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# COERCIVE DIFFERENTIAL OPERATORS AND FRACTIONALLY INTEGRATED COSINE FUNCTIONS

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**Abstract.** We show that a class of abstract differential operators generates fractionally integrated cosine functions, and give an application to partial differential operators on many function spaces.

## 1. Introduction

The main motivation of studying semigroups of operators is due to their applications to partial differential operators (PDOs). However, many PDOs such as the Schrödinger operator $i\Delta$ on $L^p(\mathbb{R}^n)$ $(p \neq 2)$ [5] cannot be treated by strongly continuous semigroups ($C_0$-semigroups). Recently, a generalization of $C_0$-semigroups, i.e., integrated semigroups and their application to PDOs have received much attention (cf. [4]). Similarly, strongly continuous cosine functions cannot deal with many PDOs, e.g., the Laplacian $\Delta$ on $L^p(\mathbb{R}^n)$ $(n > 1,\, p \neq 2)$ [6]. It thus seems to be important to study the application of integrated cosine functions to PDOs.

In this paper, we will apply fractionally integrated cosine functions to nonelliptic PDOs with real constant coefficients, and improve the corresponding results in [1, 3, 8]. To that purpose and to avoid troubles caused by different function spaces, an abstract form of PDOs with constant coefficients will be introduced.

Let $X$ be a Banach space with norm $\|\cdot\|$, and let $B(X)$ be the space of all bounded linear operators from $X$ into itself. For a linear operator $B$ on $X$, by $\mathcal{D}(B)$ and $\rho(B)$ we denote its domain and resolvent set, respectively. $Y \hookrightarrow X$ will mean that $Y$ is continuously embedded in $X$, while $B|_Y$ will denote the restriction of $B$ to $Y$. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of rapidly decreasing functions, and let $C_c^\infty(\mathbb{R}^n)$ be

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the space of $C^\infty$-functions with compact support. Moreover, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and denote by $[\alpha]$ the integral part of $\alpha \geq 0$.

The definition of (exponentially bounded) integrated cosine functions is as follows:

**Definition.** *Let $C : [0, \infty) \to B(X)$ be an exponentially bounded and strongly continuous family, and let B be a linear operator on X. If there exists $\alpha \geq 0$ such that for large $\lambda \in \mathbb{R}$, $\lambda^2 \in \rho(B)$ and $\lambda^{1-\alpha}(\lambda^2 - B)^{-1}$ is the Laplace transform of $(C(t))_{t\geq 0}$, then we say that B generates an $\alpha$-times integrated cosine function $(C(t))_{t\geq 0}$. If in addition $C(\cdot) \in C([0, \infty), B(X))$, then $(C(t))_{t\geq 0}$ is said to be norm-continuous.*

We now introduce a functional calculus for commuting generators, $iA_j$ $(1 \leq j \leq n)$, of bounded $C_0$-groups on $X$. Write $A^\mu = A_1^{\mu_1} \cdots A_n^{\mu_n}$ $(\mu \in \mathbb{N}_0^n)$. Similarly, $D^\mu = D_1^{\mu_1} \cdots D_n^{\mu_n}$, where $D_j = -i\partial/\partial x_j$ $(1 \leq j \leq n)$. For a polynomial $P(\xi) := \sum_{|\mu| \leq m} a_\mu \xi^\mu$ $(\xi \in \mathbb{R}^n)$ with real coefficients, we define $P(A) = \sum_{|\mu|\leq m} a_\mu A^\mu$ with maximal domain. Then $P(A)$ is closable. Let $\mathcal{F}$ be the Fourier transform. If $u \in \mathcal{F}L^1(\mathbb{R}^n)$, then there exists a unique $L^1$-function, written as $\mathcal{F}^{-1}u$ (i.e., the inverse Fourier transform of $u$ in the distributional sense), such that $u = \mathcal{F}(\mathcal{F}^{-1}u)$. We define $u(A) \in B(X)$ by

$$(1) \qquad u(A)x = \int_{\mathbb{R}^n} (\mathcal{F}^{-1}u)(\xi) e^{-i(\xi, A)} x d\xi \quad \text{for } x \in X,$$

where $(\xi, A) = \sum_{j=1}^n \xi_j A_j$. Note that $P(A)$ cannot be defined by (1).

The following lemma will play a key role in our proof (cf. [9]).

**Lemma.** (a) *$\mathcal{F}L^1(\mathbb{R}^n)$ is a Banach algebra under pointwise multiplication and addition with norm $\|u\|_{\mathcal{F}L^1} := \|\mathcal{F}^{-1}u\|_{L^1}$, and $u \mapsto u(A)$ is an algebra homomorphism from $\mathcal{F}L^1(\mathbb{R}^n)$ into $B(X)$ with $\|u(A)\| \leq M\|u\|_{\mathcal{F}L^1}$ for all $u \in \mathcal{F}L^1(\mathbb{R}^n)$ and some $M > 0$.*

(b) *$E := \{\phi(A)x;\ \phi \in \mathcal{S}(\mathbb{R}^n),\ x \in X\} \subset \cap_{\mu\in\mathbb{N}_0^n} \mathcal{D}(A^\mu)$, $\overline{E} = X$, $\overline{P(A)|_E} = \overline{P(A)}$, and $\phi(A)P(A) \subseteq P(A)\phi(A) = (P\phi)(A)$ for $\phi \in \mathcal{S}(\mathbb{R}^n)$.*

(c) *If $n/2 < j \in \mathbb{N}$, then $H^j(\mathbb{R}^n) \hookrightarrow \mathcal{F}L^1(\mathbb{R}^n)$ and there exists $M > 0$ such that*

$$\|u\|_{\mathcal{F}L^1} \leq M\|u\|_{L^2}^{1-n/2j} \sum_{|\mu|=j} \|D^\mu u\|_{L^2}^{n/2j} \quad \text{for } u \in H^j(\mathbb{R}^n).$$

(d) *Let $u_t \in C^\infty(\mathbb{R}^n)$ $(t \geq 0)$. If there exist $M,\ L,\ a > 0,\ b < (2a/n) - 1$ such that $|D^k u_t(\xi)| \leq M(1 + t^{|k|})|\xi|^{b|k|-a}$ $(|\xi| \geq L,\ t \geq 0,\ |k| \leq [n/2] + 1)$, then there exist $\psi \in C_c^\infty(\mathbb{R}^n)$, $M' > 0$ such that $u_t(1 - \psi) \in \mathcal{F}L^1(\mathbb{R}^n)$ and $\|u_t(1 - \psi)\|_{\mathcal{F}L^1} \leq M'(1 + t^{n/2})$ $(t \geq 0)$.*

For $r \in (0, m]$, the polynomial $P(\xi)$ is called $r$-coercive if $|P(\xi)|^{-1} = O(|\xi|^{-r})$ as $|\xi| \to \infty$. In the sequel, for convenience, we denote by $M$ a general constant independent of $t$ and $\xi$. The following is the main result of this paper.

**Theorem.** *Let $P(\xi)$ be r-coercive for some $r \in (0, m]$, $\omega := \sup\{P(\xi);\ \xi \in \mathbb{R}^n\} < \infty$, and $\alpha > n(2m-r)/2r$. If $\rho(\overline{P(A)}) \neq \emptyset$, then $\overline{P(A)}$ generates a norm-continuous, α-times integrated cosine function $(C(t))_{t\geq 0}$ such that $\|C(t)\| \leq M g_{n,\alpha}(t)$ $(t \geq 0)$, where*

$$g_{n,\alpha}(t) = \begin{cases} (1+t^{n/2})e^{\sqrt{\omega}t} & for\ \omega > 0, \\ 1+t^{\alpha+n} & for\ \omega = 0, \\ 1+t^{\alpha+n/2} & for\ \omega < 0. \end{cases}$$

*Proof.* Let $t \geq 0$ and $|k| \leq [n/2]+1$ ($k \in \mathbb{N}_0^n$). Then an induction on $|k|$ leads to

$$\text{(2)}\quad D^k U(t\sqrt{P(\xi)}) = \sum_{j=0}^{|k|} t^j U^{(j)}(t\sqrt{P(\xi)}) Q_j(\xi)(P(\xi))^{-|k|+j/2} \quad \text{for } P(\xi) \neq 0,$$

where $Q_j(\xi)$ is a polynomial of degree $\leq (m-1)|k|$, and $U(\cdot) = \cosh(\cdot)$ or $\sinh(\cdot)$. Let

$$v_t = \Gamma(\beta)^{-1}\int_0^t (t-s)^{\beta-1} U(s\sqrt{P})ds \quad \text{for } \beta \in (0,1).$$

By our assumptions on $P(\xi)$, there exists a constant $L \geq 1$ such that $P(\xi) \leq -M|\xi|^r$ $(|\xi| \geq L)$. It thus follows from (2) that

$$|D^k v_t(\xi)| \leq M \sum_{j=0}^{|k|} |\xi|^{(m-1-r/2)|k|} \Big|\int_0^t (t-s)^{\beta-1} s^j U^{(j)}(s\sqrt{P(\xi)})ds\Big| \quad \text{for } |\xi| \geq L.$$

But, from an estimate of Kummer's function (cf. [3, p.278]),

$$\Big|\int_0^1 (1-u)^{\beta-1} u^j e^{\lambda u} du\Big| \leq M(|\lambda|^{-\beta} e^{\mathrm{Re}\,\lambda} + |\lambda|^{-j-\beta}) \quad \text{for } \lambda \in \mathbb{C}\setminus\{0\}$$

and from $\mathrm{Re}\sqrt{P(\xi)} \in \mathrm{i}\mathbb{R}$ it follows that

$$\text{(3)}\quad \begin{aligned} |D^k v_t(\xi)| &\leq M \sum_{j=0}^{|k|} t^{\beta+j}(|t\sqrt{P(\xi)}|^{-\beta} + |t\sqrt{P(\xi)}|^{-j-\beta})|\xi|^{(m-1-r/2)|k|} \\ &\leq M(1+t^{|k|})|\xi|^{(m-1-r/2)|k|-r\beta/2} \quad \text{for } |\xi| \geq L. \end{aligned}$$

Let $\omega' > \omega$ and $\beta = \alpha - [\alpha]$. Define

$$u_t = \begin{cases} \Gamma(\beta)^{-1} \int_0^t (t-s)^{\beta-1} W(s\sqrt{P}) ds (\omega' - P)^{-\langle\alpha\rangle} & \text{for } \beta > 0, \\ W(t\sqrt{P})(\omega' - P)^{-\langle\alpha\rangle} & \text{for } \beta = 0, \end{cases}$$

where $\langle\alpha\rangle = [[\alpha]/2]$ and

$$W(t\sqrt{P}) = \begin{cases} \cosh(t\sqrt{P}) & \text{for even } [\alpha], \\ \sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!} P^j & \text{for odd } [\alpha]. \end{cases}$$

A direct computation yields that

$$|D^k(\omega' - P(\xi))^{-\langle\alpha\rangle}| \le M|\xi|^{(m-1-r)|k| - r\langle\alpha\rangle} \quad \text{for } |\xi| \ge L. \tag{4}$$

If $[\alpha]$ is even, then by (3), (4) and Leibniz's formula we obtain

$$|D^k u_t(\xi)| \le M(1 + t^{|k|})|\xi|^{(m-1-r/2)|k| - r\alpha/2} \quad \text{for } t \ge 0 \text{ and } |\xi| \ge L. \tag{5}$$

If $[\alpha]$ is odd, we note the facts that

$$W(t\sqrt{P(\xi)}) = \sinh(t\sqrt{P(\xi)})/\sqrt{P(\xi)} \quad \text{for } |\xi| \ge L$$

and

$$|D^k(P(\xi))^{-1/2}| \le M|\xi|^{(m-1-r)|k| - r/2} \quad \text{for } |\xi| \ge L.$$

Combining these with (3)-(4), we find that (5) is also true for odd $[\alpha]$. Hence, by $\alpha > n(2m-r)/2r$ and Lemma (d), there exists $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $u_t(1-\psi) \in \mathcal{F}L^1(\mathbb{R}^n)$ and

$$\|u_t(1-\psi)\|_{\mathcal{F}L^1} \le M(1 + t^{n/2}). \tag{6}$$

Also, one can show as in the proof of (16) in [8] that

$$|D^k(u_t(\xi)\psi(\xi))| \le \begin{cases} Mt^{\beta+1}(1 + t^{2|k|}) & \text{for } \omega = 0 \text{ and odd } [\alpha], \\ Mg_{2|k|,\beta}(t) & \text{otherwise.} \end{cases}$$

Using Lemma (c) and then combining it with (6), we obtain that $u_t \in \mathcal{F}L^1$ and

$$\|u_t\|_{\mathcal{F}L^1} \le \begin{cases} Mt^{\beta+1}(1 + t^n) & \text{for } \omega = 0 \text{ and odd } [\alpha], \\ Mg_{n,\beta}(t) & \text{otherwise.} \end{cases} \tag{7}$$

We now construct $(C(t))_{t\ge 0}$. Let $C = (r - \overline{P(A)})^{-\langle\alpha\rangle}$ for some $r \in \rho(\overline{P(A)})$. Then $C_1 := (\omega' - \overline{P(A)})^{\langle\alpha\rangle} C \in B(X)$. Define

$$\begin{aligned} C(t)x = &\sum_{j=0}^{\langle\alpha\rangle} \binom{\langle\alpha\rangle}{j} r^{\langle\alpha\rangle - j}(-1)^j \Big\{ \int_0^t \cdots \int_0^t u_t(A) C_1 x (dt)^{2\langle\alpha\rangle - 2j} \\ &- \sum_{k=0}^{j-1} \frac{t^{\alpha+2k-2j}}{\Gamma(\alpha + 2k - 2j + 1)} \overline{P(A)}^k Cx \Big\} \quad \text{for } t \ge 0 \text{ and } x \in X. \end{aligned} \tag{8}$$

Obviously, $C : [0,\infty) \to B(X)$ is strongly continuous. By (7) and a simple computation, one finds that $\|C(t)\| \le M g_{n,\alpha}(t)$ $(t \ge 0)$. Meanwhile, from (8) and Lemma (a)-(b) we deduce that

$$(9)\quad C(t)\phi(A) = \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \cosh(s\sqrt{P}) ds\phi\Big)(A) \quad \text{for } \phi \in \mathcal{S} \text{ and } t \ge 0.$$

In view of $\overline{E} = X$, one has that

$$C(t)x = \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} C_\gamma(s)x ds \quad \text{for } t \ge 0 \text{ and } x \in X,$$

where $(C_\gamma(t))_{t\ge0}$ is the strongly continuous family $(C(t))_{t\ge0}$ in which $\alpha$ is replaced by $\gamma \in (n(2m-r)/2r, \alpha)$. The norm-continuity of $(C(t))_{t\ge0}$ now follows from this. Finally, let $L_\lambda$ $(\lambda > \{\max(0,\omega)\}^{1/2})$ be the Laplace transform of $(C(t))_{t\ge0}$. Then from Lemma (a)-(b), Fubini's theorem and (9), one can deduce that

$$\begin{aligned}(\lambda^2 - P(A))L_\lambda\phi(A) &= L_\lambda(\lambda^2 - P(A))\phi(A)\\ &= \Big(\int_0^\infty e^{-\lambda t} w_t dt(\lambda^2 - P)\phi\Big)(A)\\ &= \lambda^{1-\alpha}\phi(A) \quad \text{for } \phi \in \mathcal{S},\end{aligned}$$

where

$$w_t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \cosh(s\sqrt{P}) ds \quad \text{for } t \ge 0.$$

In view of $\overline{P(A)|_E} = \overline{P(A)}$, we obtain that $\lambda^2 \in \rho(\overline{P(A)})$ and $L_\lambda = \lambda^{1-\alpha}(\lambda^2 - \overline{P(A)})^{-1}$. The desired conclusion now follows from Definition. ■

By [2, Lemma 3.1], we easily obtain that a sufficient condition for $\rho(\overline{P(A)}) \ne \emptyset$ is $r > nm/(n+2)$. In particular, $\rho(\overline{P(A)}) \ne \emptyset$ if $P(\xi)$ is elliptic (i.e., $r = m$). Moreover, we remark that $\omega < \infty$ and $r = m$ is equivalent to the strong ellipticity of $P(\xi)$.

**Corollary.** *Let $P(\xi)$ be bounded above and $r$-coercive for some $r \in (nm/(n+2), m]$. Then $\overline{P(A)}$ generates a norm-continuous, $\alpha$-times integrated cosine function for every $\alpha > n(2m-r)/2r$. In particular, if $P(\xi)$ is strongly elliptic, then $\overline{P(A)}$ generates a norm-continuous, $\alpha$-times integrated cosine function for every $\alpha > n/2$.*

In the sequel, $X$ will be $L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$ or one of the following spaces of continuous functions: $\{f \in C(\mathbb{R}^n);\ \lim_{|x|\to\infty} f(x) = 0\}$, $\{f \in C(\mathbb{R}^n);\ \lim_{|x|\to\infty} f(x) \text{ exists}\}$, $\{f \in C(\mathbb{R}^n);\ f \text{ is bounded and uniformly continuous}\}$,

$\{f \in C(\mathbb{R}^n);\ f$ is 1-periodic$\}$, $\{f \in C(\mathbb{R}^n);\ f$ is almost periodic$\}$ with sup-norms. Assume that all partial differential operators have the maximal domains in the distributional sense, and so they are closed and densely defined on $X$. Since $iD_j$ is the generator of the translation group with respect to the $j$th variate for every $1 \leq j \leq n$, the above theorem and its corollary can be applied to $P(D)$ on $X$, immediately. In particular, when $X = L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, by the method of Fourier multipliers we can show sharper results (cf. [3, 9]). Indeed, the following statements are true:

(i) Suppose $P(\xi)$ satisfies the conditions of Theorem with $\alpha = (2m-r)n_X/r$, and

$$n_X \begin{cases} = n|\frac{1}{2} - \frac{1}{p}| & \text{if } X = L^p \ (1 < p < \infty), \\ > n/2 & \text{otherwise.} \end{cases}$$

If $\rho(P(D)) \neq \emptyset$, then $P(D)$ generates a norm-continuous, $(2m-r)n_X/r$-times integrated cosine function on $X$.

(ii) If $P(D)$ is strongly elliptic, then $P(D)$ generates a norm-continuous, $n_X$-times integrated cosine function on $X$.

We remark that conclusion (ii) improves the corresponding results in [1,3,8]. In fact, only the case that $P(D)$ generates an integrally integrated cosine function is considered in [1, 8], while Theorem 6.5 in [3] is related to a special form of $P(\xi)$, i.e., $P(\xi) = -(q(\xi))^2$ for some real elliptic polynomial $q(\xi)$. By the method of Fourier multipliers we can show that conclusions (i) and (ii) are also are true on $L^\infty(\mathbb{R}^n)$ and $\{f \in C(\mathbb{R}^n);\ f$ bounded$\}$ (cf. [3]). Moreover, if $P(\xi)$ is bounded above, then $P(D)$ generates a strongly continuous cosine function on $L^2(\mathbb{R}^n)$.

**Example.** (a) It was proved by Littman [6] that $\Delta$ does not generate a cosine function on $L^p(\mathbb{R}^n)$ $(n > 1,\ p \neq 2)$. But $\Delta$ generates by conclusion (ii) an $n_X$-times integrated cosine function on $X$. We refer to Hieber [3] for an improvement of this result.

(b) Consider the semi-elliptic differential operator $P(D)$ (cf. [7, pp.70-72]). Let $P(\xi) = \sum_{|\mu/e| \leq 1} a_\mu \xi^\mu$ $(\xi \in \mathbb{R}^n)$, where $|\mu/e| = \sum_{k=1}^n \mu_k/e_k$ $(\mu \in \mathbb{N}_0^n$, $e \in \mathbb{N}^n)$. We say $P(\xi)$, and so $P(D)$, is semi-elliptic if $\sum_{|\mu/e|=1} a_\mu \xi^\mu \neq 0$ $(\xi \neq 0)$. In this case, $P(\xi)$ is an $r$-coercive polynomial of degree $m$, where $r = \min\{e_k;\ 1 \leq k \leq n\}$ and $m = \max\{e_k;\ 1 \leq k \leq n\}$. If $P(\xi)$ is real and bounded above, and $\rho(P(D)) \neq \emptyset$, then $P(D)$ generates a $(2m-r)n_X/r$-times integrated cosine function on $X$.

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