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CONCERNING THE KARATSUBA CONJECTURES

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Abstract. In this note we discuss the Karatsuba conjectures on lower bound estimates for the maximum modulus of the Riemann Zeta-Function over very short intervals of the critical line and in small domains of the critical strip.

1. Introduction

One of the interesting questions in the theory of the Riemann zeta-function is the problem of lower estimates of the function

$$F(T;\Delta;\sigma) = \max_{T\le t\le T+\Delta} |\zeta(\sigma+it)|$$

for a fixed $\sigma, \frac{1}{2}\le\sigma\le 1$ and $\Delta=\Delta(T)$ increases with T. Many important results in this topic are described in Titchmarsh's book [4] [edited by D. R. Heath-Brown] which also contains results for $\Delta = 1$. The most interesting case is $\sigma = \frac{1}{2}$. Set, following [2],

$$F(T;\Delta) = F\left(T;\Delta;\frac{1}{2}\right).$$

Let us use this notation to reformulate one conditional theorem [4, p. 357]. As usual RH means the Riemann hypothesis. By $A, A_1, A_2, \ldots, c, c_1, c_2, \ldots$ we denote some absolute positive constants, T is a large positive parameter.

Theorem A. (*Assume RH*). *For* $\Delta = 1$ *we have*

$$F(T;\Delta) > e^{-A\frac{\log T}{\log\log T}}. \tag{1}$$

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There is also an unconditional theorem [4, p. 218] for the case $\delta = 1$. We formulate only part of it.

Theorem B. *If* H *is any number greater than unity*, *then*

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| > T^{-AH}$$

for $T \le t \le T+1$, *except possibly for a set of values of* t *of measure* $1/H$.

Note that Theorem B implies a positive answer to one of the conjectures of R. Balasubramanian and K. Ramachandra [1]:

Given any constant $\delta > 0$ *there exists a constant* $a > 0$ *depending on* δ *such that*

$$F(T;\delta) > T^{-a}$$

for $T \ge T_0(\delta; a) > 0$.

The proof of it follows from Theorem B by taking $H = \max(2; \frac{2}{\delta})$.

Let now $s_0 = \frac{1}{2} + iT$, and consider the function

$$G(T;\delta) = \max_{|s-s_0|=\delta} |\zeta(s)|.$$

A. A. Karatsuba [2, 3] considered the question of behavior of the functions $F(T;\delta)$ and $G(T;\delta)$ for $\delta = \delta(T) \to 0$ as $T \to +\infty$. He stated the following conjectures.

Conjecture 1. *There exists a function* $\delta = \delta(T) \to 0$ *as* $T \to +\infty$ *such that*

$$F(T;\delta) \ge T^{-A}.$$

Conjecture 2. *Conjecture* 1 *is valid for* $\delta = (\log\log T)^{-1}$.

Conjecture 3. *Conjecture* 1 *is valid for* $\delta = (\log T)^{-1}$.

Conjecture 1′. *There exists a function* $\delta = \delta(T) \to 0$ *as* $T \to +\infty$ *such that*

$$G(T;\delta) \ge T^{-A}.$$

Conjecture 2′. *Conjecture* 1′ *is valid for* $\delta = (\log\log T)^{-1}$.

Conjecture 3′. *Conjecture* 1′ *is valid for* $\delta = (\log T)^{-1}$.

A. A. Karatsuba also obtained the following unconditional results.

Theorem C. *For* $0 < \Delta \leq (\log T)^{-1}$ *we have*

$$F(T;\Delta) \geq e^{A \log \Delta \log T}.$$

Theorem D. *For* $0 < \Delta < \frac{1}{3}$ *we have*

$$G(T;\Delta) \geq e^{6 \log \Delta \log T}.$$

Obviously for $N = 1, 2, 3$ Conjecture N implies Conjecture N', and Conjecture 3 implies all the other conjectures.

It should be pointed out that RH implies Conjecture 3, and thus all the Karatsuba conjectures. This can be proved by using the folowing two Lemmas. Lemma A is a consequence of [4, Theorem 14.13]. Lemma B follows from [4, Theorem 14.15].

Lemma A. (*Assume RH*). *We have*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O\left(\frac{\log T}{\log \log T}\right),$$

where $N(T)$ *is a number of zeros of* $\zeta(s)$ *in the rectangle* $0 < \Re s < 1; 0 < \Im s \leq T.$

Lemma B. (*Assume RH*). *For* $\frac{1}{2} \leq \sigma \leq 2;\ t \geq t_0 > 0$ *and* $s = \sigma + it$ *we have*

$$\log \zeta(s) = \sum_{|t-\gamma| < 1/\log \log t} \log(s - \rho) + O\left(\frac{\log t \, \log \log \log t}{\log \log t}\right),$$

where $\rho = \frac{1}{2} + i\gamma$ *runs through all zeros of* $\zeta(s)$ *counting with multiplicity.*

Lemma B should be applied to any value of t defined from $t \in (T; T + \Delta)$ and $|t - \gamma| > c(\log T)^{-2}$ for all γ. Such t exists for $\Delta = (\log T)^{-1}$.

Using Lemma A and the proof of Theorem A (see [4, p. 357]) we can also under RH prove that estimate (1) is valid for $\Delta = (\log \log T)^{-1}$. Shortly speaking the following conditional theorem takes place.

Theorem E. (*Assume RH*). *Then*

a) *Conjecture* 3 *is valid*

b) *Estimate* (1) *is valid for* $\Delta = (\log \log T)^{-1}$.

We will prove the following theorems.

Theorem 1. *Conjecture* $3'$ *is equivalent to Conjecture* 3.

Theorem 1 states that for $\delta = (\log T)^{-1}$ the estimate

$$\max_{|s-s_0|\le\delta} |\zeta(s)| > T^{-A}$$

with some $A > 0$ is equivalent to

$$\max_{T\le t\le T+\delta} \left|\zeta\left(\frac{1}{2}+it\right)\right| > T^{-A_1},$$

with some $A_1 > 0$.

Theorem 2. *For* $0 < \delta < \frac{1}{3}$ *we have*

$$F(T,\delta) \ge e^{A\log\delta\,\log T}.$$

Theorem 2 extends Theorem C to the range $0 < \delta < \frac{1}{3}$ and implies Theorem D (with another constant).

2. Proof of Theorem 1

To prove Theorem 1 we use the following Lemmas.

Lemma 1. (Hadamard's three-circles theorem). *Let* $0 < r_1 < r_2$ *and let the function* $f(s)$ *be an anlytic in* $r_1 \le |s| \le r_2$. *If*

$$M(r) = \max_{|s|=r} |f(s)|$$

then for $r_1 \le r \le r_2$ *we have the estimate*

$$M(r)^{\log(r_2/r_1)} \le M(r_1)^{\log(r_2/r)} M(r_2)^{\log(r/r_1)}.$$

Lemma 2. (Borel-Carathéodory theorem). *Let* $f(s)$ *be an analytic function in* $|s-s_0| \le R$, *and let* $\Re f(s) \le M$ *on* $|s-s_0| = R$. *Then for* $0 < r < R$ *we have*

$$\max_{|s-s_0|=r} |f(s)| \le \frac{2rM}{R-r} + \frac{R+r}{R-r}|f(s_0)|.$$

For the proof of these assertions see [5, pp. 172-175].

Assume now that for all large enough T_1 we have $G(T_1,\delta_1) \ge T_1^{-A}$ with $\delta_1 = (\log T_1)^{-1}$. Let T be large enough and $\delta = (\log T)^{-1}$. Obviously we can find a number δ_2, $\frac{\delta}{3} < \delta_2 < \frac{2\delta}{3} - \delta^3$ such that there are no zeros of $\zeta(s)$ inside of the horizontal strip $T + \delta_2 \leq \Im s \leq T + \delta_2 + \delta^3$. Put $T_1 = T + \delta_2 + \frac{\delta^3}{2}$, $s_1 = \frac{1}{2} + iT_1$ and consider the circle $|s - s_1| = \delta^4$. We shall now prove that

$$G(T_1, \delta^4) := \max_{|s-s_1|=\delta^4} |\zeta(s)| > T^{-A_1}$$

for some absolute constant $A_1 > 0$. In order to do that we apply Lemma 1 with

$$f(s) = \zeta(s + s_1), \ r_1 = \delta^4, \ r = \delta_1 = (\log T_1)^{-1}, \ r_2 = 2.$$

From known properties of $\zeta(s)$ we have that

$$M(r_2) < T^2.$$

According to our hypothesis

$$M(r) > T^{-A_2}.$$

From the other side

$$\log(r_2/r_1)/\log(r_2/r) < 4, \quad \log(r/r_1)/\log(r_2/r) < 4.$$

Therefore Lemma 1 gives

$$G(T_1, \delta^4) = M(r_1) > T^{-A_1}.$$

It then follows that we can choose s_2 such that $|s_2 - s_1| = \delta^4$ and $|\zeta(s_2)| > T^{-A_1}$. Since $\zeta(s)$ does not vanish on union of the strip $T + \delta_1 \leq \Im s \leq T + \delta_1 + \delta^3$ and halfplane $\Re s \geq 1$ then in this region we can define $\log \zeta(s)$ to be real for real s and analytically continued along the segment $2, 2 + it$ and thereby to $s = \sigma + it$. Let us apply Lemma 2 with the following data:

$$s_0 = s_2, R = 4\delta^4, r = 2\delta^4, \ f(s) = \log \zeta(s) - \log \zeta(s_0).$$

According to our estimate of $|\zeta(s_2)|$ for $|s - s_2| \leq R$ we have

$$\Re f(s) = \log|\zeta(s)| - \log|\zeta(s_2)| < A_3 \log T.$$

Therefore in Lemma 2 we can take $M = A_3 \log T$. It then follows from $f(s_0) = 0$ that

$$\max_{|s-s_2|=r} \left| \log \frac{\zeta(s)}{\zeta(s_2)} \right| < A_4 \log T.$$

Note that the circle $|s - s_2| = r$ intersects the interval $(1/2 + iT, 1/2 + iT + i\delta)$. Hence for the point $s_3 = \frac{1}{2} + iT_2$ of intersection we have

$$\left| \log \frac{\zeta(s_3)}{\zeta(s_2)} \right| < A_4 \log T.$$

In particular

$$\Re\left(-\log\frac{\zeta(s_3)}{\zeta(s_2)}\right) < A_4\log T$$

i.e.

$$\log\left|\frac{\zeta(s_2)}{\zeta(s_3)}\right| < A_4\log T.$$

Therefore

$$\max_{T<t<T+\delta}\left|\zeta\left(\frac{1}{2}+it\right)\right| \geq |\zeta(s_3)| > |\zeta(s_2)|T^{-A_4} > T^{-A_5}.$$

Theorem 1 is proved.

3. Proof of Theorem 2

For the proof of Theorem 2 we require the following Lemma.

Lemma 3. *Let* $p(t)$ *be a monic polynomial of degree* N *with complex coefficients. Then for any* $\delta > 0$ *we have*

$$\max_{t\in[T;T+\delta]}|p(t)| \geq 2^{3-4N}\delta^N.$$

We shall prove Lemma 3 by induction on N. For $N=1$ it follows from

$$|p(T)| + |p(T+\delta)| \geq |p(T)-p(T+\delta)| = \delta.$$

Let now K is an integer, $K\geq 2$; and suppose that Lemma 3 is valid for all $N\leq K-1$. Under this assumption we shall prove that Lemma 3 is also valid for $N=K$.

Put $N_1 = [\frac{N+1}{2}]$, $N_2 = N - N_1 = [\frac{N}{2}]$. As a polynomial of a complex variable, $p(s)$ has N complex zeros (counting with multiplicity). No less than half of these zeros either are on the half-plane $\Re s \leq T+\frac{\delta}{2}$ or on $\Re s \geq T+\frac{\delta}{2}$. Without loss of generality we may suppose that the first case takes place. Let $s_1,\ldots,s_{N_1}$ be the first N_1 roots of $p(s)$ which lie on $\Re s \leq T+\frac{\delta}{2}$; and let $\rho_1,\ldots,\rho_{N_2}$ be all other zeros of $p(s)$. Note that some of ρ_j also can lie in the region $\Re s \leq T+\frac{\delta}{2}$.

We have

$$\max_{t\in[T;T+\delta]}|p(t)| \geq \max_{t\in[T+\frac{3\delta}{4};T+\delta]}|(t-s_1)\ldots(t-s_{N_1})(t-\rho_1)\ldots(t-\rho_{N_2})|.$$

Since $|t-s_j|\geq\frac{\delta}{4}$ for $t\in[T+\frac{3\delta}{4}, T+\delta]$ and $1\leq j\leq N_1$ then

$$\max_{t\in[T;T+\delta]}|p(t)| \geq (\delta/4)^{N_1}\max_{t\in[T+\frac{3\delta}{4};T+\delta]}|(t-\rho_1)\ldots(t-\rho_{N_2})|.$$

Denoting $p_1(t) = (t-\alpha_1)\dots(t-\alpha_{N_2})$, $T_1 = T + \frac{3\Delta}{4}$, $\Delta_1 = \frac{\Delta}{4}$ we obtain

$$\max_{t\in[T,T+\Delta]}|p(t)| \geq (\Delta/4)^{N_1} \max_{t\in[T_1,T_1+\Delta_1]}|p_1(t)|.$$

Using our assumption and that $p_1(t)$ is a polynomial of degree N_2 with a unit leading coefficient, we have

$$\max_{t\in[T_1,T_1+\Delta_1]}|p_1(t)| \geq (\Delta/4)^{N_2} 2^{3-4N_2}$$

whence

$$\max_{t\in[T,T+\Delta]}|p(t)| \geq 2^{3-4N}\Delta^{N}.$$

Lemma 3 is proved.

Now we proceed to prove Theorem 2. We use an inequality (9.7.3) from [4, p. 218]. Due to this inequality for $t \in [T, T+\Delta]$ we have

$$\log|\zeta(1/2+it)| \geq \sum_{|t-\gamma|\leq 1} \log|t-\gamma| - c\log T$$

where γ runs through imaginary parts of zeros of $\zeta(s)$. Therefore

$$|\zeta(1/2+it)| \geq e^{-c\log T} \prod_{|t-\gamma|\leq 1} |t-\gamma|. \tag{2}$$

Further, since $|T-t| \leq \frac{1}{3}$ and the interval $(T-2, T+2)$ contains at most $c_1\log T$ of γ, then

$$\prod_{|t-\gamma|\leq 1} |t-\gamma| \geq 2^{-c_1\log T} \prod_{|T-\gamma|\leq 1} |t-\gamma|. \tag{3}$$

We apply Lemma 3 to the polynomial $\prod_{|T-\gamma|\leq 1}|t-\gamma|$. The degree of this polynomial is not greater than $c_1\log T$. Hence

$$\max_{t\in[T,T+\Delta]} \prod_{|T-\gamma|\leq 1} |t-\gamma| \geq \Delta^{c_1\log T} e^{-c_3\log T}. \tag{4}$$

Now using (2)-(4) we obtain

$$|\zeta(1/2+it)| \geq e^{-c_4\log T}\Delta^{c_1\log T} \geq e^{c_5\log\Delta\log T}.$$

Theorem 2 is proved.

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