)ds \right| \geq \epsilon_{a,k} \int_0^t |a(t-s)k(s)|ds.$$

The following is the main result in this section.

Theorem 2.1. *Let A be a closed and densely defined operator on a Banach space X. Suppose* (1.1) *admits a k-regularized resolvent $\{R(t)\}_{t\geq 0}$ such that $|k(t)|$ is increasing and satisfies*

$$\limsup_{t\to 0^+} \frac{\|R(t)\|}{|k(t)|} < \infty.$$

Then under assumption (H_a) *we have*

a) $D(A) = \left\{x \in X : \lim_{t\to 0^+} \dfrac{R(t)x - k(t)x}{(k*a)(t)}\ exists\right\}$

b) $\lim_{t\to 0^+} \dfrac{R(t)x - k(t)x}{(k*a)(t)} = Ax\ for\ all\ x \in D(A).$

Proof. Let $z \in D(A)$. Then by (R2)-(R3), strong continuity of $R(t)$ and the fact that $|k(t)|$ is increasing we obtain

$$\left\|\frac{R(t)z}{k(t)} - z\right\| = \frac{1}{|k(t)|}\left\|\int_0^t a(t-s)AR(s)z\,ds\right\|$$
$$\cdot \left(\int_0^t |a(t-s)|\frac{\|R(s)\|}{|k(s)|}ds\right)\|Az\|.$$

Hence, $\|\frac{R(t)z}{k(t)} - z\| \to 0$ as $t \to 0^+$ for all $z \in D(A)$. The denseness of $D(A)$ and $\limsup_{t\to 0^+} \frac{\|R(t)\|}{|k(t)|} < \infty$ imply that it actually holds for all $z \in X$. Thus, for every $z \in X$ and $\epsilon > 0$ there is $0 < t(\epsilon, z) < \min\{t_{a,k}, 1\}$ such that

$$\left\|\frac{R(t)z}{k(t)} - z\right\| < \epsilon \tag{2.1}$$

for all $t \in (0, t(\epsilon, z))$.

Next, we will prove the assertions (a) and (b).

Define the set $\widetilde{D}(A) := \{x \in X : \lim_{t\to 0^+} \frac{R(t)x - k(t)x}{(k*a)(t)}$ exists$\}$.

Let $x \in D(A)$ be given and define $z = Ax$. We get in particular from (2.1)

$$\left\|\frac{R(t)Ax}{k(t)} - Ax\right\| < \epsilon, \tag{2.2}$$

for all $t \in (0, t(\epsilon, Ax))$. Therefore, using (R3) and ($\mathrm{H_a}$) we have for all $\tau \in (0, t(\epsilon, Ax))$:

$$
\begin{aligned}
&\left\|\frac{R(\tau)x - k(\tau)x}{(k * a)(\tau)} - Ax\right\| \\
&= \frac{1}{|(k*a)(\tau)|}\left\|\int_0^\tau a(\tau - s)AR(s)xds - \int_0^\tau a(\tau - s)k(s)Axds\right\| \\
&= \frac{1}{|(k*a)(\tau)|}\left\|\int_0^\tau a(\tau - s)k(s)\Big[\frac{R(s)}{k(s)}Ax - Ax\Big]ds\right\| \\
&\cdot \frac{1}{|(k*a)(\tau)|}\int_0^\tau |a(\tau - s)k(s)|\epsilon ds = \frac{\epsilon}{\epsilon_{a,k}}.
\end{aligned}
$$

We conclude that $x \in \widetilde{D}(A)$, that is $D(A) \subseteq \widetilde{D}(A)$ and (b) holds.

On the other hand, let $x \in \widetilde{D}(A)$ be given. Then

$$
\lim_{t \to 0^+} \frac{R(t)x - k(t)x}{(k*a)(t)} = y
$$

exists and, for given $\epsilon > 0$ and all $t \in (0, t(\epsilon, x))$, we have by (2.1) and ($\mathrm{H_a}$)

$$
\begin{aligned}
&\left\|\frac{1}{(k*a)(t)}\int_0^t a(t-s)R(s)xds - x\right\| \\
&\qquad = \frac{1}{|(k*a)(t)|}\left\|\int_0^t a(t-s)R(s)xds - \int_0^t a(t-s)k(s)xds\right\| \\
&\qquad = \frac{1}{|(k*a)(t)|}\left\|\int_0^t a(t-s)k(s)\ \square\frac{R(s)}{k(s)}x - x\Big]\,ds\right\| \\
&\qquad \cdot \frac{\epsilon}{|(k*a)(t)|}\int_0^t |a(t-s)k(s)|ds \cdot \frac{\epsilon}{\epsilon_{a,k}}.
\end{aligned}
$$

This proves that $\frac{1}{(k*a)(t)}\int_0^t a(t-s)R(s)xds \longrightarrow x$ as $t \longrightarrow 0^+$.

Next, observe that by (R3)

$$
\begin{aligned}
&\left\|A\ \square\frac{1}{(k*a)(t)}\int_0^t a(t-s)R(s)xds\Big] - y\right\| \\
&= \left\|\frac{1}{(k*a)(t)}\int_0^t a(t-s)AR(s)xds - y\right\| \\
&= \left\|\frac{R(t)x - k(t)x}{(k*a)(t)} - y\right\|,
\end{aligned}
$$

where the right hand side goes to zero as $t \longrightarrow 0^+$. Since A is closed, we obtain $x \in D(A)$ and $Ax = y$. This proves the theorem.

Remarks.

1. If $a(t) = t^\beta$ and $k(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$; $\alpha > -1$ then, by making use of the formula $t^\alpha * t^\beta = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta+1}$ for $\alpha > -1$ and $\beta > -1$, we obtain

$$\frac{R(t)x - k(t)x}{(k*a)(t)} = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \left[\frac{\Gamma(\alpha+1)R(t)x - t^\alpha x}{t^{\alpha+\beta+1}}\right].$$

   Moreover, note that assumption ($\mathrm{H_a}$) is satisfied with $\epsilon_{a,k} = 1$.

2. For $k(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ and $a(t) \equiv 1$, $R(t)$ is an α-times integrated semigroup and our assumption is implied by the condition

$$\|R(t)\| \cdot \ Mt^\alpha; \qquad t \geq 0$$

   which is satisfied in a longer number of examples (see [3] Theorem 4.2).

By taking $\beta = 0$ or $\beta = 1$ in remark 1, we obtain the following results.

Corollary 2.2. *Let A be a closed and densely defined operator on a Banach space X. Assume A is the generator of an α-times integrated semigroup $\{T(t)\}_{t\geq 0}$ such that $\|T(t)\| \cdot \ Mt^\alpha$. Then*

$$Ax = \lim_{t\to 0^+} \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)} \left\{\frac{\Gamma(\alpha+1)T(t)x - t^\alpha x}{t^{\alpha+1}}\right\},$$

for all $x \in D(A)$.

Corollary 2.3. *Let A be a closed and densely defined operator on a Banach space X. Assume A is the generator of an α-times integrated cosine family $\{C(t)\}_{t\geq 0}$ such that $\|C(t)\| \cdot \ Mt^\alpha$. Then*

$$Ax = \lim_{t\to 0^+} \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)} \left\{\frac{\Gamma(\alpha+1)C(t)x - t^\alpha x}{t^{\alpha+2}}\right\},$$

for all $x \in D(A)$.

Remarks.

1. Assertion (b) of Theorem 2.1 was proved for resolvent families in Proposition 2.2(i) of J.-C. Chang and S.-Y. Shaw [1] and for n-times integrated solution families in Proposition 2.2(c) of H. Liu and S.-Y. Shaw [2].
2. Corollaries 2.2 and 2.3 were proved for the case $\alpha = n \geq 0$ in Lemmas 3.5 and 4.4 of J.-C. Chang and S.-Y. Shaw [2].

3. Perturbation

In order to settle a well formulated theory for k-regularized resolvents, we must establish three basic results; a generation theorem, an approximation theorem and a perturbation theorem. The first was given in [7] whereas the second was the objective in the paper [8]. In this section we will study the perturbation problem.

Let $k \in C(\mathbb{R}_+)$ and $a \in L^1_{loc}(\mathbb{R}_+)$ be scalar kernels which we will assume to be Laplace transformable. Our main hypothesis is the following:

(H) There exists $b \in L^1_{loc}(\mathbb{R}_+)$ such that
$$\widehat{b}(\lambda) = \frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)},$$
for $Re\,\lambda$ sufficiently large.

For example, if $\frac{1}{\hat{k}(\lambda)}$ is locally analytic in $\mathbb{C}_+^\infty$ and $k(\infty) \neq \infty$ then there is a function $c \in L^1(\mathbb{R}_+)$ such that $\frac{1}{\hat{k}(\lambda)} = k(\infty) + \hat{c}(\lambda)$ (see [10] Lemma 10.1). Hence, if we define $b(t) = (a * c)(t) + k(\infty)a(t)$ we obtain that (H) is satisfied.

Let A be a closed and densely defined operator on a complex Banach space X. Consider the following Volterra equation

$$(3.1) \quad (VE; A, a, k) \quad u(t) = k(t)x + \int_0^t a(t-s)Au(s)ds, \ t \geq 0, \ x \in D(A).$$

Suppose there exists a k-regularized resolvent family $\{R(t)\}_{t\geq 0}$ for $(VE; A, a, k)$ of type (M, ω), that is, there is constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that
$$\|R(t)\| \cdot \ Me^{\omega t}.$$

Let $B : (D(A), \|\cdot\|_A) \longrightarrow X$ be a linear operator. Our main objective is to study conditions in order to guarantee the existence of a k-regularized resolvent family for the perturbed equation $(VE; A+B, a, k)$.

The following is the main result.

Theorem 3.1. *Under hypothesis* (H), *assume* $(VE; A, a, k)$ *admits a k-regularized resolvent family* $\{R(t)\}_{t\geq 0}$ *of type* (M,ω) *and suppose that there exists constants* $\mu > \omega$ *and* $\gamma \in [0, 1)$ *such that*

$$(3.2) \qquad \int_0^\infty e^{-\mu r} \left\| B \int_0^r b(r-s)R(s)x ds \right\| dr \cdot \ \gamma\|x\|, \ x \in D(A).$$

Then $(VE; A+B, a, k)$ *admits a k-regularized resolvent family* $\{S(t)\}_{t\geq 0}$ *on* X *such that* $\|S(t)\| \cdot \ \frac{M}{1-\gamma}e^{\mu t}$. *In addition,*

$$(3.3) \qquad S(t)x = R(t)x + \int_0^t S(t-r)B \int_0^r b(r-s)R(s)x ds dr, \ x \in D(A).$$

Proof. The proof follows closely [11], Theorem 1.1.

We define inductively operators $T_n(t) \in \mathcal{B}(X)(n = 0, 1, 2, \dots)$, $t \geq 0$, with the following properties:

(a) $t \longrightarrow T_n(t)$ is strongly continuous.

(b) $\|T_n(t)\| \cdot \ \gamma^n M e^{\mu t}$, $t \geq 0$.

Let $T_0(t) := R(t)$ (clearly satisfies (a) and (b)). Assume now that the claim is true for n. For $x \in D(A)$ we define

$$T_{n+1}(t)x := \int_0^t T_n(t-r) B \int_0^r b(r-s) R(s) x ds dr.$$

Obviously $t \longrightarrow T_{n+1}(t)x$ is continuous and by (b) and (3.2), we obtain

$$\begin{aligned}
\|T_{n+1}(t)x\| &= \left\| \int_0^t T_n(t-r) B \int_0^r b(r-s) R(s) x ds dr \right\| \\
&\cdot \ \int_0^t \|T_n(t-r)\| \left\| B \int_0^r b(r-s) R(s) x ds \right\| dr \\
&\cdot \ \gamma^n M \int_0^t e^{\mu(t-r)} \left\| B \int_0^r b(r-s) R(s) x ds \right\| dr \\
&= \gamma^n M e^{\mu t} \int_0^t e^{-\mu r} \left\| B \int_0^r b(r-s) R(s) x ds \right\| dr \\
&\cdot \ \gamma^{n+1} M e^{\mu t} \|x\|.
\end{aligned}$$

Since $D(A)$ is dense, $T_{n+1}(t)$ can be extended uniquely to an operator $\widetilde{T}_{n+1}(t)$ (also denoted $T_{n+1}(t)$) which satisfies (a) and (b).

Let $S(t) := \sum_{n=0}^{\infty} T_n(t)$. We note that $S(t)$ is well defined since

$$\sum_{n=0}^{\infty} \|T_n(t)\| \cdot \ M e^{\mu t} \sum_{n=0}^{\infty} \gamma^n = \frac{M}{1-\gamma} e^{\mu t}.$$

Moreover, $\|S(t)\| \cdot \ \frac{M}{1-\gamma} e^{\mu t}$.

Using (a) and (b) we see that for each $x \in D(A)$, the map $t \longrightarrow S(t)x$ is continuous and

$$
\begin{aligned}
S(t)x &= \sum_{n=0}^{\infty} T_n(t)x \\
&= T_0(t)x + \sum_{n=1}^{\infty} T_n(t)x \\
&= R(t)x + \sum_{n=0}^{\infty} T_{n+1}(t)x \\
&= R(t)x + \sum_{n=0}^{\infty} \left(\int_0^t T_n(t-r)B \int_0^r b(r-s)R(s)xdsdr \right) \\
&= R(t)x + \int_0^t \sum_{n=0}^{\infty} T_n(t-r)B \int_0^r b(r-s)R(s)xdsdr \\
&= R(t)x + \int_0^t S(t-r)B \int_0^r b(r-s)R(s)xdsdr.
\end{aligned}
$$

In particular $S(0)x = R(0)x = k(0)x$ for all $x \in D(A)$. Since $D(A)$ is dense, we conclude $S(0) = k(0)I$.

So, by [7] Proposition 3.1, it remains to show that $(\lambda - \lambda\widehat{a}(\lambda)(A+B)) : D(A) \longrightarrow X$ is invertible for $\lambda > \mu$ and

$$(I - \widehat{a}(\lambda)(A+B))^{-1}x = \frac{1}{\widehat{k}(\lambda)} \int_0^{\infty} e^{-\lambda t} S(t)xdt; \; x \in X.$$

For this, let $x \in X$ and define

$$H(\lambda)x = \int_0^{\infty} e^{-\lambda t} S(t)xdt \; \text{ and } \; H(\lambda; A)x = \int_0^{\infty} e^{-\lambda t} R(t)xdt = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}.$$

Then we define

$$H_k(\lambda)x = \frac{1}{\lambda\widehat{k}(\lambda)} H(\lambda)x \quad \text{and} \quad H_k(\lambda; A)x = \frac{1}{\lambda\widehat{k}(\lambda)} H(\lambda; A)x.$$

Note that $H_k(\lambda)$ is a bounded operator because $S(t)$ is exponentially bounded. Moreover,

$$
\begin{aligned}
\|H_k(\lambda)\| &= \frac{1}{\lambda|\widehat{k}(\lambda)|} \left\| \int_0^{\infty} e^{-\lambda t} S(t)dt \right\| \\
&\leq \frac{1}{\lambda|\widehat{k}(\lambda)|} \int_0^{\infty} e^{-\lambda t} \|S(t)\| dt \\
&\leq \frac{1}{\lambda|\widehat{k}(\lambda)|} \frac{M}{1-\gamma} \int_0^{\infty} e^{-(\lambda-\mu)t} dt \\
&= \frac{M}{(1-\gamma)(\lambda-\mu)\lambda|\widehat{k}(\lambda)|}.
\end{aligned}
$$

Now we observe that for $x \in D(A)$,

$$H_k(\lambda)x - H_k(\lambda, A)x = \frac{1}{\lambda\widehat{k}(\lambda)}(H(\lambda)x - H(\lambda, A)x),$$

and it is easy to see that $H(\lambda) - H(\lambda, A) = \widehat{b}(\lambda)H(\lambda)BH(\lambda, A)$. Then

$$\begin{aligned} H_k(\lambda)x - H_k(\lambda, A)x &= \frac{\widehat{b}(\lambda)}{\lambda\widehat{k}(\lambda)}H(\lambda)BH(\lambda, A)x \\ &= \frac{H(\lambda)}{\lambda\widehat{k}(\lambda)}\widehat{b}(\lambda)BH(\lambda, A)x \\ &= H_k(\lambda)\widehat{b}(\lambda)BH(\lambda, A)x. \end{aligned}$$

So, since $D(A)$ is dense on X, one has

$$H_k(\lambda) - H_k(\lambda, A) = H_k(\lambda)\widehat{b}(\lambda)BH(\lambda, A),$$

equivalently

$$H_k(\lambda)(I - \widehat{b}(\lambda)BH(\lambda, A)) = H_k(\lambda, A).$$

But $H(\lambda, A) = \widehat{R}(\lambda)$, then for $x \in D(A)$ we obtain

$$\begin{aligned} \|\widehat{b}(\lambda)BH(\lambda, A)x\| &= \|B\widehat{R}(\lambda)\widehat{b}(\lambda)x\| \\ &= \|B\widehat{R * b(r)}(\lambda)x\| \\ &= \left\|B\int_0^\infty e^{-\lambda r}\int_0^r b(r-s)R(s)xdsdr\right\| \\ &\cdot \int_0^\infty e^{-\mu r}\left\|B\int_0^r b(r-s)R(s)xds\right\|dr \\ &\cdot \gamma\|x\|,\ 0 \cdot \gamma < 1. \end{aligned}$$

Then $(I - \widehat{b}(\lambda)BH(\lambda, A))^{-1}$ exist and is bounded.

So, $H_k(\lambda) = H_k(\lambda, A)(I - \widehat{b}(\lambda)BH(\lambda, A))^{-1}$ gives us that

$$\begin{aligned} &(\lambda - \lambda\widehat{a}(\lambda)(A+B))H_k(\lambda) \\ &\quad = (\lambda - \lambda\widehat{a}(\lambda)(A+B))H_k(\lambda, A)(I - \widehat{b}(\lambda)BH(\lambda, A))^{-1} \\ &\quad = ((\lambda - \lambda\widehat{a}(\lambda)A)H_k(\lambda, A) - \lambda\widehat{a}(\lambda)BH_k(\lambda, A)) \cdot (I - \widehat{b}(\lambda)BH(\lambda, A))^{-1} \\ &\quad = (I - \lambda\widehat{a}(\lambda)BH_k(\lambda, A))\left(I - \frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}BH(\lambda, A)\right)^{-1} \end{aligned}$$

$$= (I - \lambda\widehat{a}(\lambda)BH_k(\lambda, A)) \left(I - \lambda\widehat{a}(\lambda)B\frac{H(\lambda, A)}{\lambda\widehat{k}(\lambda)} \right)^{-1}$$
$$= I.$$

This proves that $(\lambda - \lambda\widehat{a}(\lambda)(A+B))$ is invertible and satisfies

$$(I - \widehat{a}(\lambda)(A+B))^{-1}x = \frac{1}{\widehat{k}(\lambda)} \int_0^\infty e^{-\lambda t} S(t)x dt, \ \ x \in X.$$

Corollary 3.2. *If $B \in \mathcal{B}(X)$ and there exists b such that $b * k = a$, then $(VE; A+B, a, k)$ admits a k-regularized resolvent family on X.*

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