TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 3, pp. 587-602, September 2001
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ON GENERALIZED FRACTIONAL INTEGRALS

Eiichi Nakai

Dedicated to Professor Kôzô Yabuta on his sixtieth birthday

Abstract. It is known that the fractional integral I_α $(0 < \alpha < n)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $p > 1$ and $n/p - \alpha = n/q > 0$, from $L^p(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ when $p > 1$ and $n/p - \alpha = 0$, from $L^p(\mathbb{R}^n)$ to $\mathrm{Lip}_\beta(\mathbb{R}^n)$ when $p > 1$ and $-1 < n/p - \alpha = -\beta < 0$, from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ when $0 < \alpha < 1$, and from $\mathrm{Lip}_\beta(\mathbb{R}^n)$ to $\mathrm{Lip}_\gamma(\mathbb{R}^n)$ when $0 < \alpha + \beta = \gamma < 1$. We introduce generalized fractional integrals and extend the above boundedness to the Orlicz spaces and BMO_ϕ.

1. Introduction

The fractional integral I_α $(0 < \alpha < n)$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

It is known that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $p > 1$ and $n/p - \alpha = n/q > 0$ as the Hardy-Littlewood-Sobolev theorem. The fractional integral was studied by many authors (see, for example, Rubin [5] or Chapter 5 in Stein [6]). The Hardy-Littlewood-Sobolev theorem is an important result in the fractional integral theory and the potential theory. We introduce generalized fractional integrals and extend the Hardy-Littlewood-Sobolev theorem to the Orlicz spaces. We show that, for example, a generalized fractional integral is bounded from $\exp L^p$ to $\exp L^q$.

Received June 1, 2000; revised July 28, 2000.
Communicated by M.-H. Shih.
2000 *Mathematics Subject Classification*: 26A33, 46E30, 42B35, 46E15.
Key words and phrases: Fractional integtal, Riesz potential, Orlicz space, BMO, Lipschitz space.
This research was partially supported by the Grant-in-Aid for Scientific Research (C), No. 11640165, 1999, the Ministry of Education, Science, Sports and Culture, Japan.

Let $B(a,r)$ be the ball $\{x \in \mathbb{R}^n : |x-a| < r\}$ with center a and of radius $r > 0$, and $B_0 = B(\mathrm{O},1)$ with center the origin and of radius 1. The modified fractional integral $\tilde{I}_\alpha$ $(0 < \alpha < n+1)$ is defined by

$$\tilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} f(y) \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1 - \chi_{B_0}(y)}{|y|^{n-\alpha}} \right) dy,$$

where χ_{B_0} is the characteristic function of B_0. It is also known that the modified fractional integral $\tilde{I}_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$ when $p > 1$ and $n/p - \alpha = 0$, from $L^p(\mathbb{R}^n)$ to $\mathrm{Lip}_\beta(\mathbb{R}^n)$ when $p > 1$ and $-1 < n/p - \alpha = -\beta < 0$, from $\mathrm{BMO}(\mathbb{R}^n)$ to $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ when $0 < \alpha < 1$, and from $\mathrm{Lip}_\beta(\mathbb{R}^n)$ to $\mathrm{Lip}_\gamma(\mathbb{R}^n)$ when $0 < \alpha + \beta = \gamma < 1$. We also investigate the boundedness of generalized fractional integrals from the Orlicz space $L^\Phi(\mathbb{R}^n)$ to $\mathrm{BMO}_\phi(\mathbb{R}^n)$ and from $\mathrm{BMO}_\phi(\mathbb{R}^n)$ to $\mathrm{BMO}_\psi(\mathbb{R}^n)$, where $\mathrm{BMO}_\phi(\mathbb{R}^n)$ is the function space defined by using the mean oscillation and a weight function $\phi : (0,+\infty) \to (0,+\infty)$. If $\phi(r) \equiv 1$, then $\mathrm{BMO}_\phi(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$. If $\phi(r) = r^\alpha$ $(0 < \alpha \le 1)$, then $\mathrm{BMO}_\phi(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

Though we state our results on the Euclidean space $\mathbb{R}^n$, these hold on spaces of homogeneous type with appropriate conditions.

2. Notations and Definitions

For a function $\rho : (0,+\infty) \to (0,+\infty)$, let

$$I_\rho f(x) = \int_{\mathbb{R}^n} f(y) \frac{\rho(|x-y|)}{|x-y|^n}\, dy.$$

We consider the following conditions on ρ:

$$\int_0^1 \frac{\rho(t)}{t}\, dt < +\infty, \tag{2.1}$$

$$\frac{1}{A_1} \le \frac{\rho(s)}{\rho(r)} \le A_1 \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{2.2}$$

$$\frac{\rho(r)}{r^n} \le A_2 \frac{\rho(s)}{s^n} \quad \text{for} \quad s \le r, \tag{2.3}$$

where $A_1, A_2 > 0$ are independent of $r, s > 0$. If $\rho(r) = r^\alpha$, $0 < \alpha < n$, then I_ρ is the fractional integral or the Riesz potential denoted by I_α.

We define the modified version of I_ρ as follows:

$$\tilde{I}_\rho f(x) = \int_{\mathbb{R}^n} f(y) \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) dy.$$

We consider the following conditions on ρ: (2.1), (2.2) and

$$\frac{\rho(r)}{r^{n+1}} \le A_2' \frac{\rho(s)}{s^{n+1}} \quad \text{for} \quad s \le r, \tag{2.4}$$

$$\int_r^{+\infty} \frac{\rho(t)}{t^2}\, dt \le A_2'' \frac{\rho(r)}{r}, \tag{2.5}$$

$$\left| \frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n} \right| \le A_3 |r-s| \frac{\rho(r)}{r^{n+1}} \quad \text{for} \quad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{2.6}$$

where $A_2', A_2'', A_3 > 0$ are independent of $r, s > 0$. If $\rho(r)r^\alpha$ is increasing for some $\alpha \ge 0$ and $\rho(r)/r^\beta$ is decreasing for some $\beta \ge 0$, then ρ satisfies (2.2) and (2.6). If $\rho(r) = r^\alpha$, $0 < \alpha < n+1$, then $\tilde{I}_\rho = \tilde{I}_\alpha$. If $\tilde{I}_\rho f$ and $I_\rho f$ are well-defined, then $\tilde{I}_\rho f - I_\rho f$ is a constant.

A function $\Phi : [0, +\infty) \to [0, +\infty]$ is called a Young function if Φ is convex, $\lim_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = +\infty$. Any Young function is increasing. For a Young function Φ, the complementary function is defined by

$$\widetilde{\Phi}(r) = \sup\{rs - \Phi(s) : s \ge 0\}, \quad r \ge 0.$$

For example, if $\Phi(r) = r^p/p$, $1 < p < \infty$, then $\widetilde{\Phi}(r) = r^{p'}/p'$, $1/p + 1/p' = 1$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ $(0 \le r \le 1)$, and $= +\infty$ $(r > 1)$.

For a Young function Φ, let

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\epsilon|f(x)|)\, dx < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_\Phi = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1 \right\},$$

$$L^\Phi_{\text{weak}}(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \sup_{r>0} \Phi(r)\, m(r, \epsilon f) < +\infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{\Phi,\text{weak}} = \inf\left\{ \lambda > 0 : \sup_{r>0} \Phi(r)\, m\left(r, \frac{f}{\lambda}\right) \le 1 \right\},$$

$$\text{where} \quad m(r, f) = |\{x \in \mathbb{R}^n : |f(x)| > r\}|.$$

If a Young function Φ satisfies

$$0 < \Phi(r) < +\infty \quad \text{for} \quad 0 < r < +\infty, \tag{2.7}$$

then Φ is continuous and bijective from $[0, +\infty)$ to itself. The inverse function Φ^{-1} is also increasing and continuous.

A function Φ is said to satisfy the ∇_2-condition, denoted $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \quad r \ge 0,$$

for some $k > 1$.

Let $Mf(x)$ be the maximal function, i.e.,

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x.

We assume that Φ satisfies (2.7). Then M is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Phi_{\text{weak}}(\mathbb{R}^n)$ and

$$\|Mf\|_{\Phi,\text{weak}} \le C_0 \|f\|_\Phi. \tag{2.8}$$

If $\Phi \in \nabla_2$, then M is bounded on $L^\Phi(\mathbb{R}^n)$ and

$$\|Mf\|_\Phi \le C_0 \|f\|_\Phi. \tag{2.9}$$

For a function $\phi : (0,+\infty) \to (0,+\infty)$, let

$$\text{BMO}_\phi(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \sup_{B=B(a,r)} \frac{1}{\phi(r)} \frac{1}{|B|} \int_B |f(x) - f_B|\, dx < +\infty \right\},$$

$$\|f\|_{\text{BMO}_\phi} = \sup_{B=B(a,r)} \frac{1}{\phi(r)} \frac{1}{|B|} \int_B |f(x) - f_B|\, dx,$$

$$\text{where} \qquad f_B = \frac{1}{|B|} \int_B f(x)\, dx.$$

If $\phi(r) \equiv 1$, then $\text{BMO}_\phi(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$. If $\phi(r) = r^\alpha$, $0 < \alpha \le 1$, then it is known that $\text{BMO}_\phi(\mathbb{R}^n) = \text{Lip}_\alpha(\mathbb{R}^n)$.

For functions $\theta, \kappa : (0,+\infty) \to (0,+\infty)$, we denote $\theta(r) \sim \kappa(r)$ if there exists a constant $C > 0$ such that

$$C^{-1}\theta(r) \le \kappa(r) \le C\theta(r), \quad r > 0.$$

A function $\theta : (0,+\infty) \to (0,+\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that $\theta(r) \le C\theta(s)$ $(\theta(r) \ge C\theta(s))$ for $r \le s$.

The letter C shall always denote a constant, not necessarily the same one.

3. Main Results

Our main results are as follows:

Theorem 3.1. *Let ρ satisfy* (2.1)$\sim$(2.3). *Let Φ and Ψ be Young functions with* (2.7). *Assume that there exist constants $A, A', A'' > 0$ such that, for all $r > 0$,*

$$\int_r^{+\infty} \widetilde{\Phi}\left(\frac{\rho(t)}{A \int_0^r (\rho(s)/s)\, ds\, \Phi^{-1}(1/r^n) t^n} \right) t^{n-1}\, dt \le A', \tag{3.1}$$

$$\int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}\left(\frac{1}{r^n}\right) \le A''\;\Psi^{-1}\left(\frac{1}{r^n}\right), \tag{3.2}$$

where $\widetilde{\Phi}$ is the complementary function with respect to Φ. Then, for any $C_0 > 0$, there exists a constant $C_1 > 0$ such that, for $f \in L^{\Phi}(\mathbb{R}^n)$,

$$\Psi\left(\frac{|I_\rho f(x)|}{C_1\|f\|_\Phi}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_\Phi}\right). \tag{3.3}$$

Therefore I_ρ is bounded from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}_{\text{weak}}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then I_ρ is bounded from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

Remark 3.1. From (2.2), it follows that

$$\rho(r) \le C\int_0^r \frac{\rho(t)}{t}\,dt. \tag{3.4}$$

If $\rho(r)/r^{\varepsilon}$ is almost increasing for some $\varepsilon > 0$ and $\rho(t)/t^n$ is almost decreasing, then ρ satisfies (2.1)∼(2.3) and $\int_0^r (\rho(t)/t)\,dt \sim \rho(r)$. Let, for example, $\rho(r) = r^{\alpha}(\log(1/r))^{-\beta}$ for small r. If $\alpha = 0$ and $\beta > 1$, then $\int_0^r (\rho(t)/t)\,dt \sim (\log(1/r))^{-\beta+1}$. If $\alpha > 0$ and $-\infty < \beta < +\infty$, then $\int_0^r (\rho(t)/t)\,dt \sim \rho(r)$.

Remark 3.2. In the case $\Phi(r) = r$, (3.1) is equivalent to

$$\frac{\rho(t)}{t^n} \le \frac{A\int_0^r (\rho(s)/s)\,ds}{r^n}, \quad 0 < r \le t.$$

This inequality follows from (2.3) and (3.4).

The following corollary is stated without the complementary function.

Corollary 3.2. *Let ρ satisfy* (2.1)∼(2.3). *Let Φ and Ψ be Young functions with* (2.7). *Assume that*

$$\int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}\left(\frac{1}{r^n}\right)$$

is almost decreasing and that there exist constants $A, A' > 0$ such that, for all $r > 0$,

$$\int_r^{+\infty} \frac{\rho(t)}{t}\Phi^{-1}\left(\frac{1}{t^n}\right)\,dt \le A\int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}\left(\frac{1}{r^n}\right), \tag{3.5}$$

$$\int_0^r \frac{\rho(t)}{t}\,dt\;\Phi^{-1}\left(\frac{1}{r^n}\right) \le A'\;\Psi^{-1}\left(\frac{1}{r^n}\right). \tag{3.6}$$

Then (3.3) *holds. Therefore* I_ρ *is bounded from* $L^\Phi(\mathbb{R}^n)$ *to* $L^\Psi_{\text{weak}}(\mathbb{R}^n)$. *Moreover, if* $\Phi \in \nabla_2$, *then* I_ρ *is bounded from* $L^\Phi(\mathbb{R}^n)$ *to* $L^\Psi(\mathbb{R}^n)$.

Remark 3.3. If $r^\varepsilon \rho(r)\Phi^{-1}(1/r^n)$ is almost decreasing for some $\varepsilon > 0$, then

$$\int_r^{+\infty} \frac{\rho(t)}{t}\Phi^{-1}\left(\frac{1}{t^n}\right) dt \le C\rho(r)\,\Phi^{-1}\left(\frac{1}{r^n}\right).$$

This inequality and (3.4) yield (3.5).

Remark 3.4. We cannot replace (3.2) or (3.6) by

$$\rho(r)\,\Phi^{-1}\left(\frac{1}{r^n}\right) \le A\,\Psi^{-1}\left(\frac{1}{r^n}\right) \quad \text{for all } r > 0.$$

Theorem 3.3. *Let* ρ *satisfy* (2.1), (2.2), (2.4) *and* (2.6). *Let* Φ *be Young function with* (2.7), ϕ *be almost increasing and* $\phi(r) \sim \phi(2r)$. *Assume that there exist constants* $A, A', A'' > 0$ *such that, for all* $r > 0$,

$$\int_r^{+\infty} \widetilde{\Phi}\left(\frac{r\rho(t)}{A\int_0^r(\rho(s)/s)\,ds\,\Phi^{-1}(1/r^n)t^{n+1}}\right) t^{n-1}\,dt \le A', \tag{3.7}$$

$$\int_0^r \frac{\rho(t)}{t}\,dt\,\Phi^{-1}\left(\frac{1}{r^n}\right) \le A''\phi(r), \tag{3.8}$$

where $\widetilde{\Phi}$ *is the complementary function with respect to* Φ. *Then* $\tilde{I}_\rho$ *is bounded from* $L^\Phi(\mathbb{R}^n)$ *to* $\mathrm{BMO}_\phi(\mathbb{R}^n)$.

Theorem 3.4. *Let* ρ *satisfy* (2.1), (2.2), (2.5) *and* (2.6). *Let* ϕ *and* ψ *be almost increasing*, $\phi(r) \sim \phi(2r)$ *and* $\psi(r) \sim \psi(2r)$. *Assume that there exist constants* $A, A' > 0$ *such that, for all* $r > 0$,

$$\int_r^{+\infty} \frac{\rho(t)\phi(t)}{t^2}\,dt \le A\frac{\rho(r)\phi(r)}{r}, \tag{3.9}$$

$$\int_0^r \frac{\rho(t)}{t}\,dt\,\phi(r) \le A'\psi(r). \tag{3.10}$$

Then $\tilde{I}_\rho$ *is bounded from* $\mathrm{BMO}_\phi(\mathbb{R}^n)$ *to* $\mathrm{BMO}_\psi(\mathbb{R}^n)$.

Remark 3.5. From Lemma 4.2, it follows that $\tilde{I}_\rho 1$ is a constant. Hence $\tilde{I}_\rho$ is well-defined as an operator from $\mathrm{BMO}_\phi(\mathbb{R}^n)$ to $\mathrm{BMO}_\psi(\mathbb{R}^n)$.

The results in Figure 1 are known. Our results contain these. Moreover, we have the results in Figure 2.

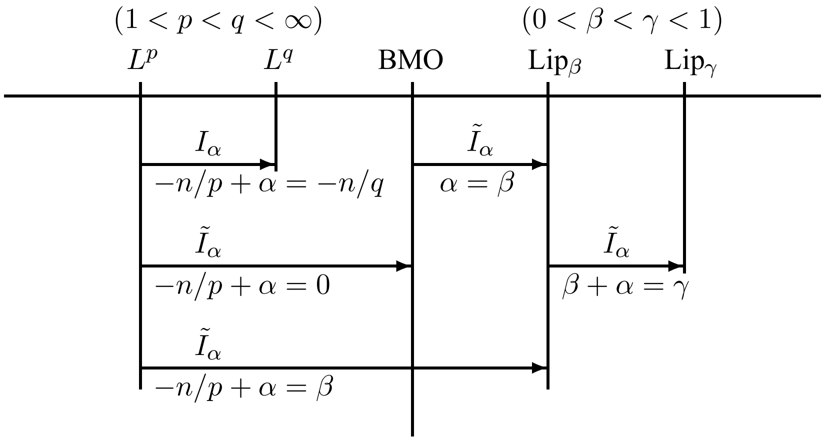

Figure 1: Boundedness of fractional integrals

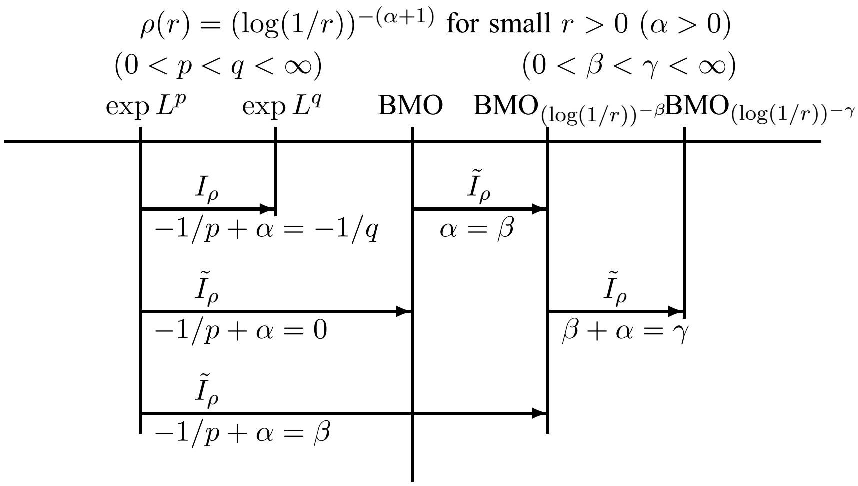

Figure 2: Boundedness of generalized fractional integrals

4. Proofs

Let Φ be a Young function. By the convexity and $\Phi(0) = 0$, we have

$$\Phi(r) \le \frac{r}{s}\Phi(s) \quad \text{for } r \le s. \tag{4.1}$$

Let $\widetilde{\Phi}$ be the complementary function with respect to Φ. Then

$$\widetilde{\Phi}\left(\frac{\Phi(r)}{r}\right) \le \Phi(r), \quad r > 0. \tag{4.2}$$

Actually,

$$\frac{\Phi(r)}{r}s - \Phi(s) \le \Phi(r) \quad \text{for } s < r$$

and

$$\frac{\Phi(r)}{r}s - \Phi(s) \le 0 \quad \text{for } s \ge r.$$

We note that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le 2\|f\|_\Phi \|g\|_{\widetilde{\Phi}} \tag{4.3}$$

(see for example [4]).

Proof of Theorem 3.1. Let

$$J_1 = \int_{|x-y|<r} f(y)\frac{\rho(|x-y|)}{|x-y|^n}\,dy \quad \text{and}$$
$$J_2 = \int_{|x-y|\ge r} f(y)\frac{\rho(|x-y|)}{|x-y|^n}\,dy.$$

Let

$$h(r) = \inf\left\{\frac{\rho(s)}{s^n} : s \le r\right\}, \quad r > 0.$$

Then h is nonincreasing. It follows that

$$\int_{|x-y|<r} |f(y)|h(|x-y|)\,dy \le Mf(x)\int_{|x-y|<r} h(|x-y|)\,dy$$

(see Stein [7, p.57])). Since $h(r) \sim \rho(r)/r^n$,

$$|J_1| \le CMf(x)\int_{|x-y|<r} \frac{\rho(|x-y|)}{|x-y|^n}\,dy \le CMf(x)\int_0^r \frac{\rho(t)}{t}\,dt. \tag{4.4}$$

Next we estimate J_2. By (4.3) we have

$$|J_2| \le 2\left\|\frac{\rho(|x-\cdot|)}{|x-\cdot|^n}\chi_{B(x,r)^{\mathrm{C}}}(\cdot)\right\|_{\widetilde{\Phi}} \|f\|_\Phi, \tag{4.5}$$

where $\chi_{B(x,r)^{\mathrm{C}}}$ is the characteristic function of the complement of $B(x,r)$. Let

$$F(r) = \int_0^r \frac{\rho(s)}{s}\,ds\, \Phi^{-1}\left(\frac{1}{r^n}\right). \tag{4.6}$$

We show

$$\left\| \frac{\rho(|x-\cdot|)}{|x-\cdot|^n} \chi_{B(x,r)^{\mathrm{C}}}(\cdot) \right\|_{\widetilde{\Phi}} \le CF(r). \tag{4.7}$$

From (2.2) and the increasingness of $\widetilde{\Phi}$ it follows that

$$\int_{|x-y|\ge r} \widetilde{\Phi}\left(\frac{\rho(|x-y|)}{\lambda|x-y|^n}\right) dy \le C_2 \int_r^{+\infty} \widetilde{\Phi}\left(\frac{\rho(t)}{\lambda t^n}\right) t^{n-1}\, dt, \tag{4.8}$$

where C_2 is independent of $\lambda > 0$ and $x \in \mathbb{R}^n$. We may assume that $C_2 A' \ge 1$. By (4.1) and (3.1) we have

$$\begin{aligned} &\int_r^{+\infty} \widetilde{\Phi}\left(\frac{\rho(t)}{C_2AA'F(r)t^n}\right) t^{n-1}\, dt \\ &\le \frac{1}{C_2A'} \int_r^{+\infty} \widetilde{\Phi}\left(\frac{\rho(t)}{AF(r)t^n}\right) t^{n-1}\, dt \le \frac{1}{C_2}. \end{aligned} \tag{4.9}$$

Let $\lambda = C_2AA'F(r)$. Then, by (4.8) and (4.9) we have

$$\int_{|x-y|\ge r} \widetilde{\Phi}\left(\frac{\rho(|x-y|)}{\lambda|x-y|^n}\right) dy \le 1,$$

and so (4.7) holds. By (4.4), (4.5) and (4.7), we have

$$|I_\rho f(x)| = |J_1 + J_2| \le C\left(Mf(x) + \|f\|_\Phi \Phi^{-1}\left(\frac{1}{r^n}\right)\right) \int_0^r \frac{\rho(t)}{t}\, dt. \tag{4.10}$$

Choose $r > 0$ so that

$$\Phi^{-1}\left(\frac{1}{r^n}\right) = \frac{Mf(x)}{C_0\|f\|_\Phi}. \tag{4.11}$$

Then

$$\int_0^r \frac{\rho(t)}{t}\, dt \le A'' \frac{\Psi^{-1}\left(\frac{1}{r^n}\right)}{\Phi^{-1}\left(\frac{1}{r^n}\right)} = A'' \frac{\Psi^{-1} \circ \Phi\left(\frac{Mf(x)}{C_0\|f\|_\Phi}\right)}{\frac{Mf(x)}{C_0\|f\|_\Phi}}. \tag{4.12}$$

By (4.10), (4.11) and (4.12), we have

$$|I_\rho f(x)| \le C_1 \|f\|_\Phi \Psi^{-1} \circ \Phi\left(\frac{Mf(x)}{C_0\|f\|_\Phi}\right).$$

Therefore we have (3.3).

Let C_0 be as in (2.8). Then

$$\sup_{r>0}\Psi(r)\,m\left(r,\tfrac{|I_\rho f(x)|}{C_1\|f\|_\Phi}\right)=\sup_{r>0}r\,m\left(r,\Psi\left(\tfrac{|I_\rho f(x)|}{C_1\|f\|_\Phi}\right)\right)$$
$$\le\sup_{r>0}r\,m\left(r,\Phi\left(\tfrac{Mf(x)}{C_0\|f\|_\Phi}\right)\right)=\sup_{r>0}\Phi(r)\,m\left(r,\tfrac{Mf(x)}{C_0\|f\|_\Phi}\right)\le 1,$$

i.e.,

$$\|I_\rho f\|_{\Psi,weak}\le C_1\|f\|_\Phi.$$

Let C_0 be as in (2.9). Then

$$\int_{\mathbb{R}^n}\Psi\left(\frac{|I_\rho f(x)|}{C_1\|f\|_\Phi}\right)dx\le\int_{\mathbb{R}^n}\Phi\left(\frac{Mf(x)}{C_0\|f\|_\Phi}\right)dx\le 1,$$

i.e.,

$$\|I_\rho f\|_\Psi\le C_1\|f\|_\Phi.$$ ■

Proof of Corollary 3.2. Let $F(r)$ be as (4.6). By the almost decreasingness of $F(r)$, we have

$$F(t)\le CF(r)\quad\text{for } 0<r\le t<+\infty.$$

By (3.4) we have

$$\frac{1}{t^n}\ge\frac{\rho(t)}{C'\int_0^t(\rho(s)/s)\,ds\,t^n}.$$

From (4.1) and (4.2), it follows that

$$\begin{aligned}\widetilde{\Phi}\left(\frac{\rho(t)}{CC'F(r)t^n}\right)&\le\frac{F(t)}{CF(r)}\widetilde{\Phi}\left(\frac{\rho(t)}{C'F(t)t^n}\right)\\&=\frac{F(t)}{CF(r)}\widetilde{\Phi}\left(\frac{\rho(t)}{C'\int_0^t(\rho(s)/s)\,ds\,\Phi^{-1}(1/t^n)\,t^n}\right)\\&\le\frac{F(t)}{CF(r)}\widetilde{\Phi}\left(\frac{\dfrac{\rho(t)}{C'\int_0^t(\rho(s)/s)\,ds\,t^n}}{\Phi^{-1}\left(\dfrac{\rho(t)}{C'\int_0^t(\rho(s)/s)\,ds\,t^n}\right)}\right)\\&\le\frac{F(t)}{CF(r)}\frac{\rho(t)}{C'\int_0^t(\rho(s)/s)\,ds\,t^n}=\frac{1}{CC'F(r)}\frac{\rho(t)}{t^n}\Phi^{-1}\left(\frac{1}{t^n}\right).\end{aligned}$$

By (3.5), we have (3.1). Therefore this corollary follows from Theorem 3.1. ■

Lemma 4.1. *Let Φ be a Young function with* (2.7) *and $\widetilde{\Phi}$ be the complementary function with respect to Φ. Then there exists a constant $C > 0$ such that, for all $a \in \mathbb{R}^n$ and $r > 0$,*

$$\|\chi_{B(a,r)}\|_{\widetilde{\Phi}} \le C\Phi^{-1}\left(\frac{1}{r^n}\right) r^n.$$

Proof. Let $\lambda = \Phi^{-1}(1/|B(a,r)|)|B(a,r)|$. Then we have, by (4.2),

$$\begin{aligned}\int_{\mathbb{R}^n} \widetilde{\Phi}\left(\frac{\chi_{B(a,r)(x)}}{\lambda}\right) dx &= \int_{B(a,r)} \widetilde{\Phi}\left(\frac{1}{\lambda}\right) dx \\ &= |B(a,r)|\widetilde{\Phi}\left(\frac{\frac{1}{|B(a,r)|}}{\Phi^{-1}\left(\frac{1}{|B(a,r)|}\right)}\right) \le 1.\end{aligned}$$ ■

Proof of Theorem 3.3. First we note that there exists a constant $C > 0$ such that, for all $a \in \mathbb{R}^n$ and $r > 0$,

$$\left\|\frac{\rho(|a-\cdot|)}{|a-\cdot|^{n+1}}\chi_{B(a,r)^C}(\cdot)\right\|_{\widetilde{\Phi}} \le C\frac{1}{r}\int_0^r \frac{\rho(t)}{t}\,dt\ \Phi^{-1}\left(\frac{1}{r^n}\right). \tag{4.13}$$

We have this inequality (4.13) by (3.7) in a way similar to the proof of (4.7),

For any ball $B = B(a,r)$, let $\tilde{B} = B(a,2r)$ and

$$\begin{aligned}E_B(x) &= \int_{\mathbb{R}^n} f(y)\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n}\right) dy,\\ C_B &= \int_{\mathbb{R}^n} f(y)\left(\frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right) dy,\\ {E_B}^1(x) &= \int_{\tilde{B}} f(y)\frac{\rho(|x-y|)}{|x-y|^n}\,dy,\\ {E_B}^2(x) &= \int_{\tilde{B}^C} f(y)\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n}\right) dy.\end{aligned}$$

Then

$$\tilde{I}_\rho f(x) - C_B = E_B(x) = {E_B}^1(x) + {E_B}^2(x) \quad \text{for } x \in B.$$

By (2.6) we have

$$\left|\frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right| \le \begin{cases} C, & |y| \le 2|a|,\\ C|a|\frac{\rho(|y|)}{|y|^{n+1}}, & |y| \ge 2|a|.\end{cases}$$

From (4.3) and (4.13), it follows that C_B is well-defined. By (4.3), Lemma 4.1 and (3.8), we have

$$\begin{aligned}\int_{\tilde{B}}\left(\int_B |f(y)|\frac{\rho(|x-y|)}{|x-y|^n}\,dx\right)\,dy &\le \int_{\tilde{B}}|f(y)|\left(\int_{B(y,3r)}\frac{\rho(|x-y|)}{|x-y|^n}\,dx\right)\,dy\\ &\le \int_{\tilde{B}}|f(y)|\,dy\int_0^{3r}\frac{\rho(t)}{t}\,dt\\ &\le C\|f\|_\Phi\|\chi_{\tilde{B}}\|_{\tilde{\Phi}}\int_0^r\frac{\rho(t)}{t}\,dt\\ &\le C\|f\|_\Phi\Phi^{-1}\left(\frac{1}{r^n}\right)r^n\int_0^r\frac{\rho(t)}{t}\,dt\\ &\le C\phi(r)r^n\|f\|_\Phi.\end{aligned}$$

From Fubini's theorem, it follows that ${E_B}^1$ is well-defined and that

$$\int_B |{E_B}^1(x)|\,dx \le C\phi(r)r^n\|f\|_\Phi. \tag{4.14}$$

By (2.6) we have

$$\left|\frac{\rho(|x-y|)}{|x-y|^n}-\frac{\rho(|a-y|)}{|a-y|^n}\right| \le C\frac{|a-x|\rho(|a-y|)}{|a-y|^{n+1}}, \quad x\in B \text{ and } y\in \tilde{B}^C.$$

From (4.3), (4.13) and (3.8), it follows that ${E_B}^2$ is well-defined and

$$|{E_B}^2(x)| \le C\phi(r)\|f\|_\Phi. \tag{4.15}$$

By (4.14) and (4.15), we have

$$\frac{1}{|B|}\int_B |\tilde{I}_\rho f(x)-C_B|\,dx \le C\phi(r)\|f\|_\Phi,$$

and

$$\|\tilde{I}_\rho f\|_{\mathrm{BMO}_\phi} \le C\|f\|_\Phi. \qquad \blacksquare$$

Lemma 4.2. *If ρ satisfies* (2.1), (2.2), (2.5) *and* (2.6), *then*

$$\frac{\rho(|x_1-y|)}{|x_1-y|^n}-\frac{\rho(|x_2-y|)}{|x_2-y|^n} \tag{4.16}$$

is integrable on $\mathbb{R}^n$ as a function of y and the value is equal to 0 for every choice of x_1 and x_2.

Proof. Let $r = |x_1 - x_2|$. For large $R > 0$, let

$$J_1 = \int_{B(x_1,R)} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} \, dy - \int_{B(x_2,R)} \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \, dy,$$

$$J_2 = \int_{B(x_1,R+r)\setminus B(x_1,R)} \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} \, dy - \int_{B(x_1,R+r)\setminus B(x_2,R)} \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \, dy,$$

$$J_3 = \int_{B(x_1,R+r)^C} \left(\frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) dy.$$

Then

$$J_1 + J_2 + J_3 = \int_{\mathbb{R}^n} \left(\frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) dy.$$

From (2.1), it follows that $\frac{\rho(|x_i - y|)}{|x_i - y|^n}$ $(i = 1, 2)$ are in $L^1_{\text{loc}}(\mathbb{R}^n)$ and that $J_1 = 0$. By (2.6) we have

$$\begin{aligned} &\int_{B(x_1,R+r)^C} \left| \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right| dy \\ &\leq \int_{B(x_1,R+r)^C} A_3 r \frac{\rho(|x_1 - y|)}{|x_1 - y|^{n+1}} \, dy = Cr \int_{R+r}^{+\infty} \frac{\rho(t)}{t^2} \, dt. \end{aligned}$$

From (2.5) it follows that (4.16) is integrable and that $|J_3| \to 0$ as $R \to +\infty$. By (2.2) and (2.5), we have

$$\begin{aligned} |J_2| &\leq \int_{B(x_1,R+r)\setminus B(x_1,R-r)} \left(\frac{\rho(|x_1 - y|)}{|x_1 - y|^n} + \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) dy \\ &\sim ((R+r)^n - (R-r)^n) \frac{\rho(R)}{R^n} \leq Cr \frac{\rho(R)}{R} \to 0 \quad \text{as } R \to +\infty. \end{aligned}$$ ■

Lemma 4.3. *Under the assumption in Theorem* 3.4, *there exists a constant* $C > 0$ *such that, for all* $a \in \mathbb{R}^n$ *and* $r > 0$,

$$\int_{B(a,r)^C} \frac{\rho(|a - y|)}{|a - y|^{n+1}} \left| f(y) - f_{B(a,r)} \right| \, dy \leq C \frac{\rho(r)\phi(r)}{r} \|f\|_{\text{BMO}_\phi}.$$

Proof. By (2.2) we have

$$\begin{aligned} |f_{B(a,2^k r)} - f_{B(a,2^{k+1} r)}| &\leq \frac{1}{|B(a, 2^k r)|} \int_{B(a,2^k r)} |f(y) - f_{B(a,2^{k+1} r)}| \, dy \\ &\leq \frac{1}{|B(a, 2^k r)|} \int_{B(a,2^{k+1} r)} |f(y) - f_{B(a,2^{k+1} r)}| \, dy \\ &\leq 2^n \phi(2^{k+1} r) \|f\|_{\text{BMO}_\phi} \\ &\leq C \int_{2^k r}^{2^{k+1} r} \frac{\phi(s)}{s} \, ds \|f\|_{\text{BMO}_\phi}, \end{aligned}$$

for $k = 0, 1, \cdots, j-1$, and so

$$\begin{aligned}
&\frac{1}{|B(a,2^j r)|}\int_{B(a,2^j r)} |f(y) - f_{B(a,r)}|\,dy \\
&\le \frac{1}{|B(a,2^j r)|}\int_{B(a,2^j r)} |f(y) - f_{B(a,2^j r)}|\,dy + |f_{B(a,r)} - f_{B(a,2^j r)}| \\
&\le C\int_r^{2^j r} \frac{\phi(s)}{s}\,ds \|f\|_{\mathrm{BMO}_\phi}.
\end{aligned}$$

Hence, using (2.5) and (3.9), we have

$$\begin{aligned}
&\int_{B(a,r)^C} \frac{\rho(|a-y|)}{|a-y|^{n+1}} \left|f(y) - f_{B(a,r)}\right|\,dy \\
&= \sum_{j=1}^{+\infty} \int_{2^{j-1}r \le |a-y| \le 2^j r} \frac{\rho(|a-y|)}{|a-y|^{n+1}} \left|f(y) - f_{B(a,r)}\right|\,dy \\
&\le C\sum_{j=1}^{+\infty} \frac{\rho(2^j r)}{(2^j r)^{n+1}} \int_{B(a,2^j r)} \left|f(y) - f_{B(a,r)}\right|\,dy \\
&\le C\sum_{j=1}^{+\infty} \frac{\rho(2^j r)}{2^j r} \int_r^{2^j r} \frac{\phi(s)}{s}\,ds\,\|f\|_{\mathrm{BMO}_\phi} \sim \int_r^{+\infty} \frac{\rho(t)}{t^2}\left(\int_r^{2t} \frac{\phi(s)}{s}\,ds\right)dt\,\|f\|_{\mathrm{BMO}_\phi} \\
&= \int_r^{+\infty}\left(\int_{s/2}^{+\infty} \frac{\rho(t)}{t^2}\,dt\right)\frac{\phi(s)}{s}\,ds\,\|f\|_{\mathrm{BMO}_\phi} \\
&\le C\int_r^{+\infty} \frac{\rho(s)}{s}\frac{\phi(s)}{s}\,ds\,\|f\|_{\mathrm{BMO}_\phi} \le C\frac{\rho(r)\phi(r)}{r}\,\|f\|_{\mathrm{BMO}_\phi}.
\end{aligned}$$

■

Proof of Theorem 3.4. For any ball $B = B(a,r)$, let $\tilde{B} = B(a,2r)$ and

$$\begin{aligned}
E_B(x) &= \int_{\mathbb{R}^n} (f(y) - f_{\tilde{B}})\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n}\right)dy, \\
{C_B}^1 &= \int_{\mathbb{R}^n} (f(y) - f_{\tilde{B}})\left(\frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right)dy, \\
{C_B}^2 &= \int_{\mathbb{R}^n} f_{\tilde{B}}\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right)dy, \\
{E_B}^1(x) &= \int_{\tilde{B}} (f(y) - f_{\tilde{B}})\frac{\rho(|x-y|)}{|x-y|^n}\,dy, \\
{E_B}^2(x) &= \int_{\tilde{B}^C} (f(y) - f_{\tilde{B}})\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n}\right)dy.
\end{aligned}$$

Then

$$\tilde{I}_\rho f(x) - ({C_B}^1 + {C_B}^2) = E_B(x) = {E_B}^1(x) + {E_B}^2(x) \quad \text{for } x \in B.$$

By (2.6) we have

$$\left| \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n} \right|$$
$$\leq \begin{cases} C, & |a-y| \leq \max(2|a|, 2r), \\ C|a| \frac{\rho(|a-y|)}{|a-y|^{n+1}}, & |a-y| \geq \max(2|a|, 2r). \end{cases}$$

From Lemma 4.3, it follows that ${C_B}^1$ is well-defined. By Lemma 4.2 and (2.1), we have

$$\int_{\mathbb{R}^n} \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n} \right) dy$$
$$= \int_{\mathbb{R}^n} \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)}{|y|^n} \right) dy + \int_{B_0} \frac{\rho(|y|)}{|y|^n} dy = C.$$

By (3.10) we have

$$\int_{\tilde{B}} \left(\int_B |f(y) - f_{\tilde{B}}| \frac{\rho(|x-y|)}{|x-y|^n} dx \right) dy$$
$$\leq \int_{\tilde{B}} |f(y) - f_{\tilde{B}}| \left(\int_{B(y,3r)} \frac{\rho(|x-y|)}{|x-y|^n} dx \right) dy$$
$$\leq \int_{\tilde{B}} |f(y) - f_{\tilde{B}}| \, dy \int_0^{3r} \frac{\rho(t)}{t} dt$$
$$\leq C\|f\|_{\mathrm{BMO}_\phi} r^n \phi(r) \int_0^r \frac{\rho(t)}{t} dt$$
$$\leq C\|f\|_{\mathrm{BMO}_\phi} r^n \psi(r).$$

From Fubini's theorem it follows that ${E_B}^1$ is well-defined and that

(4.17) $$\int_B |{E_B}^1(x)| \, dx \leq C\psi(r) r^n \|f\|_{\mathrm{BMO}_\phi}.$$

From (2.6), Lemma 4.3 and (3.10), it follows that ${E_B}^2$ is well-defined and

(4.18) $$|{E_B}^2(x)| \leq C\psi(r) \|f\|_{\mathrm{BMO}_\phi}.$$

By (4.17) and (4.18), we have

$$\frac{1}{|B|} \int_B |\tilde{I}_\rho f(x) - ({C_B}^1 + {C_B}^2)| \, dx \leq C\psi(r) \|f\|_{\mathrm{BMO}_\phi},$$

and

$$\|\tilde{I}_\rho f\|_{\mathrm{BMO}_\psi} \le C\|f\|_{\mathrm{BMO}_\phi}. \qquad \blacksquare$$

References

1. I. Genebashvili, A. Gogatishvili, V. Kokilashvili and M. Krbec, *Weight Theory for Integral Transforms on Spaces of Homogeneous Type*, Longman, Harlow, 1998.

2. A. E. Gatto and S. Vági, *Fractional integrals on spaces of homogeneous type*, in: *Analysis and Partial Differential Equations*, Cora Sadosky, ed., Marcel Dekker, New York, 1990, pp. 171-216.

3. V. Kokilashvili and M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Scientific, Singapore, New Jersey, London and Hong Kong, 1991.

4. M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, Inc., New York, Basel and Hong Kong, 1991.

5. B. Rubin, *Fractional Integrals and Potentials*, Addison Wesley Longman Limited, Essex, 1996.

6. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

7. E. M. Stein, *Harmonic Analysis, Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.

8. A. Torchinsky, *Interpolation of operations and Orlicz classes, Studia Math*. **59** (1976), 177-207.

Department of Mathematics, Osaka Kyoiku University
Kashiwara, Osaka 582-8582, Japan
E-mail: enakai@cc.osaka-kyoiku.ac.jp