TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 4, pp. 681-723, December 2001
This paper is available online at http://www.math.nthu.edu.tw/tjm/

RIEMANNIAN GEOMETRY OF LAGRANGIAN SUBMANIFOLDS

Bang-Yen Chen

Abstract. The study of Lagrangian submanifolds in Kähler manifolds and in the nearly Kähler six-sphere has been a very active field over the last quarter of century. In this article we survey the main results done during that period from Riemannian geometric point of view.

Table of Contents

Received October 28, 2000.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: Primary 50-02, 53C50, 53C52, 53B25; Secondary 53C38, 53D12, 58K25.
Key words and phrases: Lagrangian submanifolds, totally real submanifold, pinching theorem, obstruction, vanishing theorem, complex space form, index, stability, Lagrangian catenoid, finite type theory, Maslov class, ideal immersion, H-umbilical Lagrangian submanifold, parallel submanifold, basic inequality

1. Introduction

A $2n$-dimensional manifold is called a symplectic manifold if it admits a nondegenerate closed 2-form Ω, i.e., $\Omega^n \neq 0$ everywhere. An n-dimensional submanifold M of a symplectic $2n$-manifold $(\tilde{M}, \Omega)$ is called Lagrangian if the restriction of Ω on the tangent bundle of M vanishes identically. Thus, one has $\Omega(X, Y) = 0$ for X, Y tangent to M.

Symplectic manifolds and their Lagrangian submanifolds appear naturally in the context of classical mechanics and mathematical physics. For instance, the systems of partial differential equations of Hamilton-Jacobi type lead to the study of Lagrangian submanifolds and foliations in the cotangent bundle (cf., for instance, [66]). Furthermore, Lagrangian submanifolds are part of a growing list of mathematically rich special geometries that occur naturally in string theory.

The study of Lagrangian submanifolds of Kähler manifolds from Riemannian geometric point of view was initiated in the early 1970s. A submanifold M of a Kähler manifold $\tilde{M}$ is Lagrangian if the complex structure J of the ambient manifold $\tilde{M}$ carries each tangent space of M onto the corresponding normal space of M, i.e., $J(T_pM) = T_p^{\perp}M$ for any point $p \in M$. Since every curve in a Kähler curve is Lagrangian automatically, we only consider Lagrangian submanifolds of dimension greater than or equal to two.

Because the tangent bundle and the normal bundle of a Lagrangian submanifold are isomorphic via the complex structure J of the ambient manifold, the Lagrangian submanifold is a flat space if and only if the Lagrangian submanifold has flat normal connection.

The study of Lagrangian submanifolds in Kähler manifolds and in the nearly Kähler six-sphere is a very active field during the last quarter of century. Many interesting results on Lagrangian submanifolds from Riemannian point of view have been obtained by many mathematicians. In this article, we survey the main results on Lagrangian submanifolds in Kähler manifolds and also in the nearly Kähler six-sphere done during that period from this point of view.

2. Basic Properties

A result of Gromov [64] implies that every compact embedded Lagrangian submanifold of $\mathbb{C}^n$ is not simply-connected. This result is not true when the compact Lagrangian submanifolds were immersed but not embedded; for instance, an n-sphere can be immersed as a Lagrangian submanifold in $\mathbb{C}^n$.

Let h denote the second fundamental form of a Lagrangian submanifold M in $\tilde{M}$ and let $\sigma = Jh$. Another fundamental property of Lagrangian submanifolds is that $g(\sigma(X,Y),JZ)$ is totally symmetric, i.e., we have

$$g(\sigma(X,Y),JZ) = g(\sigma(Y,Z),JX) = g(\sigma(Z,X),JY) \tag{2.1}$$

for vectors X, Y, Z tangent to M.

Let $\tilde{M}^n(4c)$ denote a complex n-dimensional Kähler manifold with constant holomorphic sectional curvature $4c$. Such Kähler manifolds are called *complex space forms*. It is known that the universal covering of a complete complex space form $\tilde{M}^n(4c)$ is the complex projective n-space $CP^n(4c)$, the complex Euclidean n-space $\mathbb{C}^n$, or the complex hyperbolic space $CH^n(4c)$, according to $c > 0$, $c = 0$, or $c < 0$. A Käehler manifold is called Käehler-Einstein if its Ricci tensor is a constant multiple of its metric tensor.

The following existence and uniqueness theorems for Lagrangian isometric immersions in complex space forms are very useful [23, 38].

Theorem 2.1. *Let $(M, \langle\,.\,,.\rangle)$ be an n-dimensional simply connected Riemannian manifold. If σ is a TM-valued symmetric bilinear form on M satisfying*

(1) *$\langle\sigma(X,Y),Z\rangle$ is totally symmetric,*

(2) *$(\nabla\sigma)(X,Y,Z) = \nabla_X\sigma(Y,Z) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z)$ is totally symmetric,*

(3) *$R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + \sigma(\sigma(Y,Z),X) - \sigma(\sigma(X,Z),Y)$,*

then there exists a Lagrangian isometric immersion $L\colon (M, \langle\,.\,,.\rangle) \to \tilde{M}^n(4c)$ whose second fundamental form h is given by $h(X,Y) = J\sigma(X,Y)$.

Theorem 2.2. *Let $L_1, L_2\colon M \to \tilde{M}^n(4c)$ be two Lagrangian isometric immersions of a Riemannian manifold M with second fundamental forms h^1 and h^2. If*

$$\left\langle h^1(X,Y), JL_{1\star}Z\right\rangle = \left\langle h^2(X,Y), JL_{2\star}Z\right\rangle$$

for all vector fields X, Y, Z tangent to M, then there exists an isometry ϕ of $\tilde{M}^n(4c)$ such that $L_1 = \phi \circ L_2$.

Lagrangian submanifolds exist extensively. For instance, we have the following immersibility theorem from [31].

Theorem 2.3. *Every warped product $I \times_{f(s)} S^{n-1}$ of an open interval I and an $(n-1)$-sphere admits a Lagrangian isometric immersion into $\mathbb{C}^n$.*

In particular, Theorem 2.3 implies that every rotational hypersurface of Euclidean space and every real space form can be locally isometrically immersed in complex Euclidean space as Lagrangian submanifolds, although globally not every real space form is necessary so (cf. Section 3).

There exists a direct relationship between Lagrangian submanifolds in complex projective or complex hyperbolic space with C-totally real submanifolds of the Sasakian manifolds S^{2n+1} and H_1^{2n+1}. In fact, Reckziegel [94] proved that each horizontal lift of a Lagrangian submanifold M in $CP^n(4)$ (respectively, in $CH^n(-4)$), via the Hopf fibration $\pi : S^{2n+1} \to CP^n(4)$ (respectively, $\pi : H_1^{2n+1} \to CH^n(-4)$), is a C-totally real submanifold of S^{2n+1} (respectively, of the anti de Sitter space time H_1^{2n+1}). Conversely, the projection of an n-dimensional C-totally real submanifold of S^{2n+1} (respectively, of H_1^{2n+1}) via the Hopf fibration is a Lagrangian submanifold of $CP^n(4)$ (respectively, of $CH^n(-4)$).

Lagrangian submanifolds in a nonflat complex space form can be characterized by the curvature tensor of its ambient space in a very simple way. Namely, a result of [46] states that an n-dimensional submanifold M of a nonflat complex space form $\tilde{M}^n(4c)$ is curvature-invariant if and only if it is either a Kähler submanifold or a Lagrangian submanifold. Here, a submanifold M in a complex space form is called *curvature-invariant* if the Riemann curvature tensor $\tilde{R}$ of the ambient space satisfies $\tilde{R}(X,Y)TM \subset TM$ for X, Y tangent to M.

Harvey and Lawson [65] studied the so-called special Lagrangian submanifolds in $\mathbb{C}^n$, which are calibrated by the n-form Re$(dz_1 \wedge \cdots \wedge dz_n)$. Being calibrated implies volume minimizing in the same homology class. So, in particular, special Lagrangian submanifolds are oriented minimal Lagrangian submanifolds. In fact, a special Lagrangian submanifold M (with boundary ∂M) in $\mathbb{C}^n$ is volume minimizing in the class of all submanifolds N of $\mathbb{C}^n$ satisfying $[M] = [N] \in H_n^c(\mathbb{C}^n; \mathbb{R})$ with $\partial M = \partial N$. Harvey and Lawson constructed many examples of special Lagrangian submanifolds in $\mathbb{C}^n$.

Using the idea of calibrations, one can show that every Lagrangian minimal submanifold in an Einstein-Kähler manifold $\tilde{M}$ with $c_1(\tilde{M}) = 0$ is volume minimizing. It is false for the case $c_1 = \lambda\omega$ with $\lambda > 0$, where ω is the canonical symplectic form on $\tilde{M}$. It is unknown for the case $c_1 = \lambda\omega$ with $\lambda < 0$ (cf. [5]).

A general Kähler manifold may not have any minimal Lagrangian submanifold. In contrast, minimal Lagrangian submanifolds in an Einstein-Kähler manifold do exist in abundance, at least locally (cf. [5]). Bryant [6] constructed explicit examples of special Lagrangian tori in Ricci flat Kähler manifolds.

Minimal Lagrangian surfaces in the complex plane are well-understood. In fact, Chen and Morvan [42] proved that an orientable minimal surface M in $\mathbb{E}^4$ is a

Lagrangian surface with respect to an orthogonal complex structure on $\mathbb{E}^4$ if and only if it is a holomorphic curve with respect to some orthogonal complex structure on $\mathbb{E}^4$. It is also known from [2] that the only minimal Lagrangian immersion of a topological 2-sphere into CP^2 is the totally geodesic one.

The intrinsic and extrinsic structures of minimal Lagrangian surfaces in complex space forms are also well-understood. In fact, a simply connected Riemannian 2-manifold (M, g) with Gaussian curvature K less than a constant c admits a Lagrangian minimal immersion into a complete simply connected complex space form $\tilde{M}^2(4c)$ if and only if it satisfies the following differential equation [27, 37]:

$$\Delta \ln(c - K) = 6K, \tag{2.2}$$

where Δ is the Laplacian on M associated with the metric g.

If $f : M \to \tilde{M}^2(4c)$ is a Lagrangian minimal surface without totally geodesic points, then, w.r.t. an isothermal coordinate system satisfying $g = E(dx^2 + dy^2)$, $\Delta_0(\ln E) = 4E^{-2} - 2cE$, $\Delta_0 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, the second fundamental form of L is determined by

$$\begin{gathered} h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = -\frac{1}{E} J\left(\frac{\partial}{\partial x}\right), \quad h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{1}{E} J\left(\frac{\partial}{\partial y}\right), \\ h\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \frac{1}{E} J\left(\frac{\partial}{\partial x}\right). \end{gathered} \tag{2.3}$$

Conversely, it was proved in [27] that if E is a positive function defined on a simply connected domain U of $\mathbb{E}^2$ satisfying $\Delta_0(\ln E) = 4E^{-2} - 2cE$ and if $g = E(dx^2 + dy^2)$ is the metric tensor on U, then, up to rigid motions of $\tilde{M}^2(4c)$, there is a unique minimal Lagrangian isometric immersion of (U, g) into $\tilde{M}^2(4c)$ whose second fundamental form h is given by (2.3).

Very recently, Aiyama [114] obtains a simple proof of Chen-Morvan's result by applying a representation formula.

3. Obstructions to Lagrangian Isometric Immersions

Gromov stated in [63] that a compact n-manifold M admits a Lagrangian immersion into $\mathbb{C}^n$ if and only if the complexification of the tangent bundle, $TM \otimes \mathbb{C}$, is trivial. Since the tangent bundle of a 3-manifold is always trivial, Gromov's result implies that there do not exist topological obstructions to Lagrangian immersions for compact 3-manifolds. In views of Gromov's result, it is natural to search for *Riemannian obstructions* to Lagrangian isometric immersions.

Let $\tau = \sum_{i \neq j} K(e_i \wedge e_j)$ be the scalar curvature of a Riemannian n-manifold M, where $\{e_1, \dots, e_n\}$ is an orthonormal local frame. For an integer $k \geq 0$, denote by $\mathcal{S}(n, k)$ the finite set consisting of k-tuples $(n_1, \dots, n_k)$ of integers ≥ 2

satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Denote by $\mathcal{S}(n)$ the set of (unordered) k-tuples with $k \geq 0$ for a fixed positive integer n.

The cardinal number $\#\mathcal{S}(n)$ of $\mathcal{S}(n)$ is equal to $p(n) - 1$, where $p(n)$ denotes the number of partitions of n, which increases quite rapidly with n. For instance, for

$$n = 2,3,4,5,6,7,8,9,10, \ldots ,20, \ldots ,50, \ldots ,100, \ldots ,200,$$

the cardinal numbers $\#\mathcal{S}(n)$ are given by

$$1,2,4,6,10,14,21,29,41, \ldots , 626, \ldots , 204225, \ldots , 190569291, \ldots , 3972999029387,$$

respectively.

For each $(n_1, \ldots , n_k) \in \mathcal{S}(n)$, Chen introduced in [25, 30] the Riemannian invariant $\delta(n_1, \ldots , n_k)$ defined by

$$\delta(n_1, \ldots , n_k)(x) = \frac{1}{2}\Big(\tau(x) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}\Big), \tag{3.1}$$

where $L_1, \ldots , L_k$ run over all k mutually orthogonal subspaces of T_xM such that $\dim L_j = n_j$, $j = 1, \ldots , k$. Here $\tau(L_j)$ is the scalar curvature of the linear subspace L_j defined by $\tau(L_j) = \sum_{a\neq b} K(\epsilon_a \wedge \epsilon_b)$ and $\{\epsilon_a, a = 1, \ldots , n_j\}$ is an orthonormal basis of L_j. The invariants $\delta(n_1, \ldots , n_k)$ with $k > 0$ and the scalar curvature τ are very different in nature.

For each $(n_1, \ldots , n_k) \in \mathcal{S}(n)$, we put

$$a(n_1, \ldots , n_k) = \frac{1}{2}n(n-1) - \frac{1}{2}\sum_{j=1}^{k} n_j(n_j - 1), \tag{3.2}$$

$$b(n_1, \ldots , n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)}. \tag{3.3}$$

The following result from [25, 30] provides a sharp and simple relationship between the squared mean curvature $|H|^2$ and the invariant $\delta(n_1, \ldots , n_k)$.

Theorem 3.1. *For any n-dimensional Lagrangian submanifold M of a complex space form $\tilde{M}^n(4c)$ and any k-tuple $(n_1, \ldots , n_k) \in \mathcal{S}(n)$, we have*

$$\delta(n_1, \ldots , n_k) \leq b(n_1, \ldots , n_k)|H|^2 + a(n_1, \ldots , n_k)c. \tag{3.4}$$

The equality of (3.4) *holds at a point $x \in M$ if and only if there exists an orthonormal basis $e_1, \ldots , e_m$ at x such that the shape operator takes the following form*:

$$A_r = \begin{bmatrix} \begin{matrix} A_1^r & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_k^r \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mu_r I \end{bmatrix}, \quad r = n+1, \ldots , m, \tag{3.5}$$

where A_j^r are symmetric $n_j \times n_j$ submatrices satisfying

$$\text{trace}\,(A_1^r) = \cdots = \text{trace}\,(A_k^r) = \mu_r. \tag{3.6}$$

Inequality (3.4) is fundamental in the sense that it provides the control of the most fundamental extrinsic invariant, the squared mean curvature, through the string of Riemannian invariants $\delta(n_1, \dots, n_k)$.

By applying Theorem 3.1, we may obtain the following Riemannian obstructions to Lagrangian isometric immersions in complex space forms [30].

Theorem 3.2. *Let M be a compact Riemannian manifold with finite fundamental group $\pi_1(M)$ or null first Betti number $\beta_1(M)$. If there exists a k-tuple $(n_1, \dots, n_k)$ in $\mathcal{S}(n)$ such that*

$$\delta(n_1, \dots, n_k) > \frac{1}{2}\Big(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\Big)c, \tag{3.7}$$

then M admits no Lagrangian isometric immersion into a complex space form of constant holomorphic sectional curvature $4c$.

An immediate consequence of Theorem 3.2 is the following necessary Riemannian condition for compact Lagrangian submanifolds in $\mathbb{C}^n$; namely, the Ricci curvature of every compact Lagrangian submanifold M in $\mathbb{C}^n$ must satisfy

$$\inf_u \text{Ric}(u) \le 0, \tag{3.8}$$

where u runs over all unit tangent vectors of M.

For Lagrangian surfaces condition (3.8) means that the Gaussian curvature of every compact Lagrangian surface M in $\mathbb{C}^2$ must be nonpositive at some points on M. Another immediate consequence of (3.8) is that every compact irreducible symmetric space cannot be isometrically immersed in a complex Euclidean space as a Lagrangian submanifold.

Let $f : E^{n+1} \to \mathbb{C}^n$ be the map defined by

$$f(x_0, \dots, x_n) = \frac{1}{1 + x_0^2}(x_1, \dots, x_n, x_0x_1, \dots, x_0x_n). \tag{3.9}$$

Then f induces an immersion $w : S^n \to \mathbb{C}^n$ of S^n into $\mathbb{C}^n$ which has a unique self-intersection point $f(-1, 0, \dots, 0) = f(1, 0, \dots, 0)$.

With respect to the canonicals complex structure J on $\mathbb{C}^n$, the immersion w is a Lagrangian immersion of S^n into $\mathbb{C}^n$, which is called the Whitney immersion. The S^n endowed with the Riemannian metric induced from the Whitney immersion is called a *Whitney n-sphere.*

The example of Whitney immersion shows that the condition on the invariants given in Theorem 3.2 is sharp, since S^n has trivial fundamental group and trivial first Betti number when $n \geq 2$. Moreover, for each k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, the Whitney n-sphere satisfies $\delta(n_1, \ldots, n_k) > 0$ except at the unique point of self-intersection. Also, the assumptions on the finiteness of $\pi_1(M)$ and vanishing of $\beta_1(M)$ given above are both necessary.

For further applications of Theorem 3.1 and the invariants $\delta(n_1, \ldots, n_k)$, see for instance [25, 30, 34, 103].

4. Optimal Inequalities Between Scalar Curvature, Ricci Curvature, Shape Operator and Mean Curvature

Besides inequality (3.4), there exist other optimal general inequalities for Lagrangian submanifolds in complex space forms.

Theorem 4.1. *The scalar curvature τ and the squared mean curvature $|H|^2$ of a Lagrangian submanifold in complex space form $\tilde{M}^n(4c)$ satisfy the following general sharp inequality*:

$$\tau \leq n(n-1)c + \frac{n^2(n-1)}{n+2}|H|^2. \tag{4.1}$$

Inequality (4.1) with $c = 0$ and $n = 2$ was proved in [11]. Their proof relies on complex analysis which is not applicable to $n \geq 3$. The general inequality was established in [4] for $c = 0$ and arbitrary n; and in [21] for $c \neq 0$ and arbitrary n; and independently by [12] for $c \neq 0$ with $n = 2$, also using the method of complex analysis.

If $\tilde{M}^n(4c) = \mathbb{C}^n$, the equality of (4.1) holds identically if and only if the Lagrangian submanifold is either an open portion of a Lagrangian n-plane or, up to dilations, an open portion of the Whitney sphere [4] (see also [97] for an alternative proof).

It was proved in [21] that there exists a one-parameter family of Riemannian n-manifolds, denoted by P^n_a $(a > 1)$, which admit Lagrangian isometric immersions into $CP^n(4)$ satisfying the equality case of (4.1) for $c = 1$, and two one-parameter families of Riemannian manifolds, $C^n_a(a > 1)$, $D^n_a(0 < a < 1)$, and two exceptional n-spaces, F^n, L^n, which admit Lagrangian isometric immersion into $CH^n(-4)$ and satisfy the equality case of (4.1) for $c = -1$. It was proved in [21] that, besides the totally geodesic ones, these are the only Lagrangian submanifolds in $CP^n(4)$ and in $CH^n(-4)$ which satisfy the equality case of (4.1).

The explicit expressions of those Lagrangian immersions of P^n_a, C^n_a, D^n_a, F^n and L^n satisfying the equality case of (4.1) were completely determined in [48].

Castro and Urbano [13] showed that a Lagrangian surface in CP^2 satisfies the equality case of (4.1) for $n = 2$ and $c = 1$ if and only if the Lagrangian surface has holomorphic twistor lift. (For a most recent article related with inequality (4.1), see [8].)

Let $\overline{\mathrm{Ric}}$ denote the maximum Ricci curvature function on a Riemannian n-manifold M, which is defined by

$$\overline{\mathrm{Ric}}(p) = \max\{\mathrm{Ric}(u, u) \mid u \in U_pM\}, \quad p \in M, \tag{4.2}$$

where UM denotes the unit tangent bundle of M.

For the Ricci tensor Ric and the maximum Ricci curvature function $\overline{\mathrm{Ric}}$ of a Lagrangian submanifold in a complex space form, we have the following general results from [32].

Theorem 4.2. *If M is a Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$, then the Ricci tensor of M satisfies*

$$\mathrm{Ric} \leq \Big((n-1)c + \frac{n^2}{4}|H|^2\Big)g. \tag{4.3}$$

The equality case of (4.3) *holds identically if and only if either M is a totally geodesic submanifold or $n = 2$ and M is totally umbilical.*

Theorem 4.3. *Let M be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$. Then*

$$\overline{\mathrm{Ric}} \leq (n-1)c + \frac{n^2}{4}|H|^2. \tag{4.4}$$

If M satisfies the equality case of (4.4) *identically, then M is a minimal submanifold.*

Let M be a Riemannian n-manifold and L^k a k-plane section of T_xM^n, $x \in M$. For each unit vector X in L^k, we choose an orthonormal basis $\{e_1, \dots, e_k\}$ of L^k such that $e_1 = X$. Define the Ricci curvature Ric_{L^k} of L^k at X by

$$\mathrm{Ric}_{L^k}(X) = K_{12} + \cdots + K_{1k}, \tag{4.5}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j. We call $\mathrm{Ric}_{L^k}(X)$ a k-Ricci curvature of M at X relative to L^k. Clearly, the nth Ricci curvature is nothing but the Ricci curvature in the usual sense and the second Ricci curvature coincides with the sectional curvature.

For each integer k, $2 \leq k \leq n$, let θ_k denote the Riemannian invariant on M defined by

$$\theta_k(x) = \left(\frac{1}{k-1}\right) \inf_{L^k, X} \mathrm{Ric}_{L^k}(X), \quad X \in T_x M, \tag{4.6}$$

where L^k runs over all k-plane sections in $T_x M$ and X runs over all unit vectors in L^k.

The following results provide a sharp relationship between the k-Ricci curvature and the shape operator for Lagrangian submanifold in a complex space form.

Theorem 4.4. *Let $f : M \to \tilde{M}^n(4c)$ be a Lagrangian isometric immersion of a Riemannian manifold M into a complex space form $\tilde{M}^n(4c)$. For any integer k, $2 \leq k \leq n$, and any point $x \in M$, we have*

(1) *If $\theta_k(x) \neq c$, then the shape operator in the direction of the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}(\theta_k(x) - c)I \quad \text{at } x, \tag{4.7}$$

where I is the identity map of TM.

(2) *If $\theta_k(x) = c$, then $A_H \geq 0$ at x.*

(3) *A unit vector $X \in T_x M$ satisfies $A_H X = ((n-1)/n)(\theta_k(x) - c)X$ if and only if $\theta_k(x) = c$ and X lies in the relative null space at x.*

(4) *$A_H \equiv ((n-1)/n)(\theta_k - c)I$ at x if and only if x is a totally geodesic point.*

Since the proof of Theorem 4.4 bases on the equation of Gauss, its proof is exactly the same as the one given in [29] for submanifolds in real space form (see [76]).

5. A Vanishing Theorem for Lagrangian Isometric Immersions

Let $S^1 \subset \mathbb{C}^1$ denote a unit circle in $\mathbb{C}^1$. Then the n-torus

$$T^n = S^1 \times \cdots \times S^1 \subset \mathbb{C}^n = \mathbb{C}^1 \times \cdots \mathbb{C}^1$$

is a Lagrangian submanifold with nonzero constant mean curvature. On contrast, the following vanishing theorem from [26] implies that the mean curvature function of each compact Lagrangian submanifold M in an Einstein-Kähler manifold must have zeros under a simple topological condition.

Theorem 5.1. *Let M be a compact manifold with finite fundamental group $\pi_1(M)$ or null first Betti number $\beta_1(M)$. Then every Lagrangian immersion from M into any Einstein-Kähler manifold must have minimal points.*

Theorem 5.1 has several interesting geometric applications. For example, it implies the following.

Theorem 5.2.

(1) *There do not exist Lagrangian isometric immersions from a compact Riemannian n-manifold with positive Ricci curvature into any flat Kähler n-manifold or into any complex hyperbolic n-space.*

(2) *Up to dilations, a compact Lagrangian submanifold M in $\mathbb{C}^n$ is congruent to the Whitney sphere if and only if JH is a conformal vector field, where J is the complex structure and H the mean curvature vector field.*

(3) *Every Lagrangian isometric immersion from a spherical space form into a complex projective n-space CP^n is a totally geodesic immersion if it has constant mean curvature.*

(4) *Every Lagrangian isometric immersion of a compact Riemannian manifold with positive Ricci curvature into an Einstein-Kähler manifold is a minimal immersion if it has constant mean curvature.*

Statement (2) of Theorem 5.2 follows from Theorems 5.1 and 14.3.

Theorem 5.1 is sharp in the sense that both conditions on β_1 and π_1 cannot be removed.

6. Pinching Theorems for Lagrangian Minimal Submanifolds

It follows from the equation of Gauss that an n-dimensional totally real minimal submanifold of a complex space form $\tilde{M}^n(4c)$ satisfies the following two properties [46]:

(1) Ric$\leq (n-1)cg$, with equality holding if and only if it is totally geodesic.

(2) $\tau \leq n(n-1)c$, with equality holding if and only if it is totally geodesic.

Montiel, Ros and Urbano proved the following [81].

Theorem 6.1. *If the Ricci curvature of a compact Lagrangian submanifold M of $CP^n(4c)$ satisfies* Ric $\geq 3(n-2)c/4$, *then the second fundamental form is parallel*, *i.e.*, *M is a parallel submanifold.*

Compact Lagrangian submanifolds of $CP^n(4c)$ with parallel second fundamental form were completely classified by Naitoh in [84, 85].

By applying the result of Naitoh and Theorem 4.2, we know that an n-dimensional compact Lagrangian minimal submanifold of $CP^n(4)$ satisfies Ric $\geq 3(n-2)c/4$

is one of the following:

$$
\begin{array}{rcl}
RP^n(1) & \to & CP^n(4) \ \ (r \geq 2) \ \ \text{(totally geodesic)}, \\
T^2 & \to & CP^2(4) \ \ \text{(minimal)}, \\
SU(r)/SO(r) & \to & CP^{(r-1)(r+2)/2}(4) \ \ (r \geq 3) \ \ \text{(minimal)}, \\
SU(2r)/Sp(r) & \to & CP^{(r-1)(2r+1)}(4) \ \ (r \geq 3) \ \ \text{(minimal)}, \\
SU(r) & \to & CP^{r^2-1}(4) \ \ (r \geq 3) \ \ \text{(minimal)}, \\
E_6/F_4 & \to & CP^{26}(4) \ \ \text{(minimal)}.
\end{array} \tag{6.1}
$$

Liu [74] investigated scalar pinching for complete Lagrangian submanifolds and obtained the following.

Theorem 6.2. *Let M be a complete Lagrangian minimal submanifold in $CP^n(4)$. If the Ricci curvature of M is bounded from below, then either M is totally geodesic or the infimum of the scalar curvature of M is less than or equal to $(3n+1)(n-2)/3$.*

Compact Lagrangian minimal submanifolds in $CP^n(4)$ satisfying a pinching on scalar curvature were studied in [46, 75, 84, 85, 79, 111, 112], among others. In particular, we have the following.

Theorem 6.3 [79]. *Let M be a compact Lagrangian minimal submanifold of $CP^n(4c)$. Then M has constant scalar curvature τ satisfying $\tau \geq 3n(n-2)c/4$ if and only if one of the following conditions holds*:

(A) $\tau = n(n-1)c/4$ *and M is totally geodesic*,

(B) $\tau = 0$, $n = 2$, *and M is a finite Riemannian covering of the flat torus minimally embedded in $CP^2(4c)$ with parallel second fundamental form*, *or*

(C) $\tau = 3n(n-2)c/4$, $n > 2$, *and M is an embedded submanifold congruent to the standard embedding of*:

$$
\begin{gathered}
SU(3)/SO(3), \ \ n = 5; SU(6)/Sp(3), \ \ n = 14, \\
SU(3), \ \ n = 8, \text{or } E_6/F_4, \ \ n = 26.
\end{gathered} \tag{6.2}
$$

A submanifold M of a Riemannian manifold is called an isotropic submanifold if its second fundamental form h satisfies $|h(v,v)| = \lambda(p)$ for each point p of M and every unit vector v tangent to M at p. Moreover, if λ is constant then M is said to be constant isotropic.

Montiel and Urbano [82] proved the following.

Theorem 6.4. *If an n-dimensional $(n \geq 3)$ minimal Lagrangian submanifold M of a complex n-dimensional Kâhler manifold is isotropic, then M is totally geodesic or the dimension n is equal to* 5, 8, 14 *or* 26.

Montiel and Urbano also classified constant isotropic Lagrangian submanifolds of complex space forms.

Xia [111] and Matsuyama [78] proved the following.

Theorem 6.5. *Let M be a compact minimal Lagrangian submanifold of $CP^n(4c)$. If $|h(v,v)|^2 \leq c/2$ for any unit tangent vector v, then either*

(1) *M is totally geodesic, or*

(2) *$|h(v,v)|^2 = c/2$, $n = 2$, and M is a finite Riemannian covering of a flat torus embedded in $CP^2(4c)$ with parallel second fundamental form, or*

(3) *$|h(v,v)|^2 = c/2$, $n > 2$, and M is an embedded submanifold congruent to the standard embedding of the spaces given in* (6.2).

7. Lagrangian Submanifolds with Parallel Mean Curvature Vector

A natural extension of minimal submanifold is submanifolds with parallel mean curvature vector.

Chen, Houh and Lue proved in 1977 the following.

Theorem 7.1 [41]. *Let M be a compact Lagrangian submanifold of $\mathbb{C}^n$. If M has nonnegative sectional curvature and parallel mean curvature vector, then M is a product submanifold $M_1 \times \cdots \times M_r$, where M_j is a compact Lagrangian submanifold embedded in some complex linear subspace $\mathbb{C}^{n_j}$ and is immersed as a minimal submanifold in a hypersphere of $\mathbb{C}^{n_j}$.*

Theorem 7.1 was extended to complete Lagrangian minimal submanifolds of $\mathbb{C}^n$ by Urbano in [107] and by Ki and Kim in [68].

Ohnita [91] and Urbano [106] investigated Lagrangian submanifolds in complex projective space with $K \geq 0$ and obtained the following.

Theorem 7.2. *Let M be a compact Lagrangian submanifold of $CP^n(4)$ with parallel mean curvature vector. If M has nonnegative sectional curvature, then the second fundamental form of M is parallel.*

By applying Theorem 7.2 and the classification of parallel submanifolds of CP^n, it follows that compact Lagrangian submanifolds of $CP^n(4)$ with parallel mean curvature vector and $K \geq 0$ must be the product submanifolds: $T \times M_1 \times \cdots \times M_k$, where T is a flat torus with $\dim M \geq k-1$ and each M_i is one of the following:

$$\begin{aligned}
RP^r(1) &\rightarrow CP^r(4) \;\; (r \geq 2) \;\; (\text{totally geodesic}),\\
SU(r)/SO(r) &\rightarrow CP^{(r-1)(r+2)/2}(4) \;\; (r \geq 3) \;\; (\text{minimal}),\\
SU(2r)/Sp(r) &\rightarrow CP^{(r-1)(2r+1)}(4) \;\; (r \geq 3) \;\; (\text{minimal}),\\
SU(r) &\rightarrow CP^{r^2-1}(4) \;\; (r \geq 3) \;\; (\text{minimal}),\\
E_6/F_4 &\rightarrow CP^{26}(4) \;\; (\text{minimal}).
\end{aligned} \tag{7.2}$$

For 3-dimensional compact Lagrangian submanifolds of complex space forms, Urbano [107] proved the following.

Theorem 7.3. *If M^3 is a 3-dimensional compact Lagrangian submanifold of a complex space form $\tilde{M}^3(4c)$ with nonzero parallel mean curvature vector, then M^3 is flat and has parallel second fundamental form.*

8. Lagrangian Real Space Form $M^n(c)$ in Complex Space Form $\tilde{M}^n(4c)$

The simplest examples of Lagrangian submanifolds of complex space forms are totally geodesic Lagrangian submanifolds. A totally geodesic Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$ is a real space form of constant sectional curvature c.

The real projective n-space $RP^n(1)$ (respectively, the real hyperbolic n-space $H^n(-1)$) can be isometrically embedded in $CP^n(4)$ (respectively, in complex hyperbolic space $CH^n(-4)$) as a Lagrangian totally geodesic submanifold.

Non-totally geodesic Lagrangian isometric immersions from real space forms of constant curvature c into a complex space form $\tilde{M}^n(4c)$ were determined by Chen, Dillen, Verstraelen, and Vrancken in [39] by applying the notion of twisted products.

Let $(M_1, g_1), \ldots, (M_m, g_m)$ be m Riemannian manifolds, f_i a positive function on $M_1 \times \cdots \times M_m$ and $\pi_i : \ M_1 \times \ldots \times M_m \to M_i$ the ith canonical projection, $i = 1, \ldots, m$. Then the *twisted product*

$$ {}_{f_1}M_1 \times \cdots \times_{f_m} M_m \tag{8.1}$$

of $(M_1, g_1), \ldots, (M_m, g_m)$ is by definition the differentiable manifold $M_1 \times \ldots \times M_m$ equipped with the twisted product metric g defined by

$$ g(X, Y) = f_1 \cdot g_1(\pi_{1*}X, \pi_{1*}Y) + \cdots + f_m \cdot g_m(\pi_{m*}X, \pi_{m*}Y) \tag{8.2}$$

for vector fields X and Y on $M_1 \times \cdots \times M_m$.

Let $N^{n-k}(c)$ denote an $(n-k)$-dimensional real space form of constant sectional curvature c. Consider, for $k < n-1$, the twisted product:

$$ {}_{f_1}I_1 \times \cdots \times_{f_k} I_k \times_1 N^{n-k}(c) \tag{8.3}$$

with twisted product metric defined by

$$ g = f_1 dx_1^2 + \cdots + f_k dx_k^2 + g_0, \tag{8.4}$$

where g_0 denotes the canonical metric of $N^{n-k}(c)$ and $I_1, \ldots, I_k$ are open intervals. When $k = n-1$ (respectively, $k = n$), consider the following twisted product instead:

$$ {}_{f_1}I_1 \times \cdots \times_{f_{n-1}} I_{n-1} \times_1 I_n \quad \text{(respectively, } {}_{f_1}I_1 \times \cdots \times_{f_n} I_n). \tag{8.5}$$

If the twisted product defined by (8.3) or (8.5) is a real space form $M^n(c)$ of constant sectional curvature c, it is called a *twisted product decomposition* of $M^n(c)$. We denote such a decomposition of $M^n(c)$ by $\mathcal{TP}^n_{f_1\cdots f_k}(c)$.

Coordinates $\{x_1,\ldots,x_n\}$ on $\mathcal{TP}^n_{f_1\cdots f_k}(c)$ such that $\partial/\partial x_j$ is tangent to I_j for $j=1,\ldots,k$, the last $n-k$ coordinate vectors are tangent to $N^{n-k}(c)$ and the metric in this coordinate system takes the form (8.4) are called *adapted coordinates*.

The 1-form $\Phi(\mathcal{TP})$ defined on $\mathcal{TP}^n_{f_1\cdots f_k}(c)$ by

$$\Phi(\mathcal{TP}) = f_1 dx_1 + \cdots + f_k dx_k \tag{8.6}$$

is called the *twistor form* of $\mathcal{TP}^n_{f_1\cdots f_k}(c)$.

The twistor form $\Phi(\mathcal{TP})$ of $\mathcal{TP}^n_{f_1\cdots f_k}(c)$ is said to be *twisted closed* if

$$\sum_{i,j=1}^{k} \frac{\partial f_i}{\partial x_j} dx_j \wedge dx_i = 0. \tag{8.7}$$

It follows from (8.6) that the twistor form $\Phi(\mathcal{TP})$ is automatically twisted closed when $k=1$. If $k=n$, the twistor form is twisted closed if and only if it is a closed 1-form in the usual sense.

Chen, Dillen, Verstraelen, and Vrancken [39] proved the following theorem which determines Lagrangian real space form of constant curvature c in a complex space form $\tilde{M}^n(4c)$.

Theorem 8.1. *We have the following.*

(1) *Let $\mathcal{TP}^n_{f_1\cdots f_k}(c)$, $1\le k\le n$, be a twisted product decomposition of a simply connected real space form $M^n(c)$. If the twistor form $\Phi(\mathcal{TP})$ of $\mathcal{TP}^n_{f_1\cdots f_k}(c)$ is twisted closed, then, up to rigid motions of $\tilde{M}^n(4c)$, there is a unique Lagrangian isometric immersion*:

$$L_{f_1\cdots f_k} : \mathcal{TP}^n_{f_1\cdots f_k}(c) \to \tilde{M}^n(4c), \tag{8.8}$$

whose second fundamental form satisfies

$$\begin{aligned} &h\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j}\right) = J\frac{\partial}{\partial x_j}, \quad j=1,\ldots,k, \\ &h\left(\frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_t}\right) = 0, \quad otherwise, \end{aligned} \tag{8.9}$$

for any adapted coordinate system $\{x_1,\ldots,x_n\}$ on $\mathcal{TP}^n_{f_1\cdots f_k}(c)$.

(2) *Let M be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$. Then M is of constant sectional curvature c if and only if, for each point $p \in M$, the second fundamental form of M in $\tilde{M}^n(4c)$ at p satisfies*

$$\begin{aligned} &h(e_1, e_1) = \lambda_1 Je_1, \ldots, h(e_n, e_n) = \lambda_n Je_n, \\ &h(e_i, e_j) = 0, \quad 1 \le i \ne j \le n, \end{aligned} \tag{8.10}$$

for some $\lambda_1, \ldots, \lambda_n$ with respect to some orthonormal basis $e_1, \ldots, e_n$ of T_pM.

(3) *Let $\tilde{L} : M^n(c) \to \tilde{M}^n(4c)$ be a Lagrangian isometric immersion of a real space form $M^n(c)$ into $\tilde{M}^n(4c)$. If there is an integer $k \in \{1, \ldots, n\}$ such that the second fundamental form of $\tilde{L}$ satisfies*

$$\begin{aligned} &h(e_1, e_1) = \lambda_1 Je_1, \ldots, h(e_k, e_k) = \lambda_k Je_k, \quad and \\ &h(e_i, e_j) = 0, \quad otherwise, \end{aligned} \tag{8.11}$$

for some nowhere vanishing functions $\lambda_1, \ldots, \lambda_k$, with respect to some suitable orthonormal frame field $e_1, \ldots, e_n$, then $M^n(c)$ admits the twisted product decomposition $\mathcal{TP}^n_{\lambda_1^{-2} \cdots \lambda_k^{-2}}(c)$ whose twistor form is twisted closed.

Moreover, up to rigid motions, the Lagrangian isometric immersion $\tilde{L}$ is given by the Lagrangian isometric immersion $L_{f_1 \cdots f_k}$ given in Statement (1).

Chen *et al*. also provides in [39] the explicit construction of adapted Lagrangian isometric immersions of some natural twisted product decompositions of real space forms. Oh [90] determined in her doctoral thesis the explicit expressions of adapted Lagrangian isometric immersions for the remaining twisted product decompositions of real space forms also by applying Theorem 8.1.

9. More on Lagrangian Real Space Forms in Complex Space Forms

For Lagrangian minimal immersions in complex space forms, Chen and Ogiue [46] proved the following.

Theorem 9.1. *A Lagrangian minimal submanifold of constant sectional curvature c in a complex space form $\tilde{M}^n(4\tilde{c})$ is either totally geodesic* (*with $c = \tilde{c}$*) *or $c \le 0$.*

On the other hand, Ejiri [60] proved the following.

Theorem 9.2. *The only Lagrangian minimal submanifolds of constant sectional curvature $c \le 0$ in a complex space form $\tilde{M}^n(4\tilde{c})$ are the flat ones.*

Ejiri's result extends the corresponding result of [46] for $n = 2$ to $n \ge 2$.

A submanifold M of a Riemannian manifold is known as a Chen submanifold if $\sum_{i,j}\langle h(e_i,e_j),H\rangle h(e_i,e_j)$ is parallel to the mean curvature vector H, where $\{e_i\}$ is an orthonormal frame of the submanifold M (for general properties and examples of Chen submanifolds, see for instance [62, 98]).

Kotani studied in [69] Lagrangian Chen submanifolds of constant curvature in complex space forms and obtained the following.

Theorem 9.3. *If M is a Lagrangian Chen submanifold with constant sectional curvature in a complex space form $\tilde M^n(4\tilde c)$ with $c<\tilde c$, then either M is minimal, or locally, $M = I\times \tilde L^{n-1}$ equipped with the warped metric $g = dt^2 + f(t)\tilde g$, where I is an open interval, $(\tilde L^{n-1},\tilde g)$ is the following submanifold in $\tilde M^n(4\tilde c)$:*

$$\begin{array}{ccc} \tilde L^{n-1} & \subset S^{2n-1} & \subset \tilde M^n(4\tilde c) \\ \pi\downarrow & \downarrow\pi & \\ L^{n-1} & \subset CP^{n-1}, & \end{array} \tag{9.1}$$

S^{2n-1} is a geodesic hypersphere in $\tilde M^n(4\tilde c)$, and $\tilde L$ is the horizontal lift of a Lagrangian minimal flat torus L^{n-1} in CP^{n-1}.

Dajczer and Tojeiro [51] studied flat Lagrangian submanifolds in CP^n and proved the following.

Theorem 9.4. *A complete flat Lagrangian submanifold in CP^n with constant mean curvature is a flat torus T^n with parallel second fundamental form.*

Let $\tilde M^n_k(4\tilde c)$ denote a complex space form of constant holomorphic sectional curvature $4\tilde c$, complex dimension n and complex index k. In [70], Kriele and Vrancken studied Lagrangian minimal isometric immersions of a Lorentzian real space form $M^n_1(c)$ of constant sectional curvature c into a Lorentzian complex space form $\tilde M^n_1(4\tilde c)$. They obtained the following.

Theorem 9.5. *Let f be a Lagrangian minimal isometric immersions of a Lorentzian real space form $M^n_1(c)$ of constant sectional curvature c into a Lorentzian complex space form $\tilde M^n_1(4\tilde c)$.*

(1) *If $\tilde c<0$ and $n>3$, then $c=\tilde c$, and*

(2) *if $\tilde c>0$ and $c\neq\tilde c$, then $c=0$.*

Kriele and Vrancken also classified Lagrangian minimal isometric immersions of a Lorentzian real space form $M^n_1(c)$ into a Lagrangian complex space form $\tilde M^n_1(4\tilde c)$ with $c\neq\tilde c$ in [70]. The method used in [70] relies heavily on the assumption: $c\neq\bar c$. Hence the method of [70] does not apply to the most fundamental case; namely, minimal Lagrangian submanifolds of constant sectional curvature c in a Lorentzian complex space form $\tilde M^n_1(4c)$ of holomorphic sectional curvature $4c$.

The classification of Lagrangian minimal isometric immersions of a Lorentzian real space form $M_1^n(c)$ into a Lagrangian complex space form $\tilde{M}_1^n(4c)$ was established by Chen and Vrancken in [50] using a method different from [70].

10. Ideal Lagrangian Isometric Immersions

A Lagrangian isometric immersion from a Riemannian n-manifold into a complex space form $\tilde{M}^n(4c)$ is called *ideal* if it satisfies the equality case of (3.4) for some k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$.

Roughly speaking, an ideal isometric immersion of a Riemannian manifold into a space form is an isometric immersion which produces the least possible amount of tension from the ambient space at each point of the submanifold. Recently, many interesting results on ideal immersions have been obtained by many mathematicians (see [25] for details).

In this section, we present the results obtained in [33] concerning ideal Lagrangian isometric immersions into complex space forms.

Theorem 10.1. *Every ideal Lagrangian submanifold of a complex space form is a minimal submanifold.*

For a Lagrangian immersion $f : M \to \tilde{M}^n(4c)$, the first normal space at a point $p \in M$ is defined to be the image space, $\operatorname{Im} h_p$, of the second fundamental form h at p. A Lagrangian immersion is said to have *full first normal bundle* if the first normal space of M equals to the normal space at each point $p \in M$, i.e., $\operatorname{Im} h = T^\perp M$.

Theorem 10.2. *If $f : M \to \mathbb{C}^n$ is a Lagrangian immersion of a Riemannian n-manifold into the complex Euclidean n-space $\mathbb{C}^n$ with full first normal bundle, then f is an ideal Lagrangian immersion if and only if locally it is the product of some minimal Lagrangian immersions with full first normal bundle.*

It is known that there exist ample examples of ideal Lagrangian submanifolds in complex projective and complex hyperbolic spaces. On the contrast, we have the following two non-existence results.

Theorem 10.3. *There do not exist ideal Lagrangian submanifolds in a complex projective space with full first normal bundle.*

Theorem 10.4. *There do not exist ideal Lagrangian submanifolds in a complex hyperbolic space with full first normal bundle..*

A submanifold M in a Riemannian manifold N is called *ruled* if at each point $p \in M$, M contains a geodesic γ_P of N through p.

Theorem 10.5. *Let M be a Lagrangian submanifold of $\mathbb{C}^n$ such that* $\operatorname{Im} h_p \neq T_p^{\perp} M$ *at each point $p \in M^n$. If M is ideal, then it is a ruled minimal submanifold.*

Theorem 10.6. *Let M be a Lagrangian submanifold of a complex space form $\tilde{M}^n(4c)$ with $c \neq 0$. If M is ideal, then it is ruled minimal submanifold.*

11. (2)-ideal Lagrangian Submanifolds in Complex Space Forms

A Lagrangian minimal submanifold in $\tilde{M}^n(4c)$ is called (2)-*ideal* if it satisfies $\delta(2) = (n+1)(n-2)c/2$. It was shown in [38] that there exists a minimal (2)-ideal Lagrangian immersion $\psi : S^3 \to CP^3(4)$ of S^3 into the complex projective 3-space $CP^3(4)$ with constant scalar curvature, but not constant sectional curvature. This nonstandard Lagrangian immersion $\psi : S^3 \to CP^3(4)$ is called an exotic sphere in CP^3. Because every minimal Lagrangian immersion of a (topological) 2-sphere in CP^2 is known to be totally geodesic, $n = 3$ is the smallest dimension in which a nontrivial minimal Lagrangian immersion of S^n into CP^n can occur.

This exotic Lagrangian sphere $\psi : S^3 \to CP^3(4)$ can be realized as follows: Define two complex structures on $\mathbb{C}^4$ by

$$\begin{aligned} I(v_1, v_2, v_3, v_4) &= (iv_1, iv_2, iv_3, iv_4), \\ J(v_1, v_2, v_3, v_4) &= (-\bar{v}_4, \bar{v}_3, -\bar{v}_2, \bar{v}_1). \end{aligned} \tag{11.1}$$

Clearly, I is the standard complex structure. The corresponding Sasakian structures on $S^7(1)$ have characteristic vector fields $\xi_1 = -I(x)$ and $\xi_2 = -J(x)$. Since we consider two complex structures on $\mathbb{C}^4$, we can consider two different Hopf fibrations $\pi_j : S^7(1) \to CP^3(4)$. The characteristic vector field ξ_j on S^7 is vertical for π_j, $j = 1, 2$.

Now we consider the Calabi curve $\mathcal{C}_3$ of CP^1 into $CP^3(4)$ of constant Gauss curvature $4/3$, given by

$$\mathcal{C}_3(z) = \left[1, \sqrt{3}z, \sqrt{3}z^2, z^3\right]. \tag{11.2}$$

Since $\mathcal{C}_3$ is holomorphic with respect to I, there exists a circle bundle $\pi : M^3 \to CP^1$ over CP^1 and an isometric minimal immersion $\mathcal{I} : M^3 \to S^7(1)$ such that $\pi_1(\mathcal{I}) = \mathcal{C}_\ni(\pi)$. It can be verified that $\mathcal{I}$ is horizontal with respect to π_2, such that the immersion $\mathcal{J} : M^3 \to CP^3(4)$, defined by $\mathcal{J} = \pi_2(\mathcal{I})$, gives rise to the Lagrangian immersion $\psi : S^3 \to CP^3(4)$.

In [38], Chen, Dillen, Verstraelen and Vrancken characterized this exotic 3-sphere by applying the notion of (2)-ideal as follows.

Theorem 11.1. *Let $f : M^n \to \tilde{M}^n(4c)$, $c \in \{-1, 0, 1\}$ and $n \geq 3$ be a Lagrangian immersion with constant scalar curvature. Then M^n is* (2)*-ideal if and only if either*

(1) *M^n is a totally geodesic immersion, or*

(2) $n = 3$, $c = 1$ *and x is locally congruent to the immersion* $\psi : S^3 \to CP^3$.

We restrict ourselves to (2)-ideal Lagrangian submanifolds in $CP^n(4)$, which therefore are minimal. We define for every $p \in M$ the kernel of the second fundamental form by

$$\mathcal{D}(p) = \{X \in T_pM \mid \forall Y \in T_pM : h(X,Y) = 0\}. \tag{11.3}$$

If the dimension of $\mathcal{D}(p)$ is constant, then it follows from [20] that either M is totally geodesic or that the distribution $\mathcal{D}$ is an $(n-2)$-dimensional completely integrable distribution.

There exist many (2)-ideal Lagrangian submanifolds which satisfy the following two conditions:

(1) The dimension of $\mathcal{D}$ is constant (and hence it is a completely integrable distribution).

(2) The distribution $\mathcal{D}^\perp$ is also integrable.

The following result from [35] classifies completely such Lagrangian submanifolds.

Theorem 11.2. *Let $f : M^n \to CP^n(4)$ be a (2)-ideal Lagrangian immersion such that the dimension of $\mathcal{D}$ is constant and $\mathcal{D}^\perp$ is an integrable distribution. Then either f is totally geodesic or f has no totally geodesic points and, up to holomorphic transformations, $f(M)$ is contained in the image under the Hopf fibration $\pi : S^{2n+1}(1) \to CP^n(4)$ of the image of one of the immersions described in the next Proposition.*

Proposition 11.3. *Let $S^{2n+1}(1)$ be the unit hypersphere of $\mathbb{C}^{n+1}$ and consider the orthogonal decomposition $\mathbb{C}^{n+1} = \mathbb{C}^3 \oplus J(\mathbb{E}^{n-2}) \oplus \mathbb{E}^{n-2}$. Let $f : M^2 \to S^5(1) \subset \mathbb{C}^3$ be a minimal, isometric, C-totally real immersion and consider the hypersphere $S^{n-3}(1)$ in $\mathbb{E}^{n-2}$. Then*

$$F : (0,\pi/2) \times_{\cos t} M^2 \times_{\sin t} S^{n-3}(1) \to S^{2n+1}(1) : (t,p,q) \mapsto \cos t f(p) + \sin t q$$

is a minimal, (2)-ideal isometric and C-totally real immersion. Moreover, if f has no totally geodesic points, then the dimension of $\mathcal{D}$ is exactly $n-2$. Finally, if we extend F to a map $\tilde{F} : (-\pi/2,\pi/2) \times M^2 \times S^{n-3}(1) \to S^{2n+1}(1) : (t,p,q) \mapsto \cos t f(p) + \sin t q$. Then $\tilde{F}$ fails to be immersive at $t = 0$, but the image of $\tilde{F}$ is an immersed minimal C-totally real submanifold. If f is not totally geodesic, then this image can not be extended further.

In [1], Blair, Dillen, Verstraelen and Vrancken constructed further examples of (2)-ideal submanifolds in complex projective spaces by using Calabi curves in complex projective spaces. The classification of (2)-ideal Lagrangian submanifolds in CP^3 was obtained in [3].

A Riemannian n-manifold M whose Ricci tensor has an eigenvalue of multiplicity at least $n-1$ is called quasi-Einstein. Montealegre and Vrancken [80] determined all minimal Lagrangian quasi-Einstein submanifolds in CP^3 with $\delta(2) \neq 2$.

Consider the complex $(m+1)$-dimensional space $\mathbb{C}_1^{m+1}$ endowed with the pseudo Euclidean metric $g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^m dz_j d\bar{z}_j$, where $\bar{z}_k$ denotes the complex conjugate of z_k. On $\mathbb{C}_1^{m+1}$, we define $F(z,w) = -z_0\bar{w}_0 + \sum_{k=1}^m z_k \bar{w}_k$.

Put $H_1^{2m+1} = \{z = (z_0, z_1, \cdots, z_m) \in \mathbb{C}_1^{m+1} \,|\, \langle z, z\rangle = -1\}$, where $\langle\, ,\, \rangle$ denotes the inner product on $\mathbb{C}_1^{n+1}$ induced from g_0. Then H_1^{2m+1} is a real hypersurface of $\mathbb{C}^{m+1}$ whose tangent space at $z \in H_1^{2m+1}$ is given by

$$T_z H_1^{2m+1} = \left\{ w \in \mathbb{C}^{m+1} \,|\, \mathrm{Re}\, F(z,w) = 0 \right\}. \tag{11.4}$$

It is known that H_1^{2m+1} together with the induced metric g is a pseudo Riemannian manifold of constant sectional curvature -1, which is known as the anti de Sitter space time.

We put $H_1^1 = \{\lambda \in \mathbb{C} \,|\, \lambda\bar{\lambda} = 1\}$. Then we have an H_1^1-action on H_1^{2m+1} given by $z \mapsto \lambda z$. At each point z in H_1^{2m+1}, the vector iz is tangent to the flow of the action. Since g_0 is Hermitian, we have Re $g_0(iz, iz) = -1$. Note that the orbit is given by $\tilde{z}(t) = e^{it}z$ and $d\tilde{z}(t)/dt = i\tilde{z}(t)$. Thus the orbit lies in the negative-definite plane spanned by z and iz. The quotient space $H_1^{2m+1}/\sim$, under the identification induced from the action, is the complex hyperbolic space $CH^m(-4)$ with constant holomorphic sectional curvature -4. The almost complex structure J on $CH^m(-4)$ is induced from the canonical almost complex structure J on $\mathbb{C}_1^{m+1}$ via the totally geodesic fibration

$$\pi : H_1^{2m+1} \to CH^m(-4). \tag{11.5}$$

A submanifold M of a Kähler manifold $\tilde{M}$ is called a CR-submanifold if there exists on M a differentiable holomorphic distribution $\mathcal{F}$ such that its orthogonal complement $\mathcal{F}^\perp \subset TM$ is a totally real distribution.

The following result from [30] shows that the basic inequality (3.4) holds for arbitrary n-dimensional submanifolds in complex hyperbolic n-space as well.

Theorem 11.4. *Let M be an arbitrary n-dimensional submanifold of a complex hyperbolic space $CH^m(4c)$ of constant holomorphic sectional curvature $4c < 0$. Then for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ we have*

$$\delta(n_1, \ldots, n_k) \le b(n_1, \ldots, n_k)|H|^2 + a(n_1, \ldots, n_k)c, \tag{11.6}$$

with the equality holding at a point $p \in M$ if and only if

(a) *$n_1, \dots, n_k$ are even*,

(b) *$L_1, \dots, L_k$ at p are complex subspaces of $T_p(CH^m)$*,

(c) *the complementary orthogonal subspace of $L_1 \oplus \cdots \oplus L_k$ in T_pM is a totally real subspace*,

(d) *with respect to a suitable orthonormal basis $e_1, \dots, e_n$ of T_pM such that L_j is spanned by $e_{n_1+\cdots+n_{j-1}+1}, \dots, e_{n_1+\cdots+n_j}$ for $j = 1, \dots, k$, the shape operators of M take the form of* (3.5) *with* (3.6).

In particular, if the equality case of (11.6) *holds identically, then M is a CR-submanifold.*

For Lagrangian submanifolds in complex hyperbolic space, Chen and Vrancken [49] prove the following.

Theorem 11.5. *Let $F : M^n \to CH^n(-4)$ be a* (2)*-ideal Lagrangian immersion without geodesic points. Assume that the orthogonal complement of the nullity distribution is integrable. Then, every point p of an open dense subset of M^n has a neighborhood U_p such that either*

(i) $F(t, u, v) = \pi(\cosh t(\psi(u), 0, \cdots, 0) + \sinh t(0, 0, 0, \phi(v)))$, *where*

$$\phi : (v_1, \cdots, v_{n-3}) \mapsto \phi(v)$$

describes the standard totally real $(n-3)$-sphere S^{n-3} in $E^{n-2} \subset \mathbb{C}^{n-2}$ and $\psi : (u_1, u_2) \mapsto \psi(u)$ describes a minimal horizontal immersion in H_1^5, or

(ii) $F(t, u, v) = \pi(\cosh t(\phi(v), 0, 0, 0)) - \sinh t(0, \cdots, 0, \psi(u)))$, *where*

$$\phi : (v_1, \cdots, v_{n-3}) \mapsto \phi(v)$$

describes the standard totally real hyperbolic space H_1^{n-3} in $E_1^{n-2} \subset \mathbb{C}_1^{n-2}$ and $\psi : (u_1, u_2) \mapsto \psi(u)$ describes a minimal horizontal immersion in $S^5(1)$, or

(iii) $F(t, u, v) = \pi((\cosh t, -\sinh t, 0, \cdots, 0) + (e^{-t}/2)z(u, v)(1, -1, 0, \cdots, 0) + (e^{-t}/2)(0, 0, w_1(u), w_2(u), v_1, \cdots, v_{n-3}))$, *where*

$$w : D \subset \mathbb{R}^2 \to \mathbb{C}^2 : (u_1, u_2) \mapsto (w_1(u_1, u_2), w_2(u_1, u_2))$$

is a minimal Lagrangian immersion and z is a complex-valued function determined by the condition that

$$2(z + \bar{z}) = w_1\bar{w}_1 + w_2\bar{w}_2 + \sum_{i=1}^{n-3} v_i^2,$$

and by the condition that its imaginary part depends only on u and satisfies the following system of differential equations :

$$(z-\bar{z})_{u_1} = \frac{1}{2}\{w_1(\bar{w}_1)_{u_1} + w_2(\bar{w}_2)_{u_1} - \bar{w}_1(w_1)_{u_1} - \bar{w}_2(w_2)_{u_1}\},$$
$$(z-\bar{z})_{u_2} = \frac{1}{2}\{w_1(\bar{w}_1)_{u_2} + w_2(\bar{w}_2)_{u_2} - \bar{w}_1(w_1)_{u_2} - \bar{w}_2(w_2)_{u_2}\},$$

where $\pi : H_1^{2m+1} \to CH^m(-4)$ is the projection defined in section 2.

Theorem 11.6. *Let $f : M^3 \to CH^3(-4)$ be a Lagrangian immersion satisfying the basic equality and $p \in M^3$. If the immersion has no totally geodesic points and the distribution $\mathcal{D}^\perp$ is nowhere integrable, then there exist coordinates (u, v, t) defined in a neighborhood $D \times I$ of p and functions $h : D \to \mathbb{R} : (u, v) \mapsto h(u, v)$ and $k : D \to \mathbb{R} : (u, v) \mapsto k(u, v)$ satisfying*

$$\Delta h = e^{-2k/3}\sin(2h), \tag{11.7}$$

and

$$\Delta k = -3e^{-2k/3}(\cos(2h) + 2e^{2k}), \tag{11.8}$$

where $\Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2$. Moreover, the induced metric can be expressed by

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle = 1, &&\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = -h_u h_v \\
&\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial u} \right\rangle = h_v, &&\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = e^{-2k/3}(\cos^2 h + \sinh^2 t) + h_v^2, \\
&\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial v} \right\rangle = -h_u, &&\left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle = e^{-2k/3}(\cos^2 h + \sinh^2 t) + h_u^2,
\end{aligned} \tag{11.9}$$

and the tensor $T = -Jh$ induced from the second fundamental form satisfies

$$\begin{aligned}
T\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\right) &= T\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial t}\right) = T\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 0, \\
T\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) &= e^{2k/3}\left(1 - \frac{2\cos^2 h\cosh^2 t}{\sinh^2 t + \cos^2 h}\right)\left(\frac{\partial}{\partial u} - h_v\frac{\partial}{\partial t}\right) \\
&\quad -\frac{1}{2}e^{2k/3}\frac{\sin 2h \sinh 2t}{\sinh^2 t + \cos^2 h}\left(\frac{\partial}{\partial v} + h_u\frac{\partial}{\partial t}\right), \\
T\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) &= -e^{2k/3}\left(1 - \frac{2\cos^2 h\cosh^2 t}{\sinh^2 t + \cos^2 h}\right)\left(\frac{\partial}{\partial u} - h_v\frac{\partial}{\partial t}\right), \\
&\quad +\frac{1}{2}e^{2k/3}\frac{\sin 2h \sinh 2t}{\sinh^2 t + \cos^2 h}\left(\frac{\partial}{\partial v} + h_u\frac{\partial}{\partial t}\right), \\
T\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) &= -\frac{1}{2}e^{2k/3}\frac{\sin 2h \sinh 2t}{\sinh^2 t + \cos^2 h}\left(\frac{\partial}{\partial u} - h_v\frac{\partial}{\partial t}\right), \\
&\quad -e^{2k/3}\left(1 - \frac{2\cos^2 h\cosh^2 t}{\sinh^2 t + \cos^2 h}\right)\left(\frac{\partial}{\partial v} + h_u\frac{\partial}{\partial t}\right).
\end{aligned} \tag{11.10}$$

Conversely, let h, k *be any solutions of* (11.7) *and* (11.8) *on an open set* D *of* $\mathbb{R}^2$. *We define on* $M = D \times \mathbb{R}$ *a metric by* (11.9) *and a tensor* T *by* (11.10). *Let*

$$M_0 = \Big\{(x,t) \in M^3 \,|\, h(x) \neq \frac{1}{2}(2k+1)\pi \in \mathbb{Z} \ \text{ or } \ t \neq 0\Big\}$$

and let M_1 *be a simply connected component of* M_0. *Then, up to rigid motions, there exists a unique* (2)*-ideal Lagrangian immersion* $F : M_1 \to CH^3(-4)$ *with nonintegrable distribution* $\mathcal{D}$ *and the second fundamental form* $h = JT$.

Theorems 11.5 and 11.6 together completely determine (2)-ideal Lagrangian submanifolds in complex hyperbolic 3-space.

12. Lagrangian H-umbilical Submanifolds

It was proved in [47] that there do not exist totally umbilical Lagrangian submanifolds in complex space forms except totally geodesic ones. Thus, it is natural to look for the "simplest" Lagrangian submanifolds next to totally geodesic ones. As a result, the following notion of Lagrangian H-umbilical submanifolds was introduced in [23].

A non-totally geodesic Lagrangian submanifold M of a Kähler manifold is called a Lagrangian H-umbilical submanifold if its second fundamental form takes the following simple form:

$$\begin{aligned} &h(e_1,e_1) = \lambda J e_1, \quad h(e_2,e_2) = \cdots = h(e_n,e_n) = \mu J e_1, \\ &h(e_1,e_j) = \mu J e_j, \quad h(e_j,e_k) = 0, \ \ j \neq k, \quad j,k = 2,\ldots,n \end{aligned} \tag{12.1}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field.

Clearly, a non-minimal Lagrangian H-umbilical submanifold satisfies the following two conditions:

(a) JH is an eigenvector of the shape operator A_H and

(b) the restriction of A_H to $(JH)^\perp$ is proportional to the identity map.

In fact, Lagrangian H-umbilical submanifolds are the simplest Lagrangian submanifolds satisfying both conditions (a) and (b). In this way, Lagrangian H-umbilical submanifolds are regarded as the "simplest" Lagrangian submanifolds next to the totally geodesic ones.

There exist ample examples of Lagrangian H-umbilical submanifolds in a complex space form. The simplest examples are obtained by using the notion of complex extensors introduced in [24].

Given an immersion $G : M \to \mathbb{E}^m$ of a manifold into Euclidean m-space $\mathbb{E}^m$ and a unit speed curve $F : I \to \mathbb{C}$ in the complex plane, we may extend the immersion $G : M \to \mathbb{E}^m$ to an immersion of $I \times M$ into $\mathbb{C}^m$ by utilizing the tensor product of F and G. We call this extension the *complex extensor* of G via F. The Whitney sphere is a nice example of complex extensor of the ordinary unit hypersphere.

The following result given in [24] provides us ample examples of H-umbilical Lagrangian submanifolds.

Theorem 12.1. *Let $\iota : S^{n-1} \to \mathbb{E}^n$ be the inclusion of the unit hypersphere of $\mathbb{E}^n$ (centered at the origin). Then every complex extensor of ι via a unit speed curve F in $\mathbb{C}$ is a Lagrangian H-umbilical submanifold of $\mathbb{C}^n$, unless $F(s) = (s+a)c$ for some real number a and some unit complex number c.*

For a real number $b > 0$, let $F : \mathbb{R} \to \mathbb{C}$ denote the unit speed curve given by

$$F(s) = \frac{e^{2bsi} + 1}{2bi}. \tag{12.2}$$

With respect to the induced metric, the complex extensor $\phi = F \otimes \iota$ of the unit hypersphere of $\mathbb{E}^n$ via this F is a Lagrangian isometric immersion of an open portion of an n-sphere $S^n(b^2)$ of sectional curvature b^2 into $\mathbb{C}^n$. This isometric Lagrangian immersion of a punctured $S^n(b^2)$ is called a *Lagrangian pseudo-sphere.*

The following theorem from [24] determines H-umbilical Lagrangian submanifolds in complex Euclidean space.

Theorem 12.2. *Let $n \geq 3$ and $L : M \to \mathbb{C}^n$ be a Lagrangian H-umbilical isometric immersion. Then we have*:

(i) *If M is of constant sectional curvature, then either M is flat or, up to rigid motions of $\mathbb{C}^n$, L is a Lagrangian pseudo-sphere.*

(ii) *If M contains no open subset of constant sectional curvature, then, up to rigid motions of $\mathbb{C}^n$, L is a complex extensor of the unit hypersphere of $\mathbb{E}^n$.*

A unit speed curve $z = z(s)$ in $S^{2n-1} \subset \mathbb{C}^n$ is called a *Legendre curve* if its tangent vector $z'(s)$ is perpendicular to $J\xi$, where J is the complex structure and ξ the unit normal vector field of S^{2n-1} in $\mathbb{C}^n$. A Legendre curve $z : I \to S^{2n-1} \subset \mathbb{C}^n$ is called *special Legendre* if it satisfies

$$z''(s) = i\lambda(s)z'(s) - z(s) - \sum_{j=3}^{n} a_j(s)P_j(s), \tag{12.3}$$

for some parallel normal vector fields $P_3, \dots, P_n$ along the curve.

Clearly, every Legendre curve in S^3 is a special Legendre curve. The following result from [28] shows that special Legendre curves in S^{2n-1} do exist abundantly for each $n \geq 3$.

Theorem 12.3. *Let n be an integer ≥ 2. Then, for any given $n-1$ functions $\lambda, a_3, \ldots, a_n$ defined on an open interval I with $\lambda > 0$, there exists a special Legendre curve $z : I \to S^{2n-1} \subset \mathbb{C}^n$ which satisfies* (12.3) *for some parallel orthonormal normal vector fields $P_3, \ldots, P_n$ along the curve z.*

Example 12.1. *Let $\lambda, a_3, \ldots, a_n$ be $n-1$ real numbers with $\lambda > 0$. Put*

$$\gamma = 1 + \sum_{j=3}^{n} a_j^2, \qquad \mu = \left(\lambda^2 + 4\gamma\right)^{1/2}, \tag{12.4}$$

$$\begin{aligned} z(s) = &\frac{\mu-\lambda}{2\mu\gamma}\left(\frac{2\gamma}{\mu-\lambda}, 1, a_3, \ldots, a_n\right) e^{(\lambda+\mu)is/2} \\ &+\frac{\lambda+\mu}{2\mu\gamma}\left(-\frac{2\gamma}{\lambda+\mu}, 1, a_3, \ldots, a_n\right) e^{(\lambda-\mu)is/2} \\ &-\frac{1}{\gamma}(0, 1-\gamma, a_3, \ldots, a_n), \end{aligned} \tag{12.5}$$

$$c_3 = (0, a_3, -1, 0, \ldots, 0), \cdots, c_n = (0, a_n, 0, \ldots, 0, -1). \tag{12.6}$$

Then $z = z(s)$ is a unit speed special Legendre curve in $S^{2n-1} \subset \mathbb{C}^n$ satisfying

$$z''(s) = i\lambda z'(s) - z(s) - \sum_{j=3}^{n} a_j P_j(s), \tag{12.7}$$

where

$$P_j(s) = a_j z(s) - c_j, \quad j = 3, \ldots, n, \tag{12.8}$$

are the associated orthonormal parallel normal vector fields.

By a *Lagrangian cylinder* in $\mathbb{C}^n$ we mean a Lagrangian submanifold which is a cylinder over a curve whose rulings are $(n-1)$-planes parallel to a fixed $(n-1)$-plane.

The following result from [28] provides the explicit description of flat Lagrangian H-umbilical submanifolds in the complex Euclidean space.

Theorem 12.4. *Let $n \geq 2$ and $\lambda, b, a_3, \ldots, a_n$ be n functions defined on an open interval I with λ nowhere zero and let $z : I \to S^{2n-1} \subset \mathbb{C}^n$ be a special Legendre curve satisfying* (12.3). *Put*

$$f(t, u_2, \ldots, u_n) = b(t) + u_2 + \sum_{j=3}^{n} a_j(t) u_j. \tag{12.9}$$

Denote by $\hat{M}^n(0)$ *the twisted product manifold* ${}_fI \times \mathbb{E}^{n-1}$ *with twisted product metric given by*

$$g = f^2 dt^2 + du_2^2 + \cdots + du_n^2. \tag{12.10}$$

Then $\hat{M}^n(0)$ *is a flat Riemannian* n*-manifold and*

$$L(t, u_2, \dots, u_n) = u_2 z(t) + \sum_{j=3}^{n} u_j P_j(t) + \int^t b(t) z'(t) dt \tag{12.11}$$

defines a Lagrangian H*-umbilical isometric immersion* $L : \hat{M}^n(0) \to \mathbb{C}^n$*.*

Conversely, up to rigid motions of $\mathbb{C}^n$*, locally every flat Lagrangian* H*-umbilical submanifold in* $\mathbb{C}^n$ *without totally geodesic points is either a Lagrangian cylinder over a curve or a Lagrangian submanifold obtained in the way described above.*

Lagrangian H-umbilical submanifolds in nonflat complex space forms have been classified in [23]. In particular, it was proved that "except some exceptional cases", Lagrangian H-umbilical submanifolds of CP^n and of CH^n are obtained from Legendre curves in S^3 or in H_1^3 via warped products.

13. Lagrangian Catenoid and Its Characterizations

The Lagrangian catenoid is defined by

$$\begin{aligned} M_0 = \Big\{ &(x, y) \in \mathbb{C}^n = \mathbb{E}^n \times \mathbb{E}^n : |x|y = |y|x, \\ &\operatorname{Im}(|x| + i|y|)^n = 1,\ |y| < |x| \tan\frac{\pi}{n} \Big\}. \end{aligned} \tag{13.1}$$

Besides being a minimal Lagrangian submanifold of $\mathbb{C}^n$, M_0 is invariant under the diagonal action of $SO(n)$ on $\mathbb{C}^n = \mathbb{E}^n \times \mathbb{E}^n$.

Castro and Urbano obtained in [15] the following characterizations of Lagrangian catenoid.

Theorem 13.1. *Let* $f : M \to \mathbb{C}^n$ *be a minimal nonflat Lagrangian immersion of an* n*-manifold* M*. Then* M *is foliated by pieces of round* $(n-1)$*-spheres of* $\mathbb{C}^n$ *if and only if, up to dilations,* f *is congruent to an open subset of the Lagrangian catenoid.*

Theorem 13.2. *Let* $f : M \to \mathbb{C}^n$ *be a minimal nonflat complex immersion of a complex* n*-dimensional Kähler manifold* M*. Then* M *is foliated by pieces of round* $(2n-1)$*-spheres of* $\mathbb{C}^m$ *if and only if* $n = 1$ *and, up to dilations,* f *is congruent to an open subset of the Lagrangian catenoid.*

Castro [10] proved the following characterization of Lagrangian catenoid.

Theorem 13.3. *The Lagrangian catenoid is the only nonflat Lagrangian surface of revolution in* $\mathbb{C}^2$.

14. Whitney Sphere and Lagrangian Submanifolds with Conformal Maslov Form

A Lagrangian submanifold of a Kähler manifold is said to have conformal Maslov form if JH is a conformal vector field [96, 97]. The Whitney sphere is a nice example of Lagrangian submanifold with conformal Maslov form.

Ros and Urbano [97] studied Lagrangian submanifolds with conformal Maslov form and obtained the following results.

Theorem 14.1. *The Whitney sphere is the only compact Lagrangian submanifold in* $\mathbb{C}^n$ *with conformal Maslov form and null first Betti number* β_1.

Theorem 14.2. *The Whitney sphere is the only compact Lagrangian submanifold in* $\mathbb{C}^n$ *with conformal Maslov form such that* $\mathrm{Ric}(JH) \geq 0$.

Theorem 14.3. *Let* f *be a Lagrangian immersion of a compact* n*-manifold* M *into* $\mathbb{C}^n$ *with conformal Maslov form. If* H *has zeros on* M, *then* f *is congruent to the Whitney sphere.*

Ros and Urbano [97] also determined all the compact Lagrangian submanifolds of $\mathbb{C}^n$ with conformal Maslov form and with first Betti number one.

15. Index and Stability of Lagrangian Submanifolds

The stability of minimal Lagrangian submanifolds of a Kähler manifold was first investigated by Chen, Leung and Nagano in 1980. In particular, they proved that the second variational formula of a compact Lagrangian submanifold M in a Kähler manifold $\tilde{M}$ is given by

$$V''(\xi) = \int_M \left\{ \frac{1}{2}\|dX^{\#}\|^2 + (\delta X^{\#})^2 - \tilde{\mathrm{R}}\mathrm{ic}(X, X) \right\} dV, \tag{15.1}$$

where $JX = \xi$, $X^{\#}$ is the dual 1-form of X on M, δ is the codifferential operator, and $\tilde{\mathrm{R}}\mathrm{ic}$ is the Ricci tensor of $\tilde{M}$.

By applying (15.1), Chen, Leung and Nagano studied in 1980 the index and stability of Lagrangian minimal submanifolds in Kähler manifolds. They obtained the following (see [18, 34]).

Theorem 15.1. *Let* $f : M \to \tilde{M}$ *be a compact Lagrangian minimal submanifold of a Kaehler manifold* $\tilde{M}$.

(1) *If $\tilde{M}$ has positive Ricci curvature, then the index of f satisfies $i(f) \geq \beta_1(M)$, where $\beta_1(M)$ denotes the first Betti number of M. In particular, if the first cohomology group of M is nontrivial, i.e., $H^1(M;\mathbb{R}) \neq 0$, then M is unstable*;

(2) *If $\tilde{M}$ has nonpositive Ricci curvature, then M is stable.*

Oh [89] introduced the notion of Hamiltonian deformations in Kähler manifolds. He considered normal variations V along a minimal Lagrangian submanifold M such that the 1-form $\alpha_V = \langle JV, \cdot \rangle$ is exact and called such variations Hamiltonian variations.

A minimal Lagrangian submanifold is called Hamiltonian stable if the second variation is nonnegative in the class of Hamiltonian variations.

Oh established in [89] the following Hamiltonian stability criterion for Lagrangian submanifolds in Einstein-Kähler manifolds.

Theorem 15.2. *Let $\tilde{M}$ be an Einstein-Kähler manifold with* Ric $= cg$, *where c is a constant. Then a minimal Lagrangian submanifold M of $\tilde{M}$ is locally Hamiltonian stable if and only if $\lambda_1(M) \geq c$, where $\lambda_1(M)$ is the first nonzero eigenvalue of the Laplacian acting on $C^\infty(M)$.*

The Lagrangian totally geodesic $RP^n(1)$ in $CP^n(4)$ is unstable in the usual sense. In contrast, Oh's result implies the following.

Theorem 15.3. *Let $\tilde{M}$ be an Einstein-Käehler manifold of complex dimension n with Ricci curvature $(n+1)c$. Then a minimal Lagrangian submanifold M in $\tilde{M}$ is Hamiltonian stable if and only if $\lambda_1 \geq (n+1)c/2$, where λ_1 the first eigenvalue of Laplacian acting on $C^\infty(M)$.*

For example, Theorem 15.3 implies that the Lagrangian totally geodesic $RP^n(1)$ is Hamiltonian stable in $CP^n(4)$.

Castro and Urbano [14] constructed a family of unstable Hamiltonian minimal Lagrangian tori in $\mathbb{C}^2$ and characterized them as the only Hamiltonian-minimal Lagrangian tori in $\mathbb{C}^2$ admitting a one-parameter group of isometries. Chang [16] proved that a compact Hamiltonian stable minimal surface in CP^2 with $m(\lambda_1) \leq 6$ is either flat or totally geodesic, where $m(\lambda_1)$ denotes the multiplicity of λ_1.

Takeuchi [104] proved the following result by applying the Chen-Leung-Nagano algorithm for stability of totally geodesic submanifolds in symmetric spaces (cf. [104] and [34, p.296]).

Theorem 15.4. *Let $\tilde{M}$ be a Hermitian symmetric space of compact type and M a compact totally geodesic Lagrangian submanifold of $\tilde{M}$. Then M is a stable submanifold in the usual sense if and only if M is simply connected.*

An immersed Lagrangian submanifold M in a Kähler manifold $\tilde{M}$ is called Lagrangian stationary if the volume is stationary under Lagrangian variations. It is known that a closed immersed Lagrangian submanifold in a Kähler-Einstein manifold is stationary if and only if it is Lagrangian stationary. The question if any cycle realized by Lagrangian submanifolds in a Kähler-Einstein manifold is homologous to an integral cycle that is minimal and Lagrangian had been investigated in [102].

16. Lagrangian Immersions and Maslov Class

Let Ω denote the canonical symplectic form on $\mathbb{C}^n$ defined by

$$\Omega(X,Y) = \langle JX, Y\rangle. \tag{16.1}$$

Consider the Grassmannian $\mathcal{L}(\mathbb{C}^n)$ of all Lagrangian vector subspaces of $\mathbb{C}^n$. $\mathcal{L}(\mathbb{C}^n)$ can be identified with the symmetric space $U(n)/O(n)$ in a natural way.

$U(n)/O(n)$ is a bundle over the circle S^1 in $\mathbb{C}^1$ with the projection

$$\det{}^2 : U(n)/O(n) \to S^1, \tag{16.2}$$

where $\det^2$ is the square of the determinant.

For a Lagrangian submanifold M in $\mathbb{C}^n$, the Gauss map takes the values in $\mathcal{L}(\mathbb{C}^n)$ which yields the following sequence:

$$M \xrightarrow{G} \mathcal{L}(\mathbb{C}^n) \cong U(n)/O(n) \xrightarrow{\det^2} S^1. \tag{16.3}$$

If ds denotes the volume form of S^1, then $m_M = (\det^2 \circ G)^*(ds)$ is a closed 1-form on M. The cohomology class $[m_M] \in H^1(M;\mathbb{Z})$ is called the *Maslov class* of the Lagrangian submanifold M.

Morvan [83] proved that the Maslov form m_M and the mean curvature vector of a Lagrangian submanifold M in $\mathbb{C}^n$ are related by

$$m_M(X) = \frac{1}{\pi}\langle J\vec{H}, X\rangle, \quad X \in TM. \tag{16.4}$$

Hence, if a Lagrangian submanifold M in $\mathbb{C}^n$ is minimal, then its Maslov class is trivial.

Let ξ be a normal vector field of a Lagrangian submanifold M of a Kähler manifold $\tilde{M}$. Denote by α_ξ the 1-form on M defined by

$$\alpha_\xi(X) = \Omega(\xi, X) = \langle J\xi, X\rangle, \quad X \in TM, \tag{16.5}$$

where Ω is the Kähler form of $\tilde{M}$.

Chen and Morvan [43] introduced the notion of harmonic deformations in Kähler manifolds: A normal vector field ξ of a Lagrangian submanifold M is called *harmonic* if the 1-form α_ξ associated with ξ is a harmonic 1-form. A normal variation of a Lagrangian submanifold in a Kähler manifold is called harmonic if its variational vector field is harmonic.

A Lagrangian submanifold M of a Kähler manifold is called *harmonic minimal* if it is a critical point of the volume functional in the class of harmonic variations.

Chen and Morvan [43] proved the following.

Theorem 16.1. *The Maslov class of a Lagrangian submanifold of an Einstein-Kähler manifold vanishes if and only if it is harmonic minimal.*

This theorem provides a solution to a problem proposed by Le Khong Van and Fomenko [71], because it establishes a relationship between calculus of variations and Maslov class.

Theorem 16.1 implies the following [43].

Theorem 16.2. *A closed curve γ in a Kähler manifold $\tilde M$ with $\dim_R \tilde M = 2$ is harmonic minimal if and only if it has zero total curvature, that is,* $\int_\gamma \kappa(s)ds = 0$.

The following theorems follow from Theorems 16.1, 16.2, and Gauss-Bonnet's formula.

Theorem 16.3. *If $\tilde M$ is a Kähler surface with nonpositive Gauss curvature, then every harmonic minimal closed curve in $\tilde M$ has self-intersection points.*

Theorem 16.4. *If $\tilde M$ is a Kähler surface diffeomorphic to a 2-sphere, then an embedded closed curve C in $\tilde M$ is harmonic minimal if and only if C divides $\tilde M$ into two regions with equal total Gauss curvature.*

17. Lagrangian Submanifolds and Finite Type Theory

Let (M, g) be a compact Riemannian n-manifold. Then the eigenvalues of the Laplacian Δ form a discrete infinite sequence: $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots \nearrow \infty$. Let

$$V_k = \{f \in C^\infty(M) : \Delta f = \lambda_k f\}$$

be the eigenspace of Δ associated with eigenvalue λ_k. Then each V_k is finite-dimensional. Define an inner product $(\ ,\ )$ on $C^\infty(M)$ by $(f, h) = \int_M fh\, dV$. Then $\sum_{k=0}^\infty V_k$ is dense in $C^\infty(M)$ (in L^2-sense). If we denote by $\hat\oplus V_k$ the completion of $\sum V_k$, we have $C^\infty(M) = \hat\oplus_k V_k$.

For each function $f \in C^{\infty}(M)$, let f_t denote the projection of f onto the subspace V_t. We have the spectral resolution (or decomposition): $f = \sum_{t=0}^{\infty} f_t$ (in L^2-sense).

Because V_0 is 1-dimensional, there is a positive integer $p \geq 1$ such that $f_p \neq 0$ and $f - f_0 = \sum_{t \geq p} f_t$, where $f_0 \in V_0$ is a constant. If there are infinite many f_t's which are nonzero, put $q = +\infty$; otherwise, there is an integer $q \geq p$ such that $f_q \neq 0$ and $f - f_0 = \sum_{t=p}^{q} f_t$.

If $x : M \to \mathbb{E}^m$ is an isometric immersion of a compact Riemannian n-manifold M into $\mathbb{E}^m$, for each coordinate function x_A we have $x_A = (x_A)_0 + \sum_{t=p_A}^{q_A} (x_A)_t$. We put

$$p = \inf_A \{p_A\} \quad \text{and} \quad q = \sup_A \{q_A\}, \tag{17.1}$$

where A ranges over all A such $x_A - (x_A)_0 \neq 0$. Both p and q are well-defined geometric invariants such that p is a positive integer and q is either $+\infty$ or an integer $\geq p$. Consequently, we have the spectral decomposition of x in vector form:

$$x = x_0 + \sum_{t=p}^{q} x_t, \tag{17.2}$$

which is called the spectral resolution (or decomposition) of the immersion x. Put $T(x) = \{t \in \mathbb{Z} : x_t \neq \text{constant map}\}$. The immersion x or the submanifold M is said to be of k-type if $T(x)$ contains exactly k elements (cf. [19] for details).

For an n-dimensional submanifold M of $\mathbb{E}^m$, the Gauss map of M is defined by

$$\nu : M \to G(n, m-n) \tag{17.3}$$

which maps a point $u \in M$ into the n-dimensional linear subspace of $\mathbb{E}^m$ obtained by parallel displacement of the tangent space T_uM of M at u. Here $G(n, m-n)$ denotes the Grassmann manifold consisting of linear n-subspaces of $\mathbb{E}^m$. One may define the type number for Gauss map exactly in the same way as for immersions.

The following result of Chen, Morvan and Nore [44] provides a direct relationship between type number and topology for Lagrangian submanifolds of complex Euclidean space.

Theorem 17.1. *Let M be a compact orientable Lagrangian submanifold of $\mathbb{C}^n$. If the type number of the Gauss map of M is $\leq n/2$, then M has zero Euler number and zero self-intersection number.*

Let $\psi_1 : CP^n(4) \to \mathbb{E}^m$ denote the first standard embedding of $CP^n(4)$ into Euclidean space.

The following result provides a simple relationship between type number and Lagrangian submanifolds of CP^n.

Theorem 17.2. *Let* $f : M \to CP^n(4)$ *be a compact Lagrangian submanifold of* $CP^n(4)$*. If* M *has parallel mean curvature vector in* $CP^n(4)$*, then the composition* $\psi_1 \circ f : M \to CP^n(4) \to \mathbb{E}^m$ *is of* 1*-type.*

Theorem 17.2 was first proved by Ros in [95] when f is a minimal Lagrangian immersion. It was then extended to Lagrangian immersions with parallel mean curvature vector by Dimitric [58].

18. Lagrangian Submanifolds of the Nearly Kähler Six-Sphere

Calabi [7] proved that every oriented submanifold M^6 of the hyperplane Im $\mathcal{O}$ of the imaginary octonions carries a $U(3)$-structure (that is, an almost Hermitian structure). For instance, let $S^6 \subset \operatorname{Im}\mathcal{O}$ be the sphere of unit imaginary vectors; then the right multiplication by $u \in S^6$ induces a linear transformation $J_u : \mathcal{O} \to \mathcal{O}$ which is orthogonal and satisfies $(J_u)^2 = -I$. The operator J_u preserves the 2-plane spanned by 1 and u and therefore preserves its orthogonal 6-plane which may be identified with T_uS^6. Thus J_u induces an almost complex structure on T_uS^6 which is compatible with the inner product induced by the inner product of $\mathcal{O}$ and S^6 has an almost complex structure.

The almost complex structure J on S^6 is a nearly Kähler structure in the sense that the (2,1)-tensor field G on S^6, defined by $G(X,Y) = (\tilde{\nabla}_X J)(Y)$, is skew-symmetric, where $\tilde{\nabla}$ denotes the Riemannian connection on S^6.

The group of automorphisms of this nearly Kähler structure is the exceptional simple Lie group G_2 which acts transitively on S^6 as a group of isometries.

A 3-dimensional submanifold M of the nearly Kähler S^6 is called Lagrangian if the almost complex structure J on the nearly Kähler 6-sphere carries each tangent space $T_xM,\ x \in M,$ onto the corresponding normal space $T_x^{\perp}M.$

The following result of Ejiri [59] is fundamental for the study of Lagrangian submanifold in the nearly Kähler six-sphere.

Theorem 18.1. *Lagrangian submanifolds in the nearly Kähler* S^6 *are minimal and orientable.*

Ejiri [59] also proved the following.

Theorem 18.2. *If a Lagrangian submanifold* M *in the nearly Kähler* S^6 *has constant sectional curvature, then* M *is either totally geodesic or has constant curvature* 1/16.

The first nonhomogeneous examples of Lagrangian submanifolds in the nearly Kähler 6-sphere were described in [61].

Dillen, Opozda, Verstraelen and Vrancken [55] proved the following.

Theorem 18.2. *If M is a compact Lagrangian submanifold of S^6 with $K > 1/16$, then M is a totally geodesic submanifold.*

Mashimo [77] classified the G_2-equivariant Lagrangian submanifolds of the nearly Kähler six-sphere. It turns out that there are five models and every equivariant Lagrangian submanifold in the nearly Kähler six-sphere is G_2-congruent to one of the five models.

These five models can be distinguished by the following curvature properties:

(1) M^3 is totally geodesic ($\delta(2) = 2$),

(2) M^3 has constant curvature $1/16$ ($\delta(2) = 1/8$),

(3) the curvature of M^3 satisfies $1/16 \leq K \leq 21/16$ ($\delta(2) = 11/8$),

(4) the curvature of M^3 satisfies $-7/3 \leq K \leq 1$ ($\delta(2) = 2$),

(5) the curvature of M^3 satisfies $-1 \leq K \leq 1$ ($\delta(2) = 2$),

where $\delta(2)$ is the invariant introduced by Chen (cf. Section 3).

Dillen, Verstraelen and Vrancken [56] characterized models (1), (2) and (3) as the only compact Lagrangian submanifolds in S^6 whose sectional curvatures satisfy $K \geq 1/16$. They also obtained an explicit expression for the Lagrangian submanifold of constant curvature 1/16 in terms of harmonic homogeneous polynomials of degree 6. Using these formulas, it follows that the immersion has degree 24. Further, they also obtained an explicit expression for model (3).

It follows from Theorems 3.1 and 18.1 that the invariant $\delta(2)$ always satisfies $\delta(2) \leq 2$ for every Lagrangian submanifold of the nearly Kähler S^6. Notice that the models (1), (4) and (5) satisfy the equality $\delta(2) = 2$ identically.

Chen, Dillen, Verstraelen and Vrancken [36] are able to characterize models (1), (4) and (5) by applying the invariant $\delta(2)$ as follows.

Theorem 18.3. *Models* (1), (4) *and* (5) *of Mashimo's list are the only Lagrangian submanifolds of the nearly Kähler S^6 with constant scalar curvature that satisfy the equality $\delta(2) = 2$.*

Many further examples of Lagrangian submanifolds in the nearly Kähler S^6 satisfying the equality $\delta(2) = 2$ have been constructed in [36, 37].

Deszcz, Dillen, Verstraelen and Vrancken [54] studied quasi-Einstein space and proved the following.

Theorem 18.4. *Lagrangian submanifolds of the nearly Kähler 6-sphere satisfying $\delta(2) = 2$ are quasi-Einstein.*

Theorem 18.5. *Let $f : M \to S^6$ be a Lagrangian immersion of a 3-dimensional quasi-Einstein manifold. Then either $\delta(2) = 2$ or there exists an open dense subset V of M such that each point $p \in V$ has a neighborhood W such that either*

(1) $f(W) = \psi_\gamma(UN)$, *where* N *is a superminimal linearly full almost complex curve in* S^6, UN *is the unit tangent bundle of* N, *and* $\psi_\gamma : UN \to S^6$ *with* $\cos^2\gamma = 5/9$ *is defined by*

$$\psi_\gamma(v) = \cos\gamma\phi + \sin\gamma v \times \frac{\alpha(v,v)}{\|\alpha(v,v)\|},$$

α *being the second fundamental form of the surface* $\phi : N \to S^6$, *or*

(2) $f(W)$ *is an open portion of the quasi-Einstein submanifold* $\tilde{\psi}(S^3) \subset S^6$ *defined in* [56].

The complete classification of Lagrangian submanifolds in the nearly Kähler six-sphere satisfying the equality $\delta(2) = 2$ was established by Dillen and Vrancken [57]. More precisely, they proved the following.

Theorem 18.6.

(1) *Let* $\phi : N_1 \to CP^2(4)$ *be a holomorphic curve in* $CP^2(4)$, PN_1 *the circle bundle over* N_1 *induced by the Hopf fibration* $\pi : S^5(1) \to CP^2(4)$, *and* ψ *the isometric immersion such that the following diagram commutes*:

$$\begin{array}{ccc} PN_1 & \xrightarrow{\psi} & S^5 \\ \downarrow & & \downarrow_\pi \\ N_1 & \xrightarrow{\phi} & CP^2(4) \end{array}$$

Then, *there exists a totally geodesic embedding* i *of* S^5 *into the nearly Kähler* 6*-sphere such that the immersion* $i \circ \psi : PN_1 \to S^6$ *is a* 3*-dimensional Lagrangian immersion in* S^6 *satisfying equality* $\delta(2) = 2$.

(2) *Let* $\bar{\phi} : N_2 \to S^6$ *be an almost complex curve* (*with second fundamental form* h) *without totally geodesic points. Denote by* UN_2 *the unit tangent bundle over* N_2 *and define a map*

$$\bar{\psi} : UN_2 \to S^6 : v \mapsto \bar{\phi}_\star(v) \times \frac{h(v,v)}{\|h(v,v)\|}. \tag{18.1}$$

Then $\bar{\psi}$ *is a* (*possibly branched*) *Lagrangian immersion into* S^6 *satisfying equality* $\delta(2) = 2$. *Moreover, the immersion is linearly full in* S^6.

(3) *Let* $\bar{\phi} : N_2 \to S^6$ *be a* (*branched*) *almost complex immersion. Then*, SN_2 *is a* 3*-dimensional* (*possibly branched*) *Lagrangian submanifold of* S^6 *satisfying equality* $\delta(2) = 2$.

(4) *Let* $f : M \to S^6$ *be a Lagrangian immersion which is not linearly full in* S^6. *Then* M *automatically satisfies equality* $\delta(2) = 2$ *and there exists a totally geodesic* S^5, *and a holomorphic immersion* $\phi : N_1 \to CP^2(4)$ *such that* f *is congruent to* ψ, *which is obtained from* ϕ *as in* (1).

(5) *Let* $f : M \to S^6$ *be a linearly full Lagrangian immersion of a 3-dimensional manifold satisfying equality* $\delta(2) = 2$. *Let* p *be a non totally geodesic point of* M. *Then there exists a* (*possibly branched*) *almost complex curve* $\bar{\phi} : N_2 \to S^6$ *such that* f *is locally around* p *congruent to* $\bar{\psi}$, *which is obtained from* $\bar{\phi}$ *as in* (3).

Let $f : S \to S^6$ be an almost complex curve without totally geodesic points. Define

$$F : T_1S \to S^6(1) : v \mapsto \frac{h(v,v)}{\|h(v,v)\|}, \tag{18.2}$$

where T_1S denotes the unit tangent bundle of S.

Vrancken [109] studied locally symmetric Lagrangian submanifolds in S^6 and obtained the following.

Theorem 18.7. *A locally symmetric Lagrangian submanifold of the nearly Kähler* S^6 *has constant curvature* 1 *or* 1/16.

Vrancken [110] also investigated Lagrangian isometric immersions which admit a unit length Killing vector field and showed the following.

Theorem 18.8.

(1) F *given by* (18.2) *defines a Lagrangian immersion if and only if* f *is superminimal*, *and*

(2) *If* $\psi : M \to S^6(1)$ *is a Lagrangian immersion which admits a unit length Killing vector field whose integral curves are great circles. Then there exists an open dense subset* U *of* M *such that each point* p *of* U *has a neighborhood* V *such that* $\psi : V \to S^6$ *satisfies* $\delta(2) = 2$, *or* $\psi : V \to S^6$ *is obtained as in statement* (1).

Li [73] studied Ricci pinching problem for Lagrangian submanifolds in nearly Kähler S^6 and proved the following.

Theorem 18.9. *If the Ricci tensor of a compact Lagrangian submanifold in the nearly Kähler* S^6 *satisfies* Ric$\geq$ (53/64) g, *then either* Ric $= 2g$ *or the submanifold is totally geodesic.*

Let M be an n-dimensional submanifold of a Riemannian m-manifold. The normal scalar curvature $\rho^{\perp}$ is defined by

$$\rho^{\perp} = \frac{2}{n(n-1)}\sqrt{\sum_{i<j=1}^{n}\sum_{r<s=1}^{m-n}\langle R^{\perp}(e_i,e_j)\xi_r,\xi_s\rangle^2}, \tag{18.3}$$

where $\{e_1,\cdots,e_n\}$ and $\{\xi_1,\cdots,\xi_{m-n}\}$ are orthonormal bases of the tangent and the normal spaces at the point p, respectively, and $R^{\perp}$ is curvature tensor of the normal bundle.

de Smet, Dillen, Verstraelen and Vrancken [52] obtained the following.

Theorem 18.10. *Let $f : M \to S^6(1)$ be a Lagrangian immersion. Then we have* :

(1) *The normal scalar curvature $\rho^{\perp}$ and the normalized scalar curvature ρ of M satisfy $\rho + \rho^{\perp} \leq 1$.*

(2) *f is (2)-ideal if and only if $\rho + \rho^{\perp} = 1$ holds identically.*

Deshmukh [53] proved that the index of the Jacobi operator on a compact Lagrangian submanifold M in the nearly Kähler S^6 is $\geq 3+\beta_1$. Using the theory of calibrations, Palmer [93] gave estimates for the nullity and Morse index of compact Lagrangian submanifolds in the nearly Kähler six-sphere.

References

1. D. E. Blair, F. Dillen, L. Verstraelen and L. Vrancken, Calabi curves as holomorphic Legendre curves and Chen's inequality, *Kyungpook Math. J.* **35** (1996), 407-416.

2. J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward, On conformal minimal immersions of S^2 into CP^n, *Math. Ann.* **279** (1988), 599-620.

3. J. Bolton, C. Scharlach, L. Vrancken and L. M. Woodward, Lagrangian submanifolds of CP^3 satisfying Chen's equality, preprint.

4. V. Borrelli, B. Y. Chen and J. M. Morvan, Une caractérisation gómétrique de la sphère de Whitney, *C. R. Acad. Sci. Paris Sér. I Math.* **321** (1995), 1485-1490.

5. R. L. Bryant, Minimal Lagrangian submanifolds of Kaehler-Einstein manifolds, in: *Differential Geometry and Differential Equations* (Shanghai, 1985), Lecture Notes in Math. **1255**, 1987, Springer, New York, pp. 1-12.

6. R. L. Bryant, Some examples of special Lagrangian tori, *Adv. Theor. Math. Phys.* **3** (1999), 83-90.

7. E. Calabi, Construction and properties of some 6-dimensional almost complex manifolds, *Trans. Amer. Math. Soc.* **87** (1958), 407-438.

8. I. Castro, C. R. Montealegre and F. Urbano, Closed conformal vector fields and Lagrangian submanifolds in complex space forms, *Pacific J. Math.*, to appear.

9. I. Castro, Lagrangian spheres in the complex Euclidean space satisfying a geometric equality, *Geom. Dedicata* **70** (1998), 197-208.

10. I. Castro, The Lagrangian version of a theorem of J. B. Meusnier, in: *Proc. Summer School on Diff. Geom.*, Coimbra 1999, pp. 83-89.

11. I. Castro and F. Urbano, Lagrangian surfaces in the complex Euclidean plane with conformal Maslov form, *Tôhoku Math. J.* **45** (1993), 656-582.

12. I. Castro and F. Urbano, New examples of minimal Lagrangian tori in the complex projective plane, *Manuscripta Math.* **85** (1994), 265-281.

13. I. Castro and F. Urbano, Twistor holomorphic Lagrangian surface in the complex projective and hyperbolic planes, *Ann. Global Anal. Geom.* **13** (1995), 59-67.

14. I. Castro and F. Urbano, Examples of unstable Hamiltonian-minimal Lagrangian tori in $\mathbb{C}^2$ *Compositis Math.* **111** (1998), 1-14.

15. I. Castro and F. Urbano, On a minimal Lagrangian submanifolds of $\mathbb{C}^n$ foliated by spheres, *Michigan Math. J.* **46** (1999), 71-82.

16. S. Chang, On Hamiltonian stable minimal Lagrangian surfaces in CP^2 *J. Geom. Anal.* **10** (2000), 243-255.

17. B. Y. Chen, *Geometry of Submanifolds*, Mercel Dekker, New York, 1973.

18. B. Y. Chen, *Geometry of Submanifolds and its Applications*, Science Univ. Tokyo, Tokyo, 1981.

19. B. Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific Publisher, New Jersey, 1984.

20. B. Y. Chen, Some pinching and classification theorems for minimal submanifolds, *Arch. Math.* $(Basel)$, **60** (1993), 568-578.

21. B. Y. Chen, Jacobi's elliptic functions and Lagrangian immersions, *Proc. Roy. Soc. Edinburgh Ser. A* **126** (1996), 687-704.

22. B. Y. Chen, A general inequality for submanifolds in complex space forms and its applications, *Arch. Math.* $(Basel)$ **67** (1996), 519-528.

23. B. Y. Chen, Interaction of Legendre curves and Lagrangian submanifolds, *Israel J. Math.* **99** (1997), 69-108.

24. B. Y. Chen, Complex extensors and Lagrangian submanifolds in complex space forms, *Tôhoku Math. J.* **49** (1997), 277-297.

25. B. Y. Chen, Strings of Riemannian invariants, inequalities, ideal immersions and their applications, in: *The Third Pacific Rim Geometry Conference*, Internat Press, Cambridge, MA, 1998, pp. 7-60.

26. B. Y. Chen, A vanishing theorem for Lagrangian immersions into Einstein-Kaehler manifolds and its applications, *Soochow J. Math.* **24** (1998), 157-164.

27. B. Y. Chen, Intrinsic and extrinsic structures of Lagrangian surfaces in complex space forms, *Tsukuba J. Math.* **22** (1998), 657-680.

28. B. Y. Chen, Representation of flat Lagrangian H-umbilical submanifolds in complex Euclidean spaces, *Tôhoku Math. J.* **50** (1999), 13-20.

29. B. Y. Chen, Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions, *Glasgow Math. J.* **41** (1999), 33-41.

30. B. Y. Chen, Some new obstructions to minimal and Lagrangian isometric immersions, *Japan. J. Math.* **26** (2000), 105-127.

31. B. Y. Chen, Warped product, complex extensors and Lagrangian submanifolds, *Soochow J. Math.* **26** (2000), 1-17.

32. B. Y. Chen, On Ricci curvature of isotropic and Lagrangian submanifolds in complex space forms, *Arch. Math.* $(Basel)$ **74** (2000), 154-160.

33. B. Y. Chen, Ideal Lagrangian immersions in complex space forms, *Math. Proc. Cambridge Philos. Soc.* **128** (2000), 511-533.

34. B. Y. Chen, Riemannian submanifolds, in: *Handbook of Differential Geometry*, vol. I, F. Dillen and L. Verstraelen, eds., North Holland Publ. 2000, pp. 187-418.

35. B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, Totally real submanifolds of CP^n satisfying a basic inequality, *Arch. Math.* $(Basel)$ **63** (1994), 553-564.

36. B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, Two equivariant totally real immersions into the nearly Kähler 6-sphere and their characterization, *Japan. J. Math.* **21** (1995), 207-222.

37. B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, Characterizing a class of totally real submanifolds of S^6 by their sectional curvatures, *Tôhoku Math. J.* **47** (1995), 185-198.

38. B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, An exotic totally real minimal immersion of S^3 in CP^3 and its characterization, *Proc. Roy. Soc. Edinburgh Ser. A* **126** (1996), 153-165.

39. B. Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, Lagrangian isometric immersions of a real space form $M^n(c)$ into a complex space form $\tilde{M}^n(4c)$, *Math. Proc. Cambridge Philos. Soc.* **124** (1998), 107-125.

40. B. Y. Chen and C. S. Houh, Totally real submanifolds of a quaternion projective space, *Ann. Mat. Pura Appl.* **70** (1979), 185-199.

41. B. Y. Chen, C. S. Houh and H. S. Lue, Totally real submanifolds, *J. Differential Geom.* **12** (1977), 473-480.

42. B. Y. Chen and J. M. Morvan, Geometrie des surfaces lagrangiennes de $\mathbb{C}^2$ *J. Math. Pures Appl.* **66** (1987), 321-325.

43. B. Y. Chen and J. M. Morvan, Deformations of isotropic submanifolds in Kähler manifolds, *J. Geom. Phys.* **13** (1994), 79-104.

44. B. Y. Chen, J. M. Morvan and T. Nore, Energy, tension and finite type maps, *Kodai Math. J.* **9** (1986), 406-418.

45. B. Y. Chen and T. Nagano, Totally geodesic submanifolds of symmetric spaces, I *Duke Math. J.* **44** (1977), 745-755; —, II, *ibid* **45** (1978), 405-425.

46. B. Y. Chen and K. Ogiue, On totally real submanifolds, *Trans. Amer. Math. Soc.* **193** (1974), 257-266.

47. B. Y. Chen and K. Ogiue, Two theorems on Kaehler manifolds, *Michigan Math. J.* **21** (1974), 225-229.

48. B. Y. Chen and L. Vrancken, Lagrangian submanifolds satisfying a basic inequality, *Math. Proc. Cambridge Philos. Soc.* **120** (1996), 291-307.

49. B. Y. Chen and L. Vrancken, Lagrangian submanifolds of the complex hyperbolic space, to appear.

50. B. Y. Chen and L. Vrancken, Lagrangian minimal isometric immersions of a Lorentzian real space form $M_1^n(c)$ into a Lorentzian complex space form $\tilde{M}_1^n(4c)$, *Tôhoku Math. J.*, to appear.

51. M. Dajczer and R. Tojeiro, Flat totally real submanifolds of CP^n and the symmetric generalized wave equation, *Tôhoku Math. J.* **47** (1995), 117-123.

52. P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, The normal curvature of totally real submanifolds of $S^6(1)$ *Glasgow Math. J.* **40** (1998), 199-204.

53. S. Deshmukh, Totally real submanifolds of S^6 *Nihonkai Math. J.* **1** (1990), 193-201.

54. R. Deszcz, F. Dillen, L. Verstraelen and L. Vrancken, Quasi-Einstein totally real submanifolds of the nearly Kähler 6-sphere, *Tôhoku Math. J.* **51** (1999), 461-478.

55. F. Dillen, B. Opozda, L. Verstraelen and L. Vrancken, On totally real 3-dimensional submanifolds of the nearly Kaehler 6-sphere, *Proc. Amer. Math. Soc.* **99** (1987), 741-749.

56. F. Dillen, L. Verstraelen and L. Vrancken, Classification of totally real 3-dimensional submanifolds of $S^6(1)$ with $K \geq 1/16$, *J. Math. Soc. Japan* **42** (1990), 565-584.

57. F. Dillen and L. Vrancken, Totally real submanifolds in $S^6(1)$ satisfying Chen's equality, *Trans. Amer. Math. Soc.* **348** (1996), 1633-1646.

58. I. Dimitric, 1-type submanifolds of the complex projective space, *Kodai Math. J.* **14** (1991), 281-295.

59. N. Ejiri, Totally real submanifolds in a 6-sphere, *Proc. Amer. Math. Soc.* **83** (1981), 759-763.

60. N. Ejiri, Totally real minimal immersions of n-dimensional real space forms into n-dimensional complex space forms, *Proc. Amer. Math. Soc.* **84** (1982), 243-246.

61. N. Ejiri, Calabi lifting and surface geometry in S^4, *Tokyo J. Math.* **9** (1986), 297-324.

62. L. Gheysens, P. Verheyen and L. Verstraelen, Sur les surfaces $\mathcal{A}$ ou les surfaces de Chen, *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), 913-916.

63. M. L. Gromov, A topological technique for construction of solutions of differential equations and inequalities, in: *Actes du Congress International des Mathematiciens, Tome* 2, Gauthier-Villars, Paris, 1971, pp. 221-225.

64. M. L. Gromov, Pseudoholomorphic curves in symplectic manifolds, *Invent. Math.* **82** (1985), 307-347.

65. R. Harvey and H. B. Lawson, Calibrated geometries, *Acta Math.* **148** (1982), 47-157.

66. H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Verlag, Basel, 1994.

67. C. S. Houh and Y. Hong, Lagrangian submanifolds of quaternionic Kaehlerian manifolds satisfying Chen's equality, *Beiträge Algebra. Geom.* **39** (1998), 413-421.

68. U. H. Ki and Y. H. Kim, Totally real submanifolds of a complex space form, *Internat. J. Math. Math. Sci.* **19** (1996), 39-44.

69. M. Kotani, An immersion of an n-dimensional real space form into an n-dimensional complex space form, *Tokyo J. Math.* **9** (1986), 103-113.

70. M. Kriele and L. Vrancken, Minimal Lagrangian submanifolds of Lorentzian complex space forms with constant sectional curvature, *Arch. Math.* $(Basel)$ **72** (1999), 223-232.

71. Le Khong Van and A. T. Fomenko, Lagrangian manifolds and the Maslov index in the theory of minimal surfaces, *Soviet Math. Dokl.* **37** (1988), 330-333.

72. Y. I. Lee, Lagrangian minimal surfaces in Kaehler-Einstein surfaces of negative scalar curvature, *Comm. Anal. Geom.* **2** (1994), 579-592.

73. H. Li, The Ricci curvature of totally real 3-dimensional submanifolds of the nearly Kaehler 6-sphere, *Bull. Belg. Math. Soc. Simon Stevin* **3** (1996), 193-199.

74. X. M. Liu, Totally real submanifolds in a complex projective space, *Internat. J. Math. Math. Sci.* **22** (1999), 205-208.

75. G. D. Ludden, M. Okumura and K. Yano, A totally real surface in CP^2 that is not totally geodesic, *Proc. Amer. Math. Soc.* **53** (1975), 186-190.

76. K. Matsumoto, I. Mihai and A. Oiaga, Shape operator for slant submanifolds in complex space forms, *Bull. Yamagata Univ. Natur. Sci.* **14** (2000), 169-177.

77. K. Mashimo, Homogeneous totally real submanifolds of S^6, *Tsukuba J. Math.* **9** (1985), 185-202.

78. Y. Matsuyama, On curvature pinching for totally real submanifolds of $CP^n(c)$, *J. Ramanujan Math. Soc.* **9** (1994), 13-24.

79. Y. Matsuyama, Curvature pinching for totally real submanifolds of a complex projective space, *J. Math. Soc. Japan* **52** (2000), 51-64.

80. C. Montealegre and L. Vrancken, Three dimensional Lagrangian submanifolds of the complex projective space, to appear.

81. S. Montiel, A. Ros and F. Urbano, Curvature pinching and eigenvalue rigidity for minimal submanifolds, *Math. Z.* **191** (1986), 537-548.

82. S. Montiel and F. Urbano, Isotropic totally real submanifolds, *Math. Z.* **199** (1988), 55-60.

83. J. M. Morvan, Classe de Maslov d'une immersion Lagranienne etminimalité, *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), 633-636.

84. H. Naitoh, Totally real parallel submanifolds, *Tokyo J. Math.* **4** (1981), 279-306.

85. H. Naitoh, Parallel submanifolds of complex space forms I, *Nagoya Math. J.* **90** (1983), 85-117; II, *ibid* **91** (1983), 119-149.

86. H. Naitoh and M. Takeuchi, Totally real submanifolds and symmetric bounded domain, *Osaka J. Math.* **19** (1982), 717-731.

87. K. Ogiue, K. Positively curved totally real minimal submanifolds immersed in a complex projective space, *Proc. Amer. Math. Soc.* **56** (1976), 264-266.

88. K. Ogiue, Some recent topics in the theory of submanifolds, *Sugaku Expositions*, **4** (1991), 21-41.

89. Y. G. Oh, Second variation and stability of minimal Lagrangian submanifolds in Kaehler manifolds, *Invent. Math.* **101** (1990), 501-519.

90. Y. M. Oh, *Explicit Construction of Lagrangian Isometric Immersion of a Real Space Form $M^n(c)$ into a complex space form $\tilde{M}^n(4c)$*, Ph.D. Thesis, Michigan State University, 2000.

91. Y. Ohnita, Totally real submanifolds with nonnegative sectional curvature, *Proc. Amer. Math. Soc.* **97** (1986), 474-478.

92. B. Palmer, Buckling eigenvalues, Gauss maps and Lagrangian submanifolds, *Differential Geom. Appl.* **4** (1994), 391-403.

93. B. Palmer, Calibrations and Lagrangian submanifolds, *Tôhoku Math. J.* **50** (1998), 303-315.

94. H. Reckziegel, Horizontal lifts of isometric immersions into the bundle space of a pseudo-Riemannian submersion, in: *Global Differential Geometry and Global Analysis 1984*, Springer-Verlag, Berlin, Lecture Notes in Math., **1156**, 1985, pp. 264-279.

95. A. Ros, Spectral geometry of CR-minimal submanifolds in the complex projective space, *Kodai Math. J.* **6** (1983), 88-99.

96. A. Ros and F. Urbano, Lagrangian surface in the complex Euclidean plane with conformal Maslov form, *J. Math. Soc. Japan*, **45** (1993, 565-582,

97. A. Ros and F. Urbano, Lagrangian submanifolds of $\mathbb{C}^n$ with conformal Maslov form and the Whitney sphere, *J. Math. Soc. Japan* **50** (1998), 203-226.

98. B. Rouxel, Chen submanifolds, in: *Geometry and Topology of Submanifolds*, **VI**, World Scientific, River Edge, NJ, 1994, pp. 185-198.

99. C. U. Sánchez, A. L. Calí and J. L. Moreschi, Homogeneous totally real submanifolds of complex projective space, *Rend. Sem. Mat. Univ. Padova*, **101** (1999), 83-94.

100. N. Sato, Totally real submanifolds of a complex space form with nonzero parallel mean curvature vector, *Yokohama Math. J.* **44** (1997), 1-4.

101. Y. B. Shen, Y. X. Dong and X. Y. Guo, On scalar curvature for totally real minimal submanifolds in CP^n, *Chinese Sci. Bull.* **40** (1995), 621-626.

102. R. Schoen and J. Wolfson, Minimizing volume among Lagrangian submanifolds, in: *Differential Equations*: *La Pietra* 1996 (Florence), Proc. Sympos. Pure Math., 65, Amer. Math. Soc., Providence, RI, 1999. pp. 181-199.

103. B. Suceava, The Chen invariants of warped products of hyperbolic planes and their applications to immersibility problems, *Tsukuba J. Math.*, to appear.

104. M. Takeuchi, Stability of certain minimal submanifolds of compact Hermitian symmetric spaces, *Tôhoku Math. J.* **36** (1984), 293-314.

105. F. Urbano, Totally real minimal submanifolds of a complex projective space, *Proc. Amer. Math. Soc.* **93** (1985), 332-334.

106. F. Urbano, Negatively curved totally real submanifolds, *Math. Ann.* **273** (1986), 345-348.

107. F. Urbano, Totally real submanifolds of a complex projective space, in: *Geometry and Topology of Submanifolds*, World Scientific, Teaneck, N.J., 1989, pp. 198-208.

108. F. Urbano, Index of Lagrangian submanifolds of CP^n and the Laplacian of 1-forms, *Geom. Dedicata* **48** (1993), 309-318.

109. L. Vrancken, Locally symmetric submanifolds of the nearly Kaehler S^6, *Algebras Groups Geom.* **5** (1988), 369-394.

110. L. Vrancken, Killing vector fields and Lagrangian submanifolds of the nearly Kaehler S^6, *J. Math. Pure Appl.* **77** (1998), 631-645.

111. C. Y. Xia, On the minimal submanifolds in $CP^m(c)$ and $S^N(1)$, *Kodai Math. J.* **15** (1992), 141-153.

112. S. T. Yau, Submanifolds with constant mean curvature, I, *Amer. J. Math.* **96** (1974), 346-366.

113. K. Yano, Totally real submanifolds of a Kaehlerian manifold, *J. Differential Geom.* **11** (1976), 351-359.

114. R. Aiyama, Totally real surfaces in the complex 2-space, preprint.

Department of Mathematics, Michigan State University
East Lansing, MI 48824-1027, U.S.A.
E-mail: bychen@math.msu.edu