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NUMERICAL RANGE AND PONCELET PROPERTY

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Abstract. In this survey article, we give an expository account of the recent developments on the Poncelet property for numerical ranges of the compressions of the shift $S(\phi)$. It can be considered as an updated and more advanced edition of the recent expository article published in the American Mathematical Monthly by the second author on this topic. The new information includes: (1) a simplified approach to the main results (generalizations of Poncelet, Brianchon–Ceva and Lucas–Siebeck theorems) in this area, (2) the recent discovery of Mirman refuting a previous conjecture on the coincidence of Poncelet curves and boundaries of the numerical ranges of finite-dimensional $S(\phi)$, and (3) some partial generalizations by the present authors of the above-mentioned results from the unitary-dilation context to the normal-dilation one and also from the finite-dimensional $S(\phi)$ to the infinite-dimensional.

1. Introduction

In recent years, the research on the numerical ranges of finite matrices and bounded operators has been very active, thanks to the biennially convened WONRA (Workshop on Numerical Ranges and Numerical Radii). (For more information on this, check the webpage http://www.resnet.wm.edu/~cklixx/wonra02.html.) One area of investigations concerns the numerical ranges of the finite-dimensional compressions of the shift. It was discovered that the boundaries of their numerical ranges possess the Poncelet property, meaning that there exist infinitely many polygons with the property that each has all its vertices on the unit circle and all its sides tangent to the asserted boundary. This yields an unexpected link between the 20th century subject of numerical range and some 19th century gems of projective geometry. An expository account of this development was given in [36], which explains

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the pertinent results in a historical context. The purpose of this survey is to update this previous account by providing a simplified approach and expounding the recent discoveries. Chief among the latter is the one by Mirman that not every algebraic convex curve in the open unit disc which has the Poncelet property arises as the boundary of the numerical range of the asserted operator, thus refuting a previous conjecture on identifying such numerical ranges by the Poncelet property. We will also elaborate on our recent attempts in generalizing the main results in this area to more general contexts such as general convex polygons instead of polygons with vertices on the unit circle and general compressions of the shift instead of mere the finite-dimensional ones.

In Section 2 below, we start with a brief review of the definition and basic properties of numerical ranges of operators on a Hilbert space. We also discuss the notion of dilation and its connection with numerical ranges. Section 3 then treats numerical ranges of finite matrices. Here the extra tool of the Kippenhahn curve proves very useful. It involves the point-line duality of the projective plane. Section 4 considers the compressions of the shift, whose numerical ranges will be the main focus of this paper. Several different representations of such operators, one analytic and two matricial, are presented, each of which has its merit in exposing certain properties of their numerical ranges. Section 5 gives the main results on the Poncelet property for the numerical ranges of the compressions of the shift on finite-dimensional spaces. There are three of them: generalizations of the Poncelet porism (on the existence of infinitely many interscribing polygons between two ellipses), Brianchon–Ceva theorem (on the condition for the tangent points of an inscribing ellipse of a triangle), and Lucas–Siebeck theorem (on the relation between zeros of a polynomial and its derivative). We then move on to the partial generalizations of these results in Section 6.

2. Numerical Range

Let A be a (bounded linear) operator on a complex Hilbert space H. The *numerical range* of A is the set $W(A) \equiv \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$ in the complex plane, where $\langle\cdot,\cdot\rangle$ denotes the inner product in H. In other words, $W(A)$ is the image of the unit sphere $\{x \in H : \|x\| = 1\}$ of H under the (bounded) quadratic form $x \mapsto \langle Ax, x\rangle$. Some properties of the numerical range follow easily from the definition. For one thing, the numerical range is unchanged under the unitary equivalence of operators: $W(A) = W(U^*AU)$ for any unitary U. It also behaves nicely under the operation of taking the adjoint of an operator: $W(A^*) = \{\overline{z} : z \in W(A)\}$. More generally, this is even the case when taking the affine transformation: if

$$f(x+iy) = (a_1x + b_1y + c_1) + i(a_2x + b_2y + c_2)$$

is an affine transformation of the complex plane $\mathbb{C}$, where x, y and a_j, b_j and $c_j, j = 1, 2$, are all real and the latter satisfy $a_1 b_2 \neq a_2 b_1$, and if we define $f(A)$ to be

$$(a_1 \text{Re } A + b_1 \text{Im } A + c_1 I) + i(a_2 \text{Re } A + b_2 \text{Im } A + c_2 I),$$

where $\text{Re } A = (A + A^*)/2$ and $\text{Im } A = (A - A^*)/(2i)$ are the *real* and *imaginary parts* of A, respectively, then $W(f(A)) = f(W(A)) \equiv \{f(z) : z \in W(A)\}$. Thus the numerical range can be considered as an affine property of the operator. In the study of numerical ranges, the reduction through affine transformations is a handy tool in many situations.

The most important property of the numerical range is that $W(A)$ *is always convex*. This is the celebrated Toeplitz–Hausdorff Theorem from 1918–19 [32, 16]. Over the years, there are numerous proofs and generalizations of this fact. The usual proof is to first reduce it to the case of 2-by-2 matrices (since the definition of convexity involves only two points at a time) and show that the numerical range of the latter is a closed elliptic disc or one of its degenerate forms (circular disc, line segment or a single point). Indeed, if $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, then $W(A)$ is the elliptic disc with foci a and c and minor axis of length $|b|$. An easy proof of this is to reduce A to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ via some affine transformation and check directly that the latter has numerical range $\{z \in \mathbb{C} : |z| \cdot\ 1/2\}$ (cf. [20]).

The numerical range is a bounded set, but it is not closed in general. For example, if S is the (*simple*) *unilateral shift* on l^2:

$$S(x_0, x_1, \dots) = (0, x_0, x_1, \dots),$$

then $W(S)$ equals the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. However, if the operator A acts on a finite-dimensional space, then $W(A)$ is obviously closed and hence compact. For an arbitrary operator A, the closure of its numerical range $\overline{W(A)}$ always contains the spectrum $\sigma(A)$. Hence the numerical range gives a rough estimate of the location of the spectrum. This is one of the reasons to study the numerical range and provides its main applications. If A is normal, then $\overline{W(A)}$ equals $\sigma(A)^\wedge$, the convex hull of $\sigma(A)$. Thus, in particular, if A is a normal (finite) matrix, then its numerical range is a closed polygonal region whose vertices are some of the eigenvalues of A.

A natural question in the study of numerical ranges is to determine which nonempty bounded convex set is the numerical range of some operator on a separable Hilbert space. (Note that if nonseparable Hilbert spaces are allowed, then every such set is the numerical range of some normal operator; compare [27].) Even more intricate is to determine, for each positive integer n, the numerical ranges of operators on an n-dimensional space. Although many necessary/sufficient conditions are known, a complete characterization is beyond reach at this moment. One condition on the boundary of the numerical range is worth noting. If $\triangle$ is a closed convex subset of the plane, then every nondifferentiable point of the boundary $\partial\triangle$ of $\triangle$ has two distinct supporting lines of $\triangle$ with angle less then π such that the closed section formed by them contains $\triangle$. Such a point is called a *corner* of $\triangle$. According to this definition, the endpoints of a line segment are corners. A result of Donoghue [8, Theorem 1] says that *a corner λ of $\overline{W(A)}$ which also belongs to $W(A)$ is a reducing eigenvalue of A*. The latter means that there is a nonzero vector x such that $Ax = \lambda x$ and $A^*x = \bar{\lambda}x$. The proof of this makes use of the geometric fact that an elliptic disc which contains λ and is contained in $\overline{W(A)}$ must be reduced to a line segment. It follows that if A is an n-dimensional operator, then $W(A)$ can have at most n corners. This gives a certain constraint on the shape of the numerical range of a finite-dimensional operator. Using the condition for the equality case of the Cauchy–Schwarz inequality, we may prove the analogous result that *any point λ in $W(A)$ satisfying $|\lambda| = \|A\|$ is a reducing eigenvalue of A*.

Associated with the numerical range $W(A)$ is the quantity $w(A)$, the *numerical radius* of A, defined by $\sup\{|z| : z \in W(A)\}$. For example, if S is the unilateral shift, then $w(S) = 1$, and if A is normal, then $w(A) = \sup\{|z| : z \in \sigma(A)\}$.

We say that the operator A on space H *dilates* to B on K or B *compresses* to A if there is an isometry V from H to K such that $A = V^*BV$. It is easily seen that this is equivalent to B being unitarily equivalent to a 2-by-2 operator matrix of the form $\begin{bmatrix} A & * \\ * & * \end{bmatrix}$. The notion of dilation and compression is closely related to that of numerical range. For one thing, the numerical range itself can be described in terms of dilation. Namely, for any operator A, the numerical range of A is the same as the set of complex numbers λ for which the 1-by-1 matrix $[\lambda]$ dilates to A. On the other hand, if A is an operator which dilates to B, then $W(A)$ is contained in $W(B)$. Hence a judicious choice of a nicely behaved B can yield useful information on the numerical range of A. One type of dilation which will be fully exploited in our derivations in Sections 5 and 6 is the unitary dilation of contractions. The classical result in this respect is Halmos's dilation: *every contraction A ($\|A\| \le 1$) can be dilated to the unitary operator*

$$\begin{bmatrix} A & (I - AA^*)^{1/2} \\ (I - A^*A)^{1/2} & -A^* \end{bmatrix}$$

(cf. [15, Problem 222 (a)]). With more care, the unitary dilation can be achieved in a most economical way: *if A is a contraction on H, then A can be dilated to a unitary operator U from $H \oplus K_1$ to $H \oplus K_2$ with K_1 and K_2 of dimensions $d_{A^*} \equiv \dim \operatorname{ran} (I - AA^*)^{1/2}$ and $d_A \equiv \dim \operatorname{ran} (I - A^*A)^{1/2}$, respectively, and, moreover, in this case d_{A^*} and d_A are the smallest dimensions of such spaces K_1 and K_2*. Here d_A and d_{A^*} are called the *defect indices* of the contraction A. They provide a measure on how far A deviates from the unitary operators and play a prominent role in the unitary dilation theory.

Properties of numerical ranges of operators are discussed in [15, Chapter 22]; those for finite matrices are in [17, Chapter 1]. The two classic monographs [4] and [5] treat the numerical ranges of elements of normed algebras; the more recent [14] emphasizes applications to numerical analysis.

3. Numerical Range of Finite Matrix

For the study of numerical ranges of finite matrices, the matrix-theoretic properties can be exploited to yield special tools which are not available for general operators. One such tool is the characteristic polynomial of the pencil $x\text{Re } A + y\text{Im } A$ associated with any matrix A. This can be utilized in two different ways to yield $W(A)$ or its boundary. One is via Kippenhahn's result that the numerical range of A coincides with the convex hull of the real points of the dual curve of $\det(x\text{Re } A + y\text{Im } A + zI) = 0$. In this way, the classical algebraic curve theory can be brought to bear on the study here. On the other hand, a parametric representation of the boundary $\partial W(A)$ can also be obtained from the largest eigenvalue of $\cos\theta \text{ Re } A + \sin\theta \text{ Im } A$ yielding useful information on $W(A)$. Here we give a brief account of both approaches.

Let $\mathbb{CP}^2$ be the complex projective plane consisting of all equivalence classes $[x, y, z]$ of ordered triples of complex numbers x, y and z which are not all zero. Two such triples $[x, y, z]$ and $[x', y', z']$ are equivalent if $x = \lambda x', y = \lambda y'$ and $z = \lambda z'$ for some nonzero λ. The point $[x, y, z]$ $(z \neq 0)$ in homogeneous coordinates can be identified with $(x/z, y/z)$ in nonhomogeneous coordinates. On the other hand, the point (u, v) becomes $[u, v, 1]$ in homogeneous coordinates. In this way, $\mathbb{C}^2$ is embedded in $\mathbb{CP}^2$. If $p(x, y, z)$ is a homogeneous polynomial of degree d in x, y and z, then the set of points $[x, y, z]$ in $\mathbb{CP}^2$ satisfying the equation $p(x, y, z) = 0$ is an *algebraic curve* of *order* d. If C is such a curve, then its *dual* C^* is defined by

$$C^* = \{[u, v, w] \in \mathbb{CP}^2 : ux + vy + wz = 0 \text{ is a tangent line of } C\}.$$

In this case, C^* is also an algebraic curve of order at most $d(d-1)$ and d is called the *class* of C^*. It is known that the dual of C^* is C itself. The point $[x_0, y_0, z_0]$ is a *focus* of C if it is not equal to $[1, \pm i, 0]$ and the lines through $[x_0, y_0, z_0]$ and $[1, \pm i, 0]$ are tangent to C at points other than $[1, \pm i, 0]$. In general, if a curve is of class d and is defined by an equation with real coefficients, then it has d real foci and $d^2 - d$ complex ones, counting multiplicity.

For an n-by-n matrix A, let

$$p_A(x, y, z) = \det(x\text{Re } A + y\text{Im } A + zI_n)$$

and let $C(A)$ denote the dual curve of $p_A(x, y, z) = 0$. Since p_A is a real homogeneous polynomial of degree n, the curve $C(A)$ is given by a real polynomial of degree at most $n(n-1)$, is of class n, and has n real foci $[a_j, b_j, 1]$, $j = 1, \ldots, n$, which correspond exactly to the n eigenvalues $a_j + ib_j$ of A. The connection of $C(A)$ with the numerical range $W(A)$ is provided by a result of Kippenhahn [19]: *$W(A)$ is the convex hull of the real points of the curve $C(A)$*, namely, $W(A)$ is the convex hull of the set $\{a + ib \in \mathbb{C} : a, b \in \mathbb{R},\ ax + by + z = 0$ is tangent to $p_A(x, y, z) = 0\}$. Kippenhahn's result can be easily verified by noting that $x = \max \sigma(\mathrm{Re}\ (e^{-i\theta}A))$ is a supporting line of $W(\mathrm{Re}\ (e^{-i\theta}A))$ for any real θ. Since it can be shown that $\partial W(A)$ contains only finitely many line segments, the above result implies that $\partial W(A)$ is *piecewise algebraic*, that is, it is the union of finitely many algebraic curves.

There is another way to make the above to be more revealing. For any nonempty compact convex subset $\triangle$ of the plane, there is a *natural parametrization* of its boundary $\partial\triangle$. For any $\theta, 0 \cdot \theta \cdot 2\pi$, let L_θ be the ray from the origin which has inclination θ from the positive x-axis, and let M_θ be the supporting line of $\triangle$ which is perpendicular to L_θ. If $d(\theta)$ is the signed distance from the origin to M_θ, then $\partial\triangle$ can be "parametrized" by $\alpha(\theta) = (x(\theta), y(\theta))$, where

$$x(\theta) = d(\theta)\cos\theta - d'(\theta)\sin\theta,$$
$$y(\theta) = d(\theta)\sin\theta + d'(\theta)\cos\theta.$$

It can be shown that $d(\theta)$ is differentiable for almost all θ and is equal to $\max\{\mathrm{Re}\ (e^{-i\theta}z) : z \in \triangle\}$. In particular, if $\triangle = \overline{W(A)}$ for some operator A, then

$$\begin{aligned} d(\theta) &= \max \sigma(\mathrm{Re}\ (e^{-i\theta}A)) \\ &= \max \sigma(\cos\theta\,\mathrm{Re}\ A + \sin\theta\,\mathrm{Im}\ A) \end{aligned}$$

for all θ. This shows that, for a finite matrix A, the degree-n polynomial $p_A(\cos\theta, \sin\theta, z)$ in z has $-d(\theta)$ as a zero. As an example, if A is the 3-by-3 matrix diag $(1, i, 0)$, then

$$d(\theta) = \begin{cases} \cos\theta & \text{if } 0 \cdot \theta \cdot \frac{\pi}{4} \text{ or } \frac{3}{2}\pi \cdot \theta \cdot 2\pi, \\ \sin\theta & \text{if } \frac{\pi}{4} \cdot \theta \cdot \pi, \\ 0 & \text{if } \pi \cdot \theta \cdot \frac{3}{2}\pi, \end{cases}$$

and the natural parametrization of $\partial W(A)$ (= the triangle with vertices $1, i$ and 0) is given by

$$\alpha(\theta) = \begin{cases} 1 & \text{if } 0 < \theta < \frac{\pi}{4} \text{ or } \frac{3}{2}\pi < \theta < 2\pi, \\ i & \text{if } \frac{\pi}{4} < \theta < \pi, \\ 0 & \text{if } \pi < \theta < \frac{3}{2}\pi. \end{cases}$$

In particular, this shows that the natural parametrization is not a parametrization in the usual sense: it does not traverse the line segments on the boundary, but the convex hull of its image equals $\partial\triangle$.

4. Compression of the Shift

Compressions of the shift are a class of operators studied intensively in the 1960s and '70s. Playing a role analogous to the companion matrices in the rational form for finite matrices, they are the building blocks in the "Jordan form" (under quasisimilarity) for the class of C_0 contractions. The whole theory is subsumed under the dilation theory for contractions on Hilbert spaces developed by Sz.-Nagy and Foiaş. The standard reference is the monograph [31]; a more complete account of the theory of C_0 contractions is given in [3].

We start by noting that the unilateral shift S has another representation as $(Sf)(z) = zf(z)$ for f in H^2, the Hardy space of square-summable analytic functions on $\mathbb{D}$. This analytic model of S facilitates a complete description of its invariant subspaces. Indeed, according to the celebrated theorem of Beurling (1949), all nonzero invariant subspaces of S are of the form ϕH^2 for some inner function ϕ (ϕ is *inner* if it is bounded and analytic on $\mathbb{D}$ with $|\phi(e^{i\theta})| = 1$ for almost all real θ). The *compression of the shift* $S(\phi)$ is the operator on $H(\phi) \equiv H^2 \ominus \phi H^2$ defined by

$$S(\phi)f = P(zf(z)),$$

where P denotes the (orthogonal) projection from H^2 onto $H(\phi)$. Thus $S(\phi)$ is the operator in the lower-right corner of the 2-by-2 operator matrix representation of

$$S = \begin{bmatrix} * & * \\ 0 & S(\phi) \end{bmatrix} \quad \text{on } H^2 = \phi H^2 \oplus H(\phi).$$

This class of operators was first studied by Sarason [30] and has been under intensive investigation over the past 35 years. In particular, it is known that $\|S(\phi)\| = 1, S(\phi)$ is *cyclic* (there is a vector $f\,(= 1 - \overline{\phi(0)}\phi)$ in $H(\phi)$ such that $\bigvee\{S(\phi)^n f : n \geq 0\} = H(\phi)$), and its *commutant* $\{S(\phi)\}'$ $(\equiv \{X \text{ on } H(\phi) : XS(\phi) = S(\phi)X\})$ and *double commutant* $\{S(\phi)\}''$ $(\equiv \{Y \text{ on } H(\phi) : YX = XY \text{ for every } X \text{ in } \{S(\phi)\}'\})$ are both equal to $\{f(S(\phi)) : f \in H^\infty\}$. The inner function ϕ is the *minimal function* of $S(\phi)$ in a sense similar to the minimal polynomial of a finite matrix, that is, it is such that (a) $\phi(S(\phi)) = 0$, and (b) ϕ is a factor of any function f in H^∞ for which $f(S(\phi)) = 0$. An operator A is (unitarily equivalent to) a compression of the shift if and only if it is a contraction, both A^n and A^{*n} converge to 0 in the strong operator topology, and the defect indices d_A and d_{A^*} are both equal to one. It follows from these conditions that the compression of the shift is *irreducible*, that is, it can have no nontrivial reducing subspace.

For finite matrices, the characterization of compressions of the shift is even easier: A is such an operator if and only if it is a contraction, it has no eigenvalue of modulus one and $d_A = 1$. In this case, A is unitarily equivalent to $S(\phi)$ with ϕ the finite Blaschke product

$$\phi(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z},$$

where a_j's are the eigenvalues of A in $\mathbb{D}$. We let $\mathcal{S}_n$ denote the class of such matrices. An example in $\mathcal{S}_n$ is J_n, the n-by-n nilpotent *Jordan block*

$$\begin{bmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & 0 \end{bmatrix}$$

with the corresponding inner function $\phi(z) = z^n$. By the results in Section 2, matrices in $\mathcal{S}_n$ admit unitary dilations on an $(n+1)$-dimensional space. For this reason, $\mathcal{S}_n$-matrices are called matrices admitting unitary bordering or UB-matrices by Mirman (cf. [22, 23, 24]). Since $S(\phi)$ is defined by its minimal function ϕ, we infer that for any n points $a_1, \ldots, a_n$ in $\mathbb{D}$ (not necessarily distinct) there is a matrix in $\mathcal{S}_n$, unique up to unitary equivalence, with the a_j's as its eigenvalues. A more specific description of a matrix in $\mathcal{S}_n$ with eigenvalues the a_j's is given by $[a_{ij}]_{i,j=1}^n$, where

$$(1) \qquad a_{ij} = \begin{cases} a_j & \text{if } i = j, \\ \left[\prod_{k=i+1}^{j-1} (-\overline{a_k})\right](1 - |a_i|^2)^{1/2}(1 - |a_j|^2)^{1/2} & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$

This matricial representation was first discovered by Young [37, p. 235] (cf. also [29, p. 201], [22, Theorem 4] and [11, Corollary 1.3]). In particular, it follows that $\mathcal{S}_2$ consists of 2-by-2 matrices which are unitarily equivalent to a matrix of the form

$$\begin{bmatrix} a & (1 - |a|^2)^{1/2}(1 - |b|^2)^{1/2} \\ 0 & b \end{bmatrix}$$

with a and b in $\mathbb{D}$. There is another representation for matrices in $\mathcal{S}_n$ which is useful for our discussions in Section 5. If A is in $\mathcal{S}_n$, then it has the singular-value decomposition $A = UDW$, where U and W are unitary and D is a diagonal matrix diag $(1, \ldots, 1, a)$ with $0 \le a < 1$. The equality $WAW^* = (WU)D$ shows that A is unitarily equivalent to $(WU)D$, a matrix of the form

$$(2) \qquad [f_1 \cdots f_n]$$

whose columns f_j satisfy $\|f_j\| = 1$ for $1 \le j \le n-1$, $\|f_n\| < 1$ and $f_j \perp f_k$ for $1 \le j \neq k \le n$. Conversely, a matrix of the above form with no eigenvalue of modulus one is in $\mathcal{S}_n$.

From the information we have so far on the compressions of the shift, we can already deduce certain properties of their numerical ranges. Let A be a matrix in $\mathcal{S}_n$. Then $W(A)$ must be contained in the open unit disc $\mathbb{D}$. This is because if λ in $W(A)$ is such that $|\lambda| = 1$ $(= \|A\|)$, then it will be a reducing eigenvalue of A, which contradicts the irreducibility of A. On the other hand, by Donoghue's result and the irreducibility of A, we may deduce that the boundary of $W(A)$ is a differentiable curve. In the subsequent sections, we will discuss other finer properties of $W(A)$.

5. Poncelet Property

The recent establishment of a link between the numerical ranges of matrices in $\mathcal{S}_n$ and some classical geometric results from the 19th century was achieved by Mirman [22, 24, 23, 25] and the present authors [9, 10, 11, 12]. Here we give a brief account of this development.

Our first result has to do with a geometric theorem of Poncelet. In his treatise of 1822 [28], there is contained the following result, called *Poncelet's porism* or *Poncelet's closure theorem*: if C and D are ellipses in the plane with C inside D, and if there is one n-gon circumscribed about C and inscribed in D, that is, the n-gon has n sides all tangent to C and n vertices on D, then for any point λ on D there is one such circumscribing-inscribing n-gon with λ as a vertex. This is a porism because the assertion says that some property (the existence of a circumscribing-inscribing n-gon) either fails or, if it holds for one instance, succeeds infinitely many times. It is a closure theorem since, from any point λ on D, we draw a tangent line to C, which intersects D at another point, then repeat this process by drawing tangent lines from successive points obtained in this fashion, and obtain the resulting closed n-gon when the nth tangent line reaches back to λ. Viewed dynamically, this gives a configuration of rotating n-gons with different shapes but all sharing this circumscribing-inscribing property. Since the appearance of this result, a huge literature has been developed to the explanation, exposition and generalization of it. A comprehensive survey of this topic can be found in [6]. In the above situation, we may normalize the outer ellipse D as the unit circle $\partial\mathbb{D}$ via some affine transformation and the inner ellipse C is transformed into one in $\mathbb{D}$ with the n-Poncelet property. More precisely, for $n \geq 3$, we say that a curve Γ in

$\mathbb{D}$ has the *n-Poncelet property* if for every point λ on $\partial\mathbb{D}$ there is an n-gon which circumscribes about Γ, inscribes in $\partial\mathbb{D}$ and has λ as a vertex. It is natural to ask whether there are curves other than ellipses in $\mathbb{D}$ which also have the n-Poncelet property. The next theorem provides more of such examples.

Theorem 5.1. *For any matrix A in $\mathcal{S}_n$ and any point λ on $\partial\mathbb{D}$, there is a unique $(n+1)$-gon which circumscribes about $\partial W(A)$, inscribes in $\partial\mathbb{D}$ and has λ as a vertex. In fact, such $(n+1)$-gons P are in one-to-one correspondence with the (unitary-equivalence classes of) unitary dilations U of A on an $(n+1)$-dimensional space, under which the $n+1$ vertices of P are exactly the eigenvalues of the corresponding U.*

This theorem appeared in [22, Theorem 1] and [9, Theorem 2.1]. The easy part of the proof is to show that every $(n+1)$-dimensional unitary dilation of a matrix A in $\mathcal{S}_n$ has distinct eigenvalues which form an $(n+1)$-gon inscribed in $\partial\mathbb{D}$ and circumscribed about $W(A)$ with each side tangent to $\partial W(A)$ at exactly one point. To show that such $(n+1)$-gons run over every point of $\partial\mathbb{D}$ takes more work. Instead of outlining its details, we resort to the matrix representations (1) and (2) for A to give the specific $(n+1)$-dimensional unitary dilations U. If A is represented as in (1), then U can be taken as $[b_{ij}]_{i,j=1}^{n+1}$, where

$$(3)\quad b_{ij} = \begin{cases} a_{ij} & \text{if } 1 \le i,j \le n, \\ \lambda[\prod_{k=1}^{j-1}(-\overline{a_k})](1-|a_j|^2)^{1/2} & \text{if } i=n+1 \text{ and } 1 \le j \le n, \\ [\prod_{k=i+1}^{n}(-\overline{a_k})](1-|a_i|^2)^{1/2} & \text{if } j=n+1 \text{ and } 1 \le i \le n, \\ \lambda\prod_{k=1}^{n}(-\overline{a_k}) & \text{if } i=j=n+1 \end{cases}$$

for some λ in $\partial\mathbb{D}$. Here λ acts as a parameter for the unitary dilations U. On the other hand, if A is as (2), then U can be

$$(4)\qquad \begin{bmatrix} f_1 & \dots & f_{n-1} & f_n & g \\ 0 & \dots & 0 & \lambda a & \lambda\|f_n\| \end{bmatrix},$$

where $|\lambda|=1, a=(1-\|f_n\|^2)^{1/2}>0$ and

$$g = \begin{cases} -(a/\|f_n\|)f_n & \text{if } f_n \neq 0, \\ \text{any unit vector orthogonal to } f_1,\dots,f_{n-1} & \text{if } f_n = 0. \end{cases}$$

Both (3) and (4) can be used to prove that the $(n+1)$-gons with vertices the eigenvalues of U cover all points of $\partial\mathbb{D}$ (the latter is in [9, Theorem 2.1]).

Theorem 5.1 yields additional properties for the numerical ranges of matrices in $\mathcal{S}_n$.

Corollary 5.2. *Let A be a matrix in $\mathcal{S}_n$. Then*

(a) $W(A)$ *is contained in no m-gon inscribed in $\partial\mathbb{D}$ for $m \cdot\ n$,*

(b) $w(A) > \cos(\pi/n)$,

(c) $\mathrm{Re}\, A$ *and* $\mathrm{Im}\, A$ *have simple eigenvalues*, *and*

(d) *the boundary of $W(A)$ contains no line segment and is an algebraic curve.*

Here (a) is an easy consequence of the $(n+1)$-Poncelet property of $\partial W(A)$, (b) follows from (a), (c) is a consequence of, besides the Poncelet property, the interlacing of the eigenvalues of $\mathrm{Re}\, A$ and $\mathrm{Re}\, U$ for any $(n+1)$-dimensional unitary dilation U of A, and finally (d) follows from (c) by way of Kippenhahn's result. All assertions except (d) are in [9].

If A is in $\mathcal{S}_n$, so is $e^{-i\theta}A$ for any real θ. Hence the eigenvalues of $\mathrm{Re}\,(e^{-i\theta}A)$ are all distinct by Corollary 5.2 (c). The curves $\Gamma_j, j = 1, \dots, n$, described by $\alpha_j(\theta) = (x_j(\theta), y_j(\theta))$ with

$$\begin{aligned} x_j(\theta) &= \lambda_j(\theta)\cos\theta - \lambda_j'(\theta)\sin\theta, \\ y_j(\theta) &= \lambda_j(\theta)\sin\theta + \lambda_j'(\theta)\cos\theta, \end{aligned}$$

where $\lambda_j(\theta)$ is the jth largest eigenvalue of $\mathrm{Re}\,(e^{-i\theta}A)$, are expected to have a Poncelet-type property just as $\Gamma_1{}^\wedge = \partial W(A)$ does. This is indeed the case and is proved in [22, Theorem 8]. Note that, in this case, Γ_j and Γ_{n-j+1} coincide for any j, and if $U = \mathrm{diag}\,(b_1, \dots, b_{n+1})$ is a unitary dilation of A, where the b_j's are arranged counterclockwise around $\partial\mathbb{D}$, then, for each j, the not-necessarily-convex $(n+1)$-gon $b_1\ b_{j+1}\ b_{2j+1}\ \dots\ b_{nj+1}$ $(b_p = b_q$ if $p \equiv q \pmod{n+1})$ has all its sides $[b_{kj+1}, b_{(k+1)j+1}]$ tangent to Γ_j. A detailed analysis of such curves, called *a package of Poncelet curves*, has been carried out by Mirman [22, 24]. Among other things, he showed that in this situation there is associated a probability measure μ on the unit circle $\partial\mathbb{D}$ given by

$$\mu(\triangle) = \frac{1}{2\pi(n+1)} \int_\triangle 1 + \sum_{j=1}^{n} \frac{1 - |a_j|^2}{|e^{i\theta} - a_j|^2} d\theta$$

for any Borel subset $\triangle$ of $\partial\mathbb{D}$, where a_j's are the eigenvalues of A. The measure μ is invariant under the function

$f(e^{i\theta}) =$ intersection of $\partial\mathbb{D}$ and the "right-hand" tangent line from $e^{i\theta}$ to $\partial W(A)$

on $\partial\mathbb{D}$ in the sense that $\mu(\triangle) = \mu(f^{-1}(\triangle))$ for any $\triangle$. In particular, this implies that if the chord $[e^{i\theta_1}, e^{i\theta_2}]$ $(\theta_1 < \theta_2)$ of $\partial\mathbb{D}$ is tangent to Γ_j, then

$$\frac{1}{2\pi(n+1)}\int_{\theta_1}^{\theta_2} 1+\sum_{j=1}^{n}\frac{1-|a_j|^2}{|e^{i\theta}-a_j|^2}d\theta = \frac{j}{n+1}, \quad j=1,\ldots,n.$$

Reversing this procedure, we may start with a number ρ, $0<\rho<1$, and a probability measure μ on $\partial\mathbb{D}$ with density h and consider the function f on $\partial\mathbb{D}$ which sends $e^{i\theta_1}$ to $e^{i\theta_2}$ if

$$\int_{\theta_1}^{\theta_2} h(e^{i\theta})d\theta = \rho.$$

Then the measure μ is f-invariant for the envelope Γ of the chords $[e^{i\theta_1}, e^{i\theta_2}]$. The curve Γ is in general not convex and the interscribing polygons between Γ and $\partial\mathbb{D}$ will "close" exactly when ρ is a rational number. (The idea of the invariant measure for the Poncelet property originates from a paper of King [18].) Based on this, Mirman obtained an example with $\rho = 1/3$ and

$$h(e^{i\theta}) = \frac{1}{2\pi}\Big(1+\frac{1}{2}\cos\theta\Big)$$

of a nonalgebraic convex curve Γ which has the 3-Poncelet property (cf. [24, Section 1.3.1]). This means that not every convex curve in $\mathbb{D}$ with the $(n+1)$-Poncelet property arises from the boundary of $W(A)$ for some A in $\mathcal{S}_n$. A further exploitation of this idea yields even an example of an algebraic curve with such properties (cf. [23, Example 1]). This refutes a conjecture made in [24, Problem 2] and [9, Conjecture 5.1]. We note on the side that an $(n+1)$-Poncelet curve in $\mathbb{D}$ which is also an ellipse must be the boundary of $W(A)$ for some $\mathcal{S}_n$-matrix A (cf. [22, Theorem 10b] and [12, Corollary]). There remains the problem of completely characterizing the numerical ranges of $\mathcal{S}_n$-matrices. The modified from of [9, Conjecture 5.1] that a nonempty closed convex subset of $\mathbb{D}$ whose boundary is an algebraic curve of class at most n and has the $(n+1)$-Poncelet property is the numerical range of some matrix in $\mathcal{S}_n$ seems a safe bet.

We now move on to the next theorem, which gives the condition on the tangent points of the $(n+1)$-gon with the boundary of the numerical range of a matrix in $\mathcal{S}_n$.

Theorem 5.3. *Let $b_j, j=1,\ldots,n+1$, be $n+1$ distinct points arranged counterclockwise around the unit circle and let $c_j, j=1,\ldots,n+1$, be points in the open chords (b_j, b_{j+1}), respectively ($b_{n+2}\equiv b_1$). Then a necessary and sufficient condition for the existence of a matrix A in $\mathcal{S}_n$ with $W(A)$ circumscribed* *by the* $(n+1)$*-gon* $b_1 \dots b_{n+1}$ *at the tangent points* $c_1, \dots, c_{n+1}$ *is that*

$$\prod_{j=1}^{n+1} |c_j - b_j| = \prod_{j=1}^{n+1} |c_j - b_{j+1}|. \tag{5}$$

Moreover, in this case the matrix A *is unique up to unitary equivalence.*

The case $n = 2$ can be proved by using the classical Brianchon and Ceva theorems in projective and affine geometries. It was also treated by Williams [33, Theorem 1] by way of numerical range. The general case is in [22, Theorem 7] and [9, Theorem 3.1], though not stated explicitly as above. An analogous result holds for the tangent points of each of the curves $\Gamma_k, k = 1, \dots, n$, in which case c_j is required to lie on (b_j, b_{j+k}) and (5) is replaced by

$$\prod_{j=1}^{n+1} |c_j - b_j| = \prod_{j=1}^{n+1} |c_j - b_{j+k}|$$

(cf. [22, Theorem 9]).

The proof of this theorem can be based on another representation of matrices in $\mathcal{S}_n$.

Lemma 5.4. $\mathcal{S}_n$ *consists of matrices which are compressions of some unitary matrix* diag $(b_1, \dots, b_{n+1})$ *with distinct* b_j*'s to an* n*-dimensional subspace whose orthogonal complement in* $\mathbb{C}^{n+1}$ *is generated by a unit vector* $x = (x_1, \dots, x_{n+1})^T$ *with nonzero* x_j*'s.*

This lemma can be proved easily. In contrast to the representation (1) for A in $\mathcal{S}_n$, the present one takes advantage of the diagonal form of its unitary dilation while (1) yields explicitly the upper-triangular form of A.

If A in $\mathcal{S}_n$ is represented as in Lemma 5.4, then it is not difficult to show that the tangent points c_j of the $(n+1)$-gon $b_1 \dots b_{n+1}$ with $\partial W(A)$ are given by $\langle Ay_j, y_j \rangle$ with the unit vectors

$$y_j = \frac{\overline{x_{j+1}} e_j - \overline{x_j} e_{j+1}}{(|x_j|^2 + |x_{j+1}|^2)^{1/2}}, \quad j = 1, \dots, n,$$

where e_j is the vector in $\mathbb{C}^n$ whose jth component is 1 and all other components are 0, and $e_{n+1} \equiv e_1$. A simple computation then yields that

$$\frac{|c_j - b_j|}{|c_j - b_{j+1}|} = \frac{|x_j|^2}{|x_{j+1}|^2},$$

from which (5) follows easily. Reversing the above recipe, we can prove the converse by defining

$$x_j = \frac{(\prod_{k=j}^n t_k)^{1/2}}{(\sum_{i=1}^n \prod_{k=i}^n t_k)^{1/2}}, \quad j = 1, \ldots, n,$$

where

$$t_k = \frac{|c_k - b_k|}{|c_k - b_{k+1}|}$$

for each k, and x and A accordingly. The uniqueness of A can be proved along this line.

The following corollary is an easy consequence of the uniqueness proof above.

Corollary 5.5. *Matrices A and B in $\mathcal{S}_n$ are unitarily equivalent if and only if $W(A) = W(B)$.*

This approach to Theorem 5.3 and Corollary 5.5 via Lemma 5.4 is much simpler than the one we adopted in [9]. It also has the advantage of being easily adapted to the generalizations to the normal-dilation case, which we will discuss in Section 6.

Theorem 5.3 concerns the determination of A in $\mathcal{S}_n$ by the circumscribing $(n+1)$-gon of $W(A)$ and the tangent points. The next theorem does the same with tangent points replaced by foci of $\partial W(A)$.

Theorem 5.6. *Let $a_j, j = 1, \ldots, n$, be points in $\mathbb{D}$ and let $b_j, j = 1, \ldots, n+1$, be distinct points arranged counterclockwise around $\partial\mathbb{D}$. Then a matrix in $\mathcal{S}_n$ with eigenvalues a_j's has its numerical range circumscribed about by the $(n+1)$-gon $b_1 \ldots b_{n+1}$ if and only if*

$$\alpha_j + \beta_{n+1}\overline{\alpha_{n+1-j}} = \beta_j \ \text{ for } \ j = 1, \ldots, \lceil n/2 \rceil, \tag{6}$$

where α_j and β_j denote the jth elementary symmetric functions of the a's and b's given by

$$\sum_{j=0}^{n} (-1)^j \alpha_j z^{n-j} = \prod_{k=1}^{n} (z - a_k), \ \alpha_{n+1} = 0$$

and

$$\sum_{j=0}^{n+1} (-1)^j \beta_j z^{n+1-j} = \prod_{k=1}^{n+1} (z - b_k),$$

respectively, and $\lceil n/2 \rceil$ denotes the smallest integer $\geq n/2$.

This is proved in [11, Theorem 2.5] by computing the characteristic polynomial

$$z\prod_{j=1}^{n}(z - a_j) - \lambda \prod_{j=1}^{n} (1 - \overline{a_j} z)$$

of the matrix representation (3) for the $(n+1)$-dimensional unitary dilations of an A in $\mathcal{S}_n$. A special case of the preceding theorem is the following generalization of the Lucas–Siebeck result.

Theorem 5.7. *Let p be a degree-$(n+1)$ polynomial whose distinct zeros $b_1, \ldots, b_{n+1}$ all have modulus one. Arrange the b_j's counterclockwise around $\partial\mathbb{D}$. If A is a matrix in $\mathcal{S}_n$ with eigenvalues the zeros of the derivative p', then the numerical range of A is circumscribed about by the $(n+1)$-gon $b_1 \ldots b_{n+1}$ at the tangent points $(b_j + b_{j+1})/2$, $j = 1, \ldots, n+1$ $(b_{n+2} \equiv b_1)$.*

This is a refinement of both Lucas's theorem that zeros of the derivative of a polynomial are contained in the convex hull of the zeros of the polynomial and Siebeck's result (1864) that if b_1, b_2 and b_3 are the distinct zeros of a cubic polynomial and a_1 and a_2 are zeros of its derivative, then there is an ellipse with foci a_1 and a_2 which is inscribed in the triangle $\triangle b_1 b_2 b_3$ at the midpoints of its three sides (cf. [21, p. 9]).

This theorem, originally appearing in [11, Theorem 2.1] as a corollary of Theorem 5.6, has now a simpler proof (for the assertion on the tangent points), due to Mirman [25], based on the representation in Lemma 5.4. Indeed, if $U = \operatorname{diag}(b_1, \ldots, b_{n+1})$ and $x = (1/\sqrt{n+1}, \ldots, 1/\sqrt{n+1})^T$, then the compression B of U to the orthogonal complement of x in $\mathbb{C}^{n+1}$ is in $\mathcal{S}_n$ and has numerical range circumscribed about by the $(n+1)$-gon at the midpoints $(b_j + b_{j+1})/2$, $j = 1, \ldots, n+1$. It remains then to show that B has the same eigenvalues as A.

Information on the numerical ranges of matrices in $\mathcal{S}_n$ can be carried over to that on numerical ranges of general matrices. We conclude this section with one such example. This carrying-over depends on an extension theorem of C_0 contractions. For convenience, we adapt it here for the finite-dimensional case.

Theorem 5.8. *Let A be an n-by-n contraction with all eigenvalues in $\mathbb{D}$, let m be the degree of the minimal polynomial of A, and let d be the defect index* rank $(I_n - A^*A)$. *Then A can be extended to a matrix of the form $B \oplus \cdots \oplus B$ (d times), where B is in $\mathcal{S}_m$ whose minimal polynomial is the same as A. Moreover, if $\partial W(A) \cap \partial W(B) \neq \emptyset$, then A has B as a direct summand.*

Here "extension" and "direct summand" are meant to be under a unitary equivalence.

The first part (even its C_0 version) is essentially contained in [26, Lemma 4] (cf. also [34, Theorem 1.4]). A simpler proof can be obtained along the line of [15, Solution 152]. The second assertion is in [10, Lemma 3.3]; whether its C_0 version is true seems unknown.

An application of the preceding theorem is the following generalization of Corollary 5.5 and [35, Theorem 1].

Theorem 5.9. *Let A be in $\mathcal{S}_n$ and B be an n-by-n matrix. Then A and B are unitarily equivalent if and only if B is a contraction and $W(A) = W(B)$.*

This theorem can in turn be applied to obtain a characterization of matrices in $\mathcal{S}_n$ in terms of the $(n+1)$-Poncelet property for their numerical ranges.

Theorem 5.10. *An n-by-n contraction A belongs to $\mathcal{S}_n$ if and only if its numerical range $W(A)$ is contained in $\mathbb{D}$ and has the $(n+1)$-Poncelet property.*

Finally, the promised result on the numerical ranges of general matrices. The next theorem gives a sharp estimate on the inradius of the numerical range. Recall that the *inradius* of a compact convex subset $\triangle$ of the plane is the radius of the largest circular disc contained in $\triangle$.

Theorem 5.11. *For any n-by-n matrix A, the inradius of $W(A)$ is less than or equal to $\|A\| \cos(\pi/(n+1))$. It is equal to this quantity if and only if A is unitarily equivalent to $\|A\| J_n$.*

This is a consequence of Theorem 5.8 and the fact that the inradius of an $(n+1)$-polygonal region inscribed in $\partial\mathbb{D}$ is less than or equal to $\cos(\pi/(n+1))$.

The previous three theorems all appear in [9].

6. Generalizations

Two lines of generalizations of the results in Section 5 will be reported here. One concerns replacing the unitary dilation by the more general normal dilation. It turns out that two of the three major results in Section 5 (Theorems 5.3 and 5.7) can be generalized to this setting. Another generalization concerns the general compressions of the shift instead of the finite-dimensional ones. In this case, the situation is more intricate and our endeavor is less successful.

We start with the normal-dilation generalization. Emulating the representation of $\mathcal{S}_n$-matrices in Lemma 5.4, we consider a normal matrix $N = \operatorname{diag}(b_1, \ldots, b_{n+1})$ with distinct b_j's which are such that each is a corner of their convex hull and are arranged in a counterclockwise direction. The following is the analogue of Theorem 5.3 in the present setting.

Theorem 6.1. *Let N be as above and let c_j be a point on the open line segment (b_j, b_{j+1}), $j = 1, \ldots, n+1$ ($b_{n+2} \equiv b_1$). Then there is a vector x in $\mathbb{C}^{n+1}$ with all components nonzero such that the compression A of N to the orthogonal complement of x in $\mathbb{C}^{n+1}$ has numerical range $W(A)$ tangent to (b_j, b_{j+1}) at c_j for all j if and only if*

$$\prod_{j=1}^{n+1} |c_j - b_j| = \prod_{j=1}^{n+1} |c_j - b_{j+1}|.$$

Moreover, in this case such an A is unique up to unitary equivalence.

This can be proved along the line of arguments provided in Section 5 for Theorem 5.3. The proof of necessity has recently appeared in [1, Theorem 1].

Condition (6) in Theorem 5.6 relating the eigenvalues of a matrix A in $\mathcal{S}_n$ and those of its $(n+1)$-dimensional unitary dilations U are equivalent to the one derivable from (7) in the next theorem. The difference is that before it is obtained from the upper-triangular form of A while now it is from the diagonal form of U.

Theorem 6.2. *Let N be as before, let $x = (x_1, \ldots, x_{n+1})^T$ be a unit vector in $\mathbb{C}^{n+1}$ with all x_j's nonzero, and let A be the compression of N to the orthogonal complement of x in $\mathbb{C}^{n+1}$. Then the characteristic polynomial of A is given by*

$$\sum_{j=1}^{n+1} |x_j|^2 (z-b_1)\ldots(z-b_{j-1})(z-b_{j+1})\ldots(z-b_{n+1}). \tag{7}$$

This theorem can be proved by reducing it to the computation of the characteristic polynomial of a rank-one perturbation of N. The latter was discovered by Anderson [2, Theorem 0]. Theorem 5.7 corresponds to the special case of Theorem 6.2 when all the x_j's are equal to $1/\sqrt{n+1}$ (cf. [25]).

The above normal-dilation generalizations will appear in [13].

We now move to the second line of generalizations of the results in Section 5. In this respect, we only have some limited success. Recall that a general inner function ϕ has a canonical factorization as $c\phi_1\phi_2$, where c is a complex number with $|c| = 1$, ϕ_1 is a Blaschke product

$$\phi_1(z) = \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{z-a_n}{1-\overline{a_n}z}$$

with zeros a_n in $\mathbb{D}$ satisfying $\sum_n (1-|a_n|) < \infty$ and ϕ_2 is a singular function

$$\phi_2(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta) \right),$$

where μ is a positive measure on $\partial\mathbb{D}$ which is singular with respect to the Lebesgue measure on $\partial\mathbb{D}$. If $A = S(\phi)$ is the compression of the shift on $H = H^2 \ominus \phi H^2$, then the unitary dilations of A on a space K containing H with $\dim K \ominus H = 1$ can be identified and parametrized via results of Clark [7] on rank-one perturbations of $S(\phi)$. Indeed, in [7, Section I], there is defined, for any (nonconstant) inner function ϕ and any λ in $\partial\mathbb{D}$, a unitary operator U_λ on $H^2 \ominus \psi H^2$ by

$$(U_\lambda f)(z) = \begin{cases} zf(z) & \text{if } f \perp \phi, \\ \lambda & \text{if } f = \phi, \end{cases} \tag{8}$$

where ψ is the inner function $\psi(z) = z\phi(z)$. Such U_λ's are singular unitary operators and are exactly the rank-one perturbations of $S(\psi)$.

Theorem 6.3. *Let $A = S(\phi)$ on $H = H^2 \ominus \phi H^2$ for some (nonconstant) inner function ϕ. Then every unitary dilation U of A on a space K containing H with $\dim K \ominus H = 1$ is unitarily equivalent to U_λ in (8) for some λ in $\partial\mathbb{D}$, and, conversely, for any λ in $\partial\mathbb{D}$, there is a unitary dilation U of A of the above type which is unitarily equivalent to U_λ. Moreover, if the set*

$$E = \left\{ z \in \partial\mathbb{D} : \sum_n \frac{1-|a_n|^2}{|z-a_n|^2} + \int_0^{2\pi} \frac{d\mu(\theta)}{|z-e^{i\theta}|^2} = \infty \right\}$$

is countable, where a_n's are zeros of ϕ in $\mathbb{D}$ and μ is the singular measure on $\partial\mathbb{D}$ associated with ϕ, then U_λ is unitarily equivalent to $\operatorname{diag}(d_n)$ with d_n not in E and satisfying $d_n\phi(d_n) = \lambda$ for all n. In this case, each side of the (infinite) polygon formed by the d_n's intersects $W(A)$ at a single point.

In the situation of countable E, we expect that the numerical range of A be equal to the intersection of the numerical ranges of the U_λ's. If this is indeed the case, then we would obtain an infinite-dimensional analogue of Theorem 5.1. Unfortunately, we haven't been able to prove this due to some technical difficulties. The points in E act as the accumulation points of the vertices d_n of the (infinite) polygons associated with the various U_λ's. For the Blaschke product

$$\phi_1(z) = \prod_{n=1}^{\infty} \frac{z-(1-\frac{1}{n^2})}{1-(1-\frac{1}{n^2})z}$$

and the singular function

$$\phi_2(z) = \exp\left(\frac{z+1}{z-1}\right),$$

the corresponding E's are both equal to the singleton $\{1\}$. If ϕ is a finite Blaschke product

$$\phi(z) = \prod_{j=1}^{n} \frac{z-a_j}{1-\overline{a_j}z}, \quad |a_j| < 1 \text{ for all } j,$$

then using the relation $a_j\phi(a_j) = \lambda$ for all j we can easily derive condition (6), thus yielding an alternative proof of Theorem 5.6. On the other hand, it may happen that the numerical range of $S(\phi)$ is equal to $\mathbb{D}$, in which case the Poncelet property of $\partial W(S(\phi))$ does not make sense. One example of such a phenomenon is for ϕ to be the Blaschke product with zeros $(1-(1/n^2))\exp(in\theta_0)$, where θ_0 is a fixed irrational multiple of 2π.

This is what we have so far on this fascinating topic. The connection between the numerical range and Poncelet property forms a fertile soil, which has already nourished some very interesting developments. Its further progress is highly anticipated.

Added in proof. After the acceptance of this article, there appeared in the literature several other papers with pertinent information on the topics discussed here: [38] gives a much simpler proof of Theorem 5.8, [39] generalizes the unitary dilations for $\mathcal{S}_n$-matrices to invertible dilations, which still capture the main features of the Poncelet property, and, finally, [40] gives some auxiliary results which become more revealing when viewed through the prism of the Poncelet property.

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